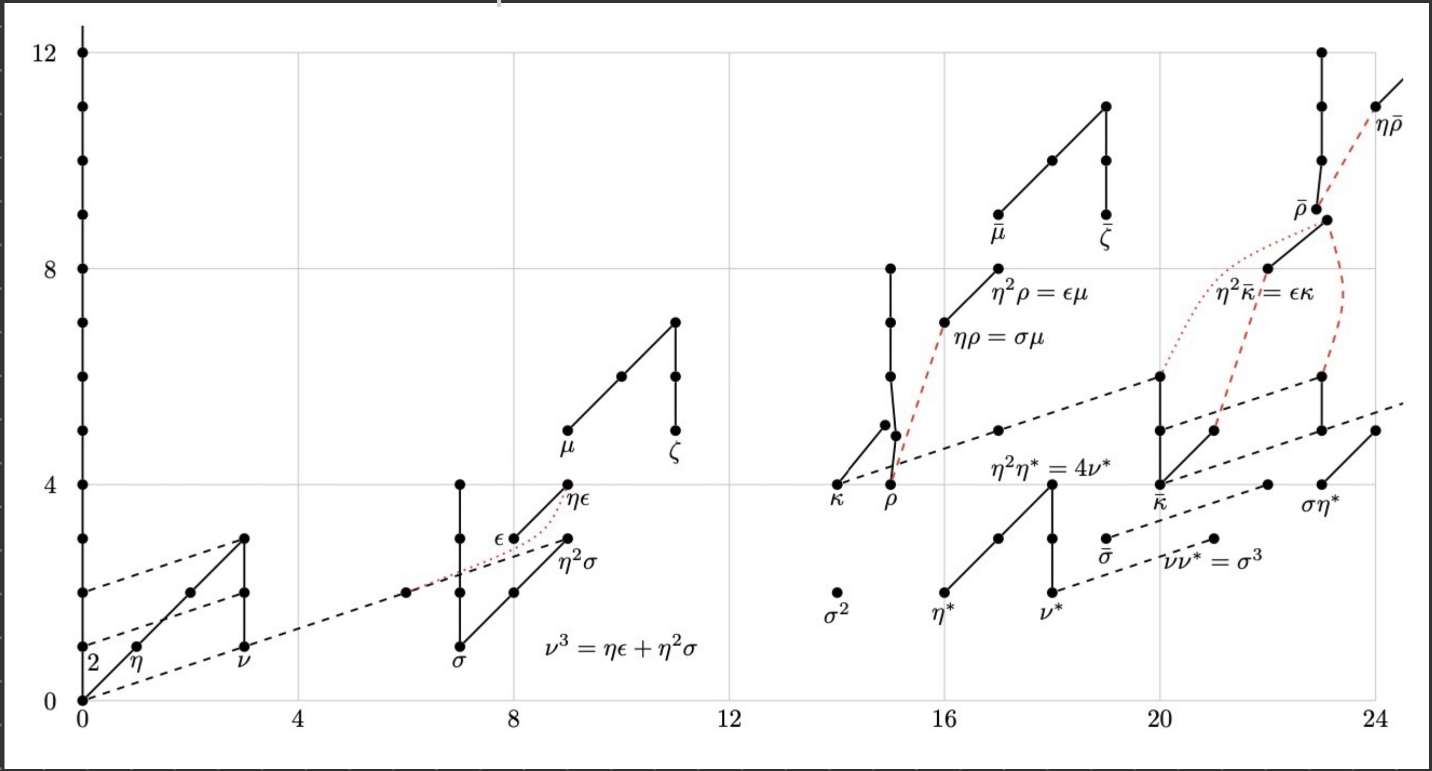
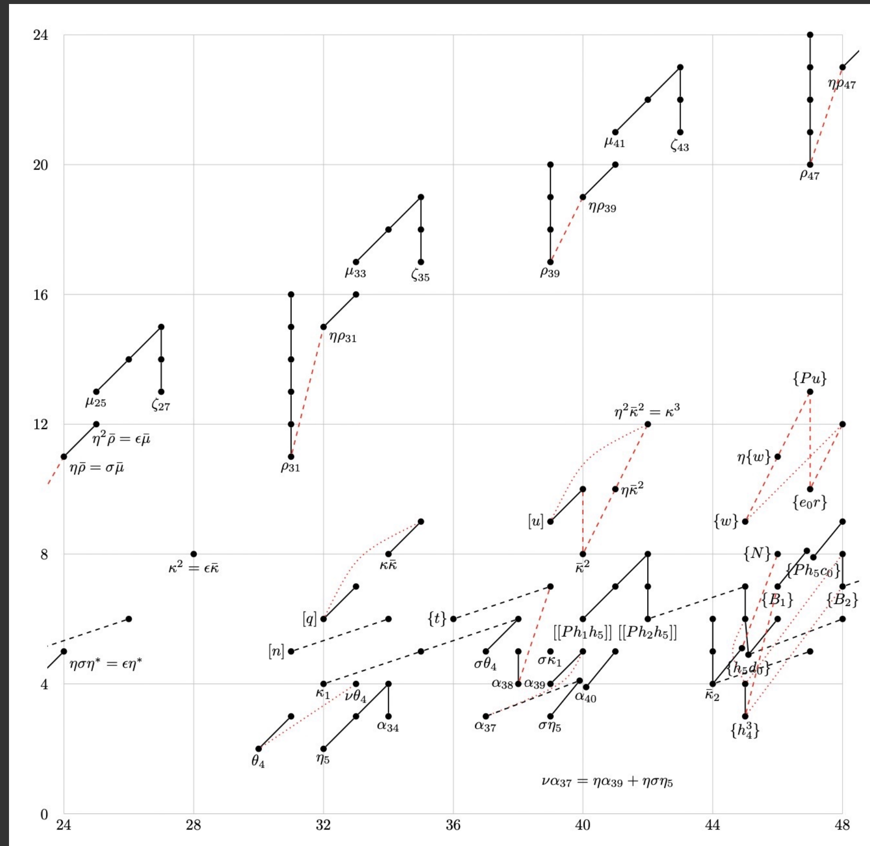
Some homotopy groups of $\mathbb{S}$

18. IV. 23

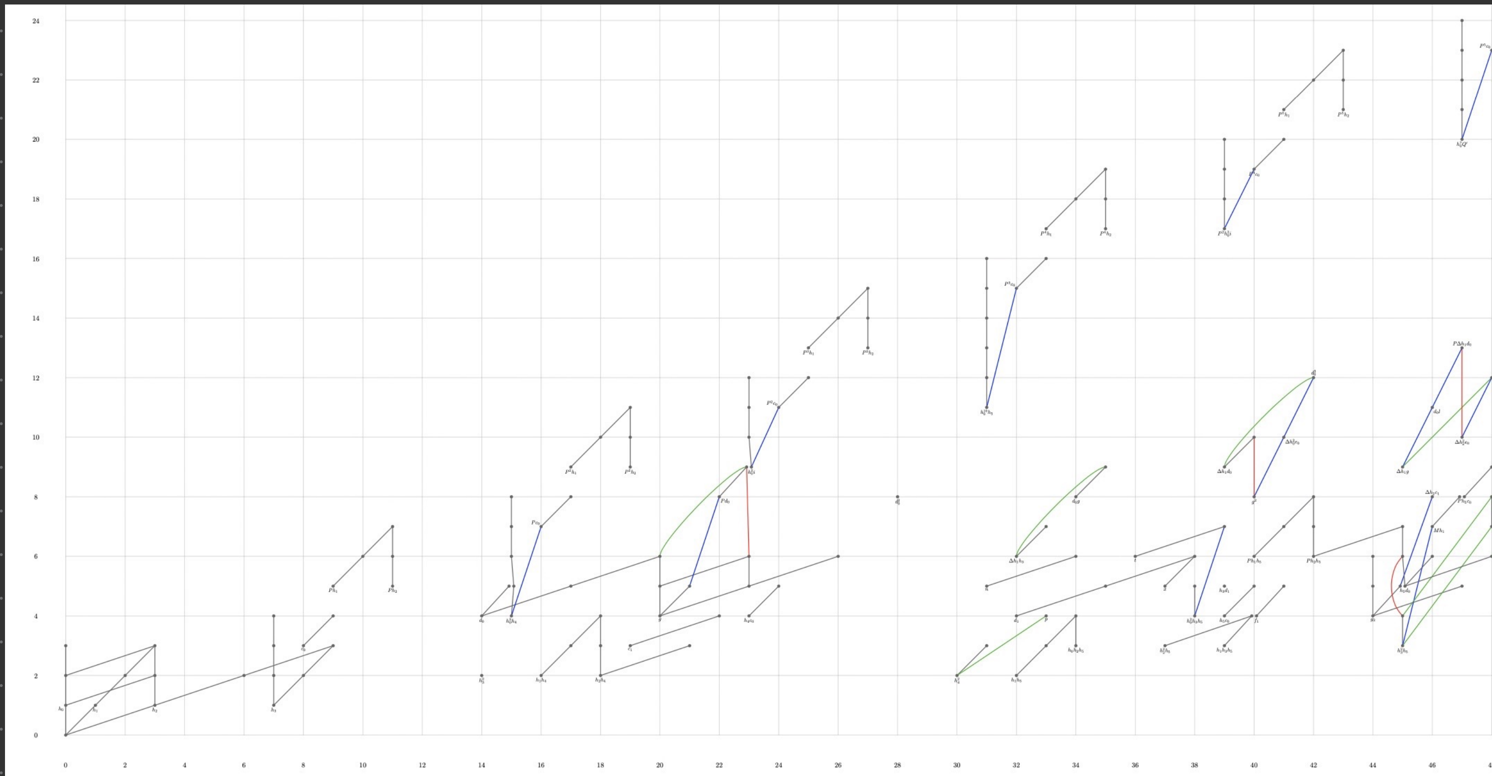
I. Understand this picture:



II. And this one:

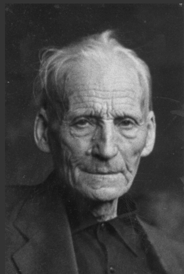


III. So that we ultimately understand



and its ring structure.

0. History



LEJ Brouwer:

1911

degree thy

Heinz Hopf:

1927

$\pi_m(S^m) = \mathbb{Z}$

1931

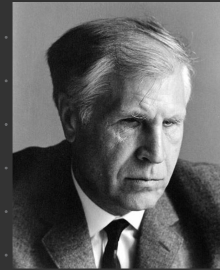
η, ν, σ essential

Hans Freudenthal:

1938

stable stems

$\pi_1 S = \mathbb{Z}/2$

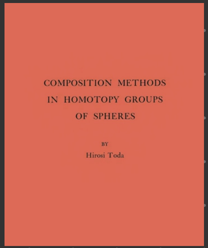


Lev Pontryagin,

George W. Whitehead:

1950

$\pi_2 S = \mathbb{Z}/2$



Hiroshi Toda,

JP Serre

1952-58

$\pi_n S$, $3 \le n \le 13$

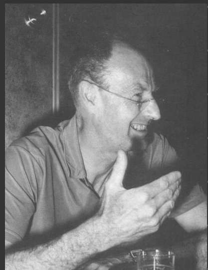
(Toda 1962: $n \le 19$)

Mamoru Mimura

1963 w/ Toda $\pi_{20} S$

1965

$\pi_{21} S, \pi_{22} S$ $(\varepsilon\kappa \neq 0)$



J. Frank Adams

1958

Adams spectral sequence

J. Peter May

1965

$n \le 28$ except $n = 23$

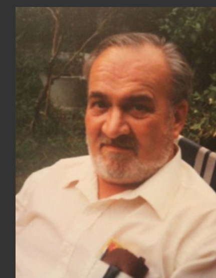
MT MM

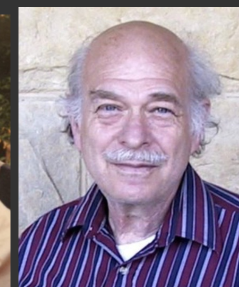
Mark Mahowald, Martin Tangora

1967

$n \le 37$, $n \in \{39, 42, 43, 44\}$

but....





Differentials corrected w/ Michael Barratt (1970) and by R. James Milgram (1972) $\Rightarrow n \le 44$ except 37, 38

Bob Bruner

1984

New Adams differential

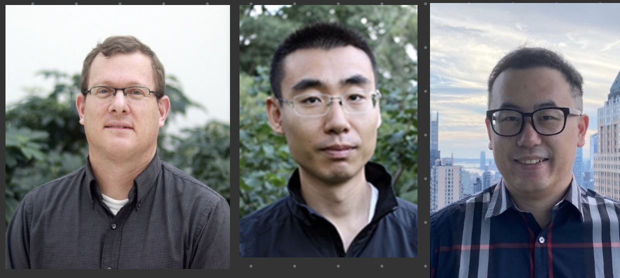
$\Rightarrow \pi_{37} S, \pi_{38} S$

Stanley Kochman

1990

$n \le 53$ except 51, $58 \le n \le 60$;

$n = 55$ corrected with Mahowald (1995)

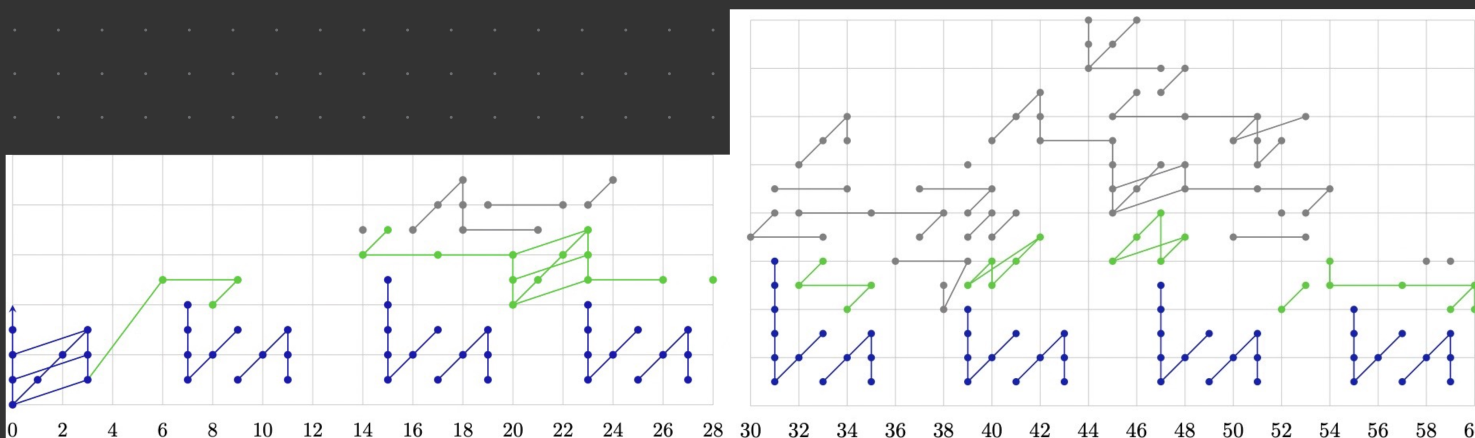


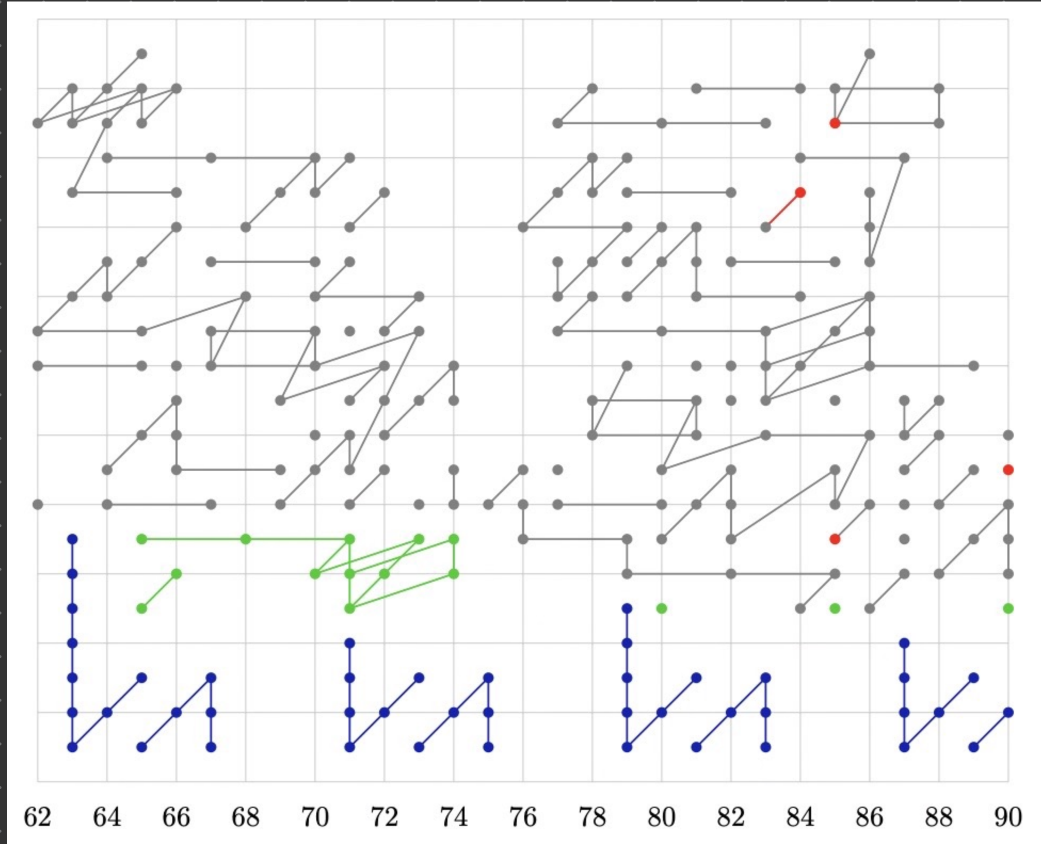
Dan Isaksen, Guozhen Wang, Zhouli Xu

2015
new Adams diff'ls
for 51, 56 stems

2017–18
$\pi_{51}S$, $\pi_{61}S$
so $n \leq 61$

All three, arXiv 2020: $n \leq 90$ except for some very precise uncertainties in $n = 84, 85, 90$

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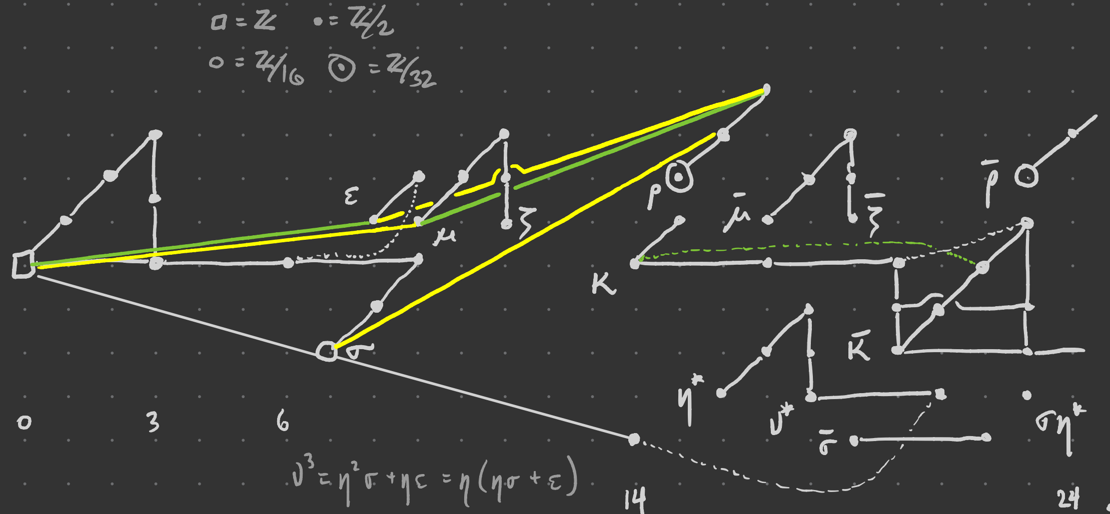
I. Thm

The ring structure of $\pi_* S$ for $* \leq 24$ is given by



$$\nu^3 = \eta^2\sigma + \eta\varepsilon = \eta(\eta\sigma + \varepsilon)$$

Method

(a) We already know $E_5(S) = E_\infty(S)$ for the Adams spectral sequence in this range.

(b) We know $\pi_* j$ and $\pi_* tmf$ in this range.

(c) Consider the ring maps $\begin{array}{c}\pi_* j\\ {\scriptstyle e}\uparrow\\ \pi_* S\\ {\scriptstyle \iota}\downarrow\\ \pi_* tmf\end{array}$ to deduce product structure on $\pi_* S$.

Recall Fiber sequence $j \longrightarrow ko \xrightarrow{\psi^3-1} bspin$

LEMMA 11.46. *The map $e\colon S \to j$ is (at least) 2-connected, and for $n \geq 2$*

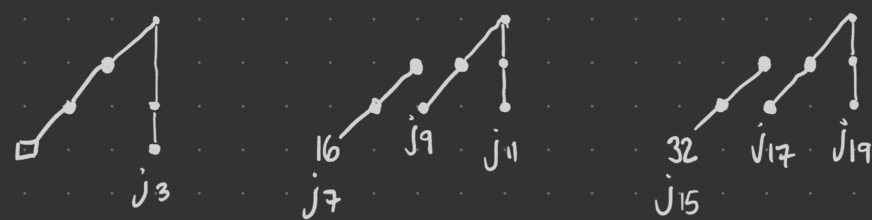
$$\pi_n(j) = \begin{cases} \mathbb{Z}_2/(16k)\{j_{8k-1}\} & \text{for } n = 8k-1, \\ \mathbb{Z}/2\{\eta j_{8k-1}\} & \text{for } n = 8k, \\ \mathbb{Z}/2\{\eta^2 j_{8k-1}\} \oplus \mathbb{Z}/2\{j_{8k+1}\} & \text{for } n = 8k+1, \\ \mathbb{Z}/2\{\eta j_{8k+1}\} & \text{for } n = 8k+2, \\ \mathbb{Z}/8\{j_{8k+3}\} & \text{for } n = 8k+3, \\ 0 & \text{otherwise}, \end{cases}$$

with $\nu j_{8k-1} = 0$ and $\eta^2 j_{8k+1} = 4 j_{8k+3}$.

Furthermore,
$$\begin{tikzcd} & S \arrow[dl, "e"'] \arrow[d] \arrow[dr, "0"] & \\ j \arrow[r] & ko \arrow[r, "\psi^3 - 1"'] & bspin \end{tikzcd}$$
induces

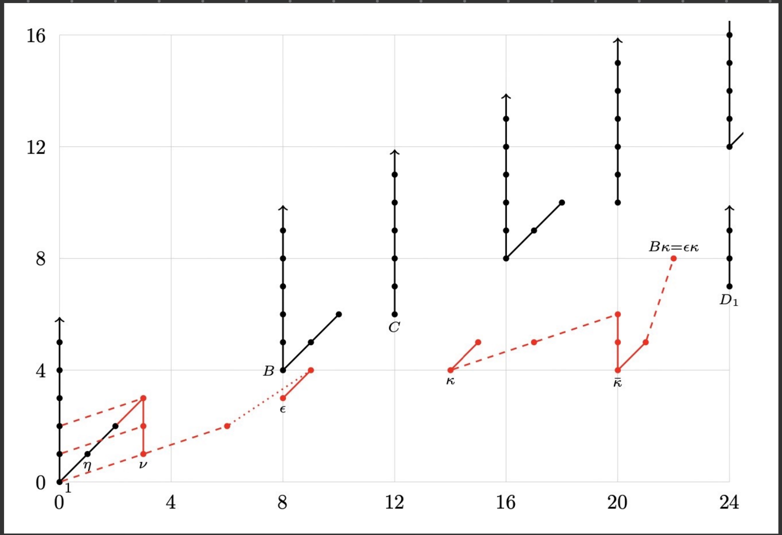
$e : \pi_* S \longrightarrow \pi_* j$ surjective with additive section, and:

PROPOSITION 11.49 (cf. [8, Prop. 12.14 and Ex. 12.15]). *The products of the j_n are given as follows: $j_{8k-1} \cdot j_{8\ell+1} = j_{8k+1} \cdot j_{8\ell-1} = \eta j_{8(k+\ell)-1}$, $j_{8k+1} \cdot j_{8\ell+1} = \eta j_{8(k+\ell)+1}$, and the remaining products are zero.*



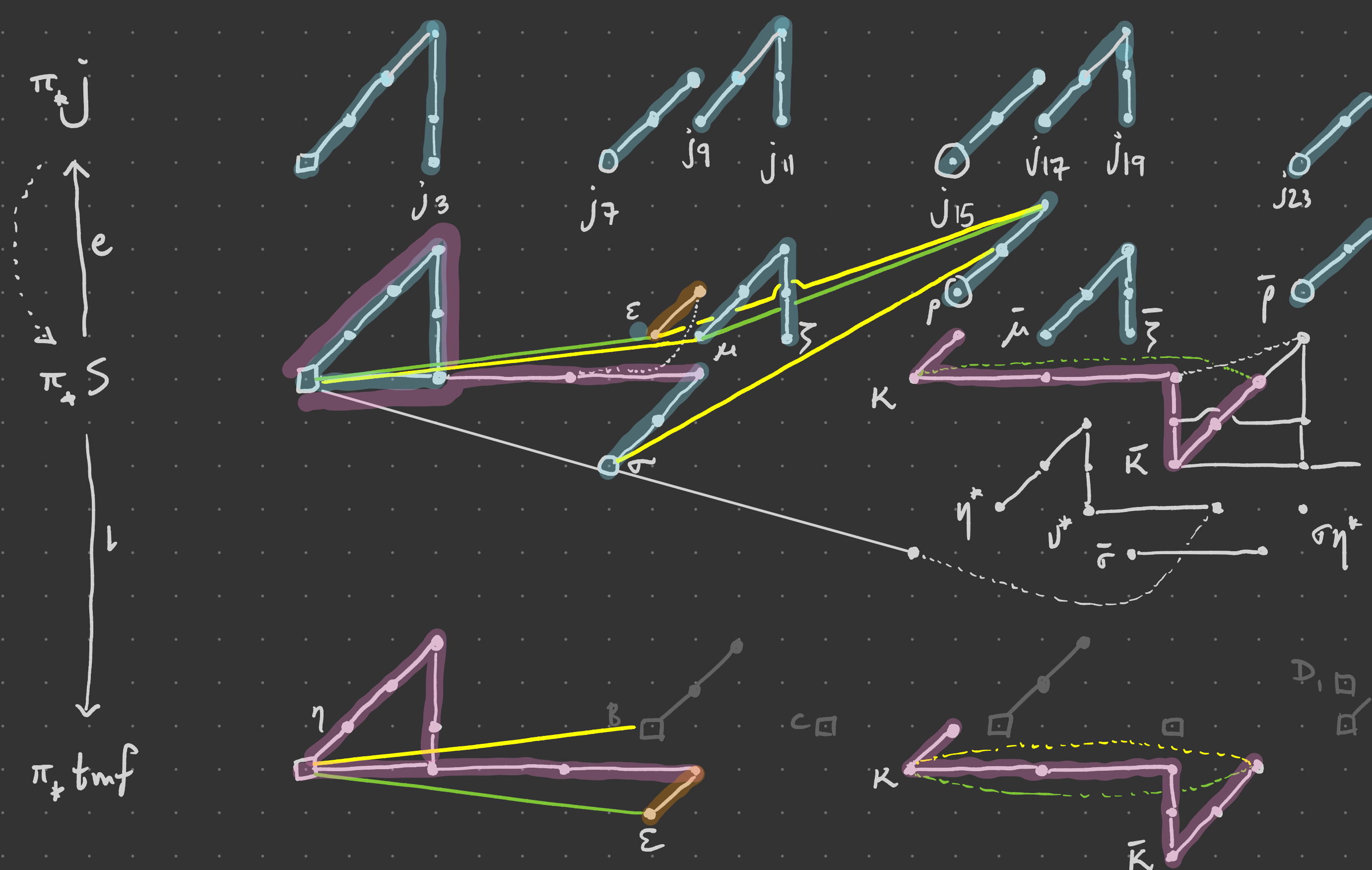
Slogan: $j_7 j_1 = j_1 j_7 = \eta j_7$, $j_1^2 = \eta j_1$ (indices mod 8)

Also recall $\pi_* tmf$ ring structure:

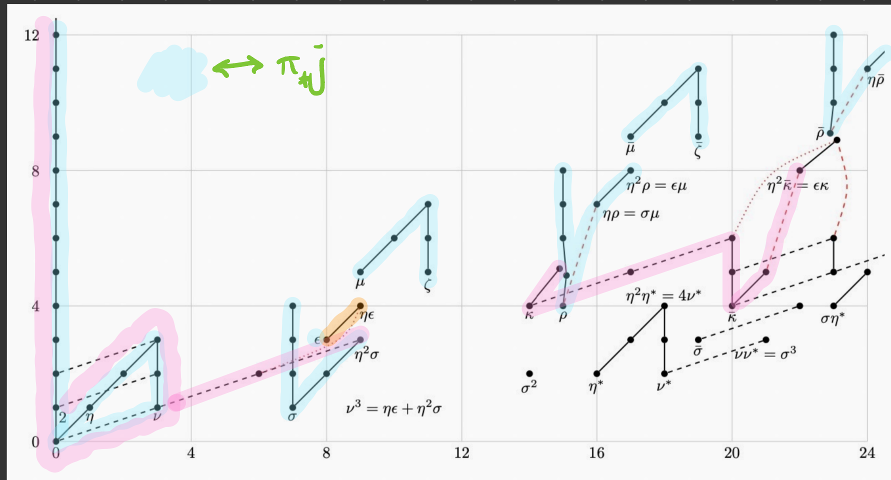


In "constellation form", this becomes:

Now compare $\pi_* j \underset{e}{\hookleftarrow} \pi_* S \longrightarrow \pi_* tmf$:



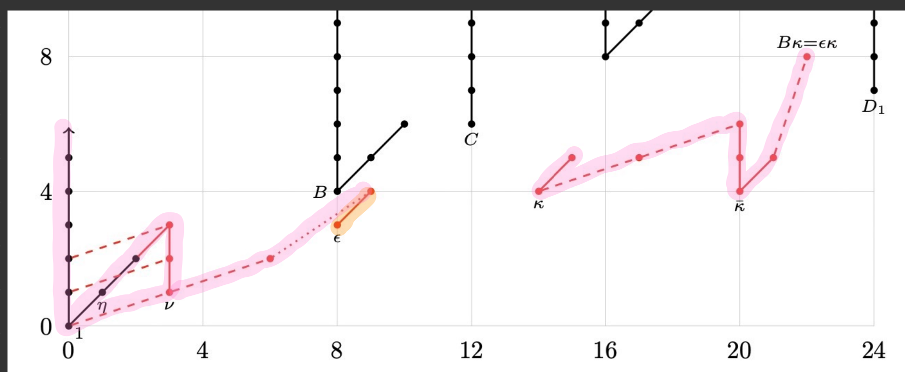
In terms of Adams chart:



$\pi_* S$

$\downarrow \iota$

$\pi_* tmf$



$\pi_9 S = \mathbb{Z}/2\{\mu, \eta\varepsilon, \eta^2\sigma\}$

with $e(\mu) = j_9$

$e(\eta^2\sigma) = \eta^2 j_7$.

By $E_2(S)$ ring structure,

$\nu^3 = \eta^2\sigma + \text{h.o.t.}$

Thus $\nu^3 = x\mu + y\eta\varepsilon + \eta^2\sigma$ for some $x, y \in \{0,1\}$.

$e(\nu^3) = \nu e(\nu^2) = 0$ so

$0 = e(x\mu + y\eta\varepsilon + \eta^2\sigma)$

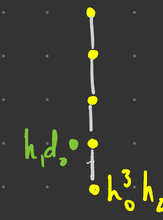
$= x j_9 + y\eta e(\varepsilon) + \eta^2 j_7$

$\Rightarrow x = 0, y = 1,$

$\eta e(\varepsilon) = \eta^2 j_7$.

Thus $\nu^3 = \eta\varepsilon + \eta^2\sigma$. $\square$

$\pi_{15}S$: For $* \le 14$, classical arguments with j suffice for ring structure. tmf starts to shine at $\pi_{15}S$.



- $E_\infty(S)\big|_{t-s=15} = \mathbb{F}_2\{h_0^k h_4 \mid 3 \le k \le 7\} \oplus \mathbb{F}_2\{h_1 d_0\}$
- $h_0^3 h_4$ detects ρ, $h_1 d_0 = \eta\kappa$ with $2 \cdot \eta\kappa = 0$

$\implies \pi_{15}S = \mathbb{Z}/2\{\eta\kappa\} \oplus \mathbb{Z}/32\{\rho\}$ with ρ determined modulo $(\eta\kappa, 2\rho)$

- Fix a choice of ρ with the following equivalent conditions:
  - $\varepsilon \cdot \rho = 0 \in \pi_{23}S$
  - $\iota(\rho) = 0 \in \pi_{15}\,\mathrm{tmf} = \mathbb{Z}/2\{\eta\kappa\}$

  $\Big\}$ determine ρ up to an odd multiple

  (Argument with $\mathrm{tmf}/S \implies (\iota(\rho) = 0 \Rightarrow \varepsilon\rho = 0)$)

  ($\eta\varepsilon\kappa \neq 0 \in \pi_{23}S \implies (\varepsilon\rho = 0 \Rightarrow \iota(\rho) = 0)$)

  - $\rho := J(\text{gen } \pi_{15} SO)$

- $e(\rho) \doteq j_{15}$ generates $\pi_{15} j = \mathbb{Z}/32\{j_{15}\}$
- Now check ring structure: $\sigma\varepsilon = 0$
  - $\eta\sigma^2 = 0$ by quadratic construction on $\sigma : S^7 \to S^4$.
  - $e(\sigma\varepsilon) = \eta\sigma j_7 = 0 \in \pi_{15}(j) \Rightarrow \sigma\varepsilon \in \ker(e) = \{0, \eta\kappa\}$.
  - Have $\iota(\sigma) = 0$ and $\iota(\eta\kappa) \neq 0$, so $\sigma\varepsilon \neq \eta\kappa$. $\square$

Fun with $\varepsilon\kappa \in \pi_{22} S$:

- $\nu^2\kappa = 4\bar{\kappa} \Rightarrow \nu^3\kappa = 4\nu\bar{\kappa}$ detected by $h_1 Pd_0$
- $\nu^3 = \eta\varepsilon + \eta^2\sigma$ & $\eta^2\sigma\kappa = 0 \Rightarrow \nu^3\kappa = \eta\varepsilon\kappa = 4\nu\bar{\kappa} \neq 0$
- In particular, $\varepsilon\kappa \neq 0$ detected by Pd_0, hence $\varepsilon\kappa = \eta^2\bar{\kappa}$.