NOTES ON THE THEORY OF REPRESENTATIONS

OF FINITE GROUPS

Part I: The Burnside Ring of a Finite Group and some AGN-applications

by

Andreas W.M. Dress

(with the aid of lecture notes, taken by Manfred Küchler)

Bielefeld, December 1971

Content

Preface

This is the first of a series of altogether three or four parts of lecture notes on some aspects of the representationtheory of finite groups. It contains a study of the prime ideals of the Burnside ring (Grothendieck ring of permutation-representations) of a finite group (Chapter I) and applications of its results to the study of induced representations, which are developed in the (categorical) framework of the theory of Mackey-functors (Chapter II). As an appendix I have included two papers, in which our theory is applied to the study of Witt rings. They have been written already in spring 1971 (about half a year before these notes, on the occasion of the 60^th birthday of Ernst Witt) and thus can be read more or less independently of Chapter I and II. It may even be reasonable to look at them first, to get a first impression of some more concrete aspects of our theory. After having developed the fundamental concepts in the first part we will study a new technique of multiplicative inductionprocesses in the second part - and this study will indeed be rather technical. In the third part we will apply our fundamental concepts (Part I) and our technique of multiplicative induction maps (Part II) to the study of (especially integral and modular) group respresentations. In a possible fourth part we may incorporate in our study also the theory of monomial permutation representations and its applications, as well as some further special results on various aspects of the structure of the Burnside ring.

Bielefeld, den 1.12.1971 Andreas W.M. Dress

§ 1 G-sets

Generally groups occur as symmetry- or automorphism-groups of various (intra- or extramathematically) structured objects. Thus they occur together with a natural action on something. In the most simple case of course this something is a finite set. This is the situation, we want to consider now, but from a slightly different point of view, starting with an abstractly given group and considering arbitrary finite sets, on which this group acts.

So let G be a finite group. A G-set is a finite set S, on which G acts from the left by permutations, i.e. we have a map $G \times S \to S : (g, s) \mapsto gs$ with $es = s$ (e the neutral element in G) and $g(hs) = (gh)s$ for all $g, h \in G, s \in S$.

Because for any $g \in G$ the map $g: S \to S: s \mapsto gs$ has an (right- and left-) inverse, given by $g^{-1}: S \to S: s \mapsto g^{-1}s$, all these maps are automorphisms of S.

There are many natural examples of G-sets:

For instance the group G itself can be considered as a G-set in two different ways, either by leftmultiplication: $G \times G \to G : (g, x) \mapsto gx$ or by conjugation: $G \times G \to G : (g, x) \mapsto g x g^{-1}$.

The first case can be generalized by considering for

any subgroup $U \leqslant G$ the set G/U of left cosets $xU \subseteq G$ $(x \in G)$ with $G \times G/U \to G/U$ defined by $(g, xU) \mapsto gxU$.
For $U = E$ we can identify G/E with G, where G is considered as a G-set via left-multiplication. Instead of G/G we write also $*$ or $*_G$ and call this the trivial G-set. We also write $*_U$ for the subgroup $U \leqslant G$ considered as an element in G/U.

Furthermore there are many natural ways, to build up new G-sets out of given ones:
For any two G-sets S and T the disjoint union $S \dot{\cup} T$ is a G-set in a natural way as well as the cartesian product $S \times T$ (with diagonal G-action: $g(s, t) = (gs, gt)$).
Next looking at the definition of the G-set G/U as a set of subsets of G $(= G/E)$ one may be led to consider quite generally the set $\mathfrak{P}(S)$ of all subsets of a given G-set S or the set $\Lambda^i(S)$ of all subsets $T \subseteq S$ with exactly i elements $(|T| = i)$ as G-sets by defining $gT = \{gt | t \in T\}$ for any $g \in G$, $T \subseteq S$. (Of course $|T| = |gT|$, so $\Lambda^i(S)$ is indeed a welldefined G-set).
Finally for any two G-sets S and T one can consider the set S^T of all set-theoretic maps of T into S as a G-set, if one defines $G \times S^T \to S^T : (g,f) \mapsto gf$ by $(gf)(t) = g(f(g^{-1}t))$ for all $g \in G$, $f \in S^T$, $t \in T$.

Of course all these constructions are related to each other in many ways. For instance the canonical interpretation of S^T as a subset of $\mathfrak{P}(T \times S)$ (identify $f \in S^T$ with its "graph" $\Gamma_f = \{(t, f(t)) \in T \times S \mid t \in T\} \subseteq T \times S$) is compatible with the G-structures on both sets and thus makes the last case appear as a special case of foregoing one , on the other hand for $S = * \dot{\cup} *$ one can identify S^T with $\mathfrak{P}(T)$, so the definition of S^T seems to be the more general one.

But instead of playing around like this, to find out the most fundamental constructions and examples, it seems more appropriate to study the properties of the above examples and constructions and to clarify their relations systematically by considering G-sets as objects in a category. This will be done in the next section.

But before we come to this, I want to prove at least one proposition in this section. For this purpose I have to introduce one more notion: the orbitspace $G\backslash S$ of a G-set S is the set of equivalence-classes with respect to the equivalence-relation $\overset{G}{\sim}$ on S, defined by: $s \overset{G}{\sim} s' \Leftrightarrow \exists g \in G : gs = s'$ (prove, that this is indeed an equivalence-relation!). Mapping any element $s \in S$ onto its equivalence-class in $G\backslash S$ defines a map $p : S \to G\backslash S$ with $p(gs) = p(s)$ for all $g \in G$ and it is easy to see, that any G-invariant map $f : S \to M$ into some set M (i.e. a map with $f(gs) = f(s)$ for all $g \in G$) factors uniquely through $p : S \to G\backslash S$.

We want to prove:

Proposition 1.1: The orbitspace of $G/U \times G/V$ (U, V arbitrary subgroups of G) can be naturally identified with the set $D(U, V) = \{UgV \mid g \in G\}$ of double cosets of U and V in G.

Proof: Define $f : G/U \times G/V \to D(U, V)$ by $f(gU, hV) = Ug^{-1}hV$. f is well defined, G-invariant ($f(xgU, xhV) = Ug^{-1}x^{-1}xhV = Ug^{-1}hV = f(gU, hV)$) and obviously surjective, thus it defines a surjective map

$$G\backslash(G/U \times G/V) \to D(U, V)$$

which remains to be shown injective, i.e. we have to show:

$f(gU, hV) = f(g'U, h'V) \Rightarrow \exists\ x \in G : xgU = g'U,\ xhV = h'V.$

But we have:

$f(gU, hV) = f(g'U, h'V) \Rightarrow Ug^{-1}hV = Ug'^{-1}h'V \Rightarrow$

$\exists\ u \in U,\ v \in V$ with $ug^{-1}hv = g'^{-1}h'$. Choose

$x = g'ug^{-1} = h'v^{-1}h^{-1}$, then one has

$xgU = g'ug^{-1}gU = g'U$, $xhV = h'v^{-1}h^{-1}hV = h'V$ q.e.d.

Remark: In almost all cases in group-theory, where double coset-sets are considered (e.g. the Mackey-subgroup-theorem in representation-theory), the interpretation of those sets as G-orbits in G-sets of type $G/U \times G/V$ can be quite useful. Moreover this observation was one of the starting points for the development of the "axiomatic representation-theory", considered in chapter 2 .

§ 2 The category of G-sets

For a finite group G let $G^\wedge$ be the category, whose objects are just the G-sets as defined in § 1 and whose morphisms are the G-equivariant or just G-maps, i.e. for any two G-sets S and T we have

$[S, T]_{G^\wedge} = \mathrm{Hom}_G(S, T) = \{\varphi : S \to T \mid \varphi(gs) = g\varphi(s)$ for all $g \in G, s \in S\}$ for the set of $G^\wedge$-morphisms from S into T with obvious composition and identities. We call $G^\wedge$ the category of G-sets.

We can consider $G^\wedge$ also as the category of G-objects in the category of finite sets, i.e. the category, whose objects are pairs (S, α_S) with S a finite set and $\alpha_S : G \to \mathrm{Aut}\, S$ a grouphomomorphism and whose morphisms for any two objects (S, α_S), (T, α_T) are those settheoretic maps $\varphi : S \to T$, for which the diagramm

$$\begin{array}{ccc} S & \xrightarrow{\varphi} & T \\ \downarrow \alpha_S(g) & & \downarrow \alpha_T(g) \\ S & \xrightarrow{\varphi} & T \end{array}$$

commutes for any $g \in G$, or as the category of covariant functors from the category $\underline{G}$ into the category of finite sets, where $\underline{G}$ has exactly one object, whose endomorphism-semigroup is just the group G itself.

Especially this last interpretation of $G^\wedge$ as a category of functors from a small (even finite!) category into the category of finite sets allows easily, to deduce the following proposition from wellknown facts about categories of functors, especially that such categories preserve many properties of the image-category.

<u>Proposition 2.1:</u> $G^\wedge$ contains an initial and a final object (the empty G-set $\emptyset$ and the trivial G-set $*$). $G^\wedge$ contains finite projective and injective limits. A G-map $\varphi : S \to T$ between two G-sets S and T is injective, resp. surjective in $G^\wedge$ if and only if it is monomorph, resp. epimorph as a settheoretic map. Any object in $G^\wedge$ can be decomposed uniquely into a sum of indecomposable objects ($\neq \emptyset$).

For non-categorists and also for the sake of explicitness I want to state the most relevant parts of this proposition in a more concrete form:

<u>Proposition 2.1':</u>

a) For any object S there exists a unique G-map $\eta_S : S \to *$.

b) For any two objects S and T the disjoint union $S \dot{\cup} T$ together with the two imbeddings $i : S \to S \dot{\cup} T$ and $j : T \to S \dot{\cup} T$ is the sum of S and T in $G^\wedge$.

c) For any two G-maps $\varphi_i : S_i \to T$ $(i = 1,2)$ there exists the pullback $S_1 \underset{T}{\times} S_2$ in $G^\wedge$:

$S_1 \underset{T}{\times} S_2 = \{(s_1, s_2) \in S_1 \times S_2 \mid \varphi_1(s_1) = \varphi_2(s_2)\}$ as a set with obvious (diagonal) G-action and projections $p_i : S_1 \underset{T}{\times} S_2 \to S_i$. For $T = *$ (and thus $\varphi_i = \eta_{S_i}$) the pullback $S_1 \underset{T}{\times} S_2$ is just the usual product of S_1 and S_2 in $G^\wedge$ and equals the cartesian product $S_1 \times S_2$, defined in § 1.

Remark: So far we have found a categorical interpretation of disjoint sums and cartesian products. One can also give an interpretation of the construction $(S, T) \mapsto S^T$ in the same framework: For a fixed G-set T taking the product with any G-set, resp. G-map defines a covariant functor $p_T: \hat{G} \to \hat{G}: S \mapsto T \times S, \varphi \mapsto Id_T \times \varphi$. This functor now has a right adjoint e_T, which on the objects is precisely the map: $S \mapsto S^T$ (and maps a morphism $\varphi: S_1 \to S_2$ onto the morphism $\varphi^T: S_1{}^T \to S_2{}^T: f \mapsto \varphi f$ $(f \in S_1{}^T)$), i.e. one has a natural isomorphism: $Hom_G(T \times S, Y) \tilde{\to} Hom_G(S, Y^T)$ for any three G-sets T, S, Y, mapping any G-map $\psi: T \times S \to Y$ onto the G-map $\psi': S \to Y^T: s \mapsto \psi(°, s)$ with $\psi(°, s): T \to Y: t \mapsto \psi(t, s)$.

But I want to postpone a more thorough study of these (and related) facts, until this becomes necessary for the further development of the theory, and first concentrate on studying the relations between sums and products (or more generally pull-backs) and their consequences.

Just let us remark, that by standard-methods of category-theory the adjointness of p_T and e_T implies for instance, that p_T commutes with sums (or more generally inductive limits), whereas e_T commutes with products, pull-backs and projective limits.

For a more explicit form of the last statement of proposition 2.1 one has to use the following definition:

A subset $T \subseteq S$ of a G-set S is called a G-subset, if $gT \subseteq T$ for any $g \in G$. Any G-subset $T \subseteq S$ can and of course will be considered as a G-set with the G-action on T given just by restricting the G-action on S to T.

A G-set S is called simple, resp. indecomposable, if $S \neq \emptyset$ and any G-subset of S is either empty or S, resp. if for any two G-sets S_1, S_2 with $S_1 \dot{\cup} S_2 \cong S$, one has either $S_1 = \emptyset$, $S_2 \cong S$ or $S_1 \cong S$, $S_2 = \emptyset$.

Then one can state:

<u>Proposition 2.1':</u> d) Any G-set S can be decomposed uniquely into a disjoint union of indecomposable G-subsets.

Proof: If $S = \dot{\bigcup}_{i=1}^{n} S_i$ is one decomposition and $S = \dot{\bigcup}_{j=1}^{m} T_j$ another one, then $S = \dot{\bigcup}_{\substack{i=1,\dots,n \\ j=1,\dots,m}} (S_i \cap T_j)$ is a common refinement. Thus by the finiteness of S there exists a unique finest decomposition of S, all of its components must be indecomposable.

On the other hand a decomposition of S into indecomposable subsets cannot have a proper refinement, thus must be the unique finest decomposition of S.

We know this unique finest decomposition of S already: it is exactly the decomposition of S into G-orbits ($\overset{G}{\sim}$ - equivalence-classes), considered in § 1. We state this in:

<u>Proposition 2.2:</u> A subset $T \subseteq S$ of a G-set S is a G-subset, if and only if T contains with any element $s \in S$ also the full G-orbit ($\overset{G}{\sim}$ - equivalence-class) of s, i.e. if and only if T is a (disjoint) union of full G-orbits in S.

Especially

a) Any G-orbit of S is a G-subset and the decomposition of S into its G-orbits is the unique finest decomposition of S into a disjoint union of indecomposable G-subsets.

b) For any G-subset $T \subseteq S$ the complement S-T is also a G-subset.

c) For a G-set S the following three statements are equivalent:

(1) S is indecomposable.

(2) S contains exactly one G-orbit, (i.e. S is transitive: for any $s, s' \in S$ there exists $g \in G$ with $gs = s'$).

(3) S is simple.

Proof: Trivial verification, left to the reader.

We go back to proposition 2.1', d) once more and observe that it states some kind of a Krull-Schmidt-Theorem for G-sets. Thus by standard-arguments it can be rephrased as:

<u>Proposition 2.3:</u> If $S \cong \bigcup_{i=1}^{n} S_i \cong \bigcup_{j=1}^{m} T_j$ with S_i, T_j indecomposable, then $n = m$ and - after eventually renumbering -

$S_i \cong T_i$ $(i = 1, \ldots, n)$.

This again can be rephrased in the following (more canonical and less classical) form: For a G-set S and a natural number n write $n \cdot S$ or just nS for the disjoint union of n copies of S, resp. for the empty G-set $\emptyset$, if $n = 0$, and $\sum_{i=1}^{r} S_i$ for the disjoint union of r G-sets S_1, $S_2, \ldots, S_r$. Then we have

Proposition 2.3': Any G-set S can be written in the form $S = \sum_{i=1}^{r} n_i S_i$ with S_i indecomposable and for two G-sets S,S' with decompositions $S = \sum_{i=1}^{r} n_i S_i$, $S' = \sum_{i=1}^{r} n'_i S_i$ we have $S \cong S' \Leftrightarrow n_i = n'_i$ $(i=1, \ldots, r)$.

Again by standard-arguments this implies:

Corollary (P. 2.3') 1: If $S \dot{\cup} T \cong S' \dot{\cup} T$ for three G-sets S, S', T, then $S \cong S'$.

Remark: Cor. (P. 2.3) 1 is true only for finite G-sets. Considering infinite G-sets one would always have

$S \dot{\cup} (S \dot{\cup} T \dot{\cup} S \dot{\cup} T \dot{\cup} \ldots) \cong T \dot{\cup} (S \dot{\cup} T \dot{\cup} S \dot{\cup} T \dot{\cup} \ldots)$

without having $S \cong T$:

Of course one now has to try to determine the indecomposable or simple G-sets. In the next section these will easily turn out to be - up to isomorphism - just the coset-sets G/U, considered in § 1, - especially there are only finitely many! To prepare this, we close this section with the following variation of "Schur's Lemma" for simple G-sets:

Lemma 2.4:

a) The image $\varphi(T) \subseteq S$ of a G-map $\varphi : T \to S$ is a G-subset of S.

b) Any G-map $\varphi : T \to S$ with $T \neq \emptyset$ and S simple is surjective.

c) Any G-endomorphism of a simple G-set is an automorphism.

The proof is left to the reader.

§ 3 Simple G-sets

Let us start with a definition, relating subgroups of G with elements of G-sets:

For a G-set S and a subgroup $U \leqslant G$ of G let $S^U = \{s \in S \mid us = s \text{ for all } u \in U\}$ be the set of U-invariant elements of S and for an element $s \in S$ let

$$G_s = \{g \in G \mid gs = s\}$$

be the stabilizer-subgroup of s.

(Prove, that G_s is indeed a (sub-)group!).

Example:

(1) For any two G-sets S and T the set $Hom_G(T, S)$ can be identified with the set $(S^T)^G$ of G-invariant ele - ments in S^T:

$f \in (S^T)^G \Leftrightarrow gf = f$ for all $g \in G \Leftrightarrow$

$g(f(g^{-1}t)) = t$ for all $g \in G$, $t \in T \Leftrightarrow$

$f(g^{-1}t) = g^{-1}f(t)$ for all $g \in G$, $t \in T \Leftrightarrow$

$f(gt) = gf(t)$ for all $g \in G$, $t \in T \Leftrightarrow$

$f \in Hom_G(T, S)$.

(2) For $s = gU \in G/U$ we have

$$G_s = \{h \in G \mid hgU = gU\} = gUg^{-1}.$$

We state some elementary properties of S^U:

<u>Lemma 3.1:</u> For any two G-sets S and T one has $(S \,\dot\cup\, T)^U = S^U \,\dot\cup\, T^U$, $(S \times T)^U = S^U \times T^U$. Any G-map $\varphi: S \to T$ maps S^U into T^U.

Proof: Trivial verification.

Remark: Lemma 3.1 can be interpreted as saying that the map $S \mapsto S^U$ can be naturally extended to a covariant functor from $G^\wedge$ into the category of finite sets, which commutes with sums and products. The next lemma shows, that this functor is representable and represented by G/U.

<u>Lemma 3.2:</u> The map $\mathrm{Hom}_G(G/U, S) \to S: \varphi \mapsto \varphi(*_U)$ induces a bijection

$$\mathrm{Hom}_G(G/U, S) \xrightarrow{\sim} S^U,$$

the inverse map given by $s \mapsto \varphi_s = \varphi_s^U$ with $\varphi_s(gU) = gs$.

Moreover φ_s is injective if and only if $G_s = U$.

Proof: Because $*_U \in (G/U)^U$, we have $\varphi(*_U) \in S^U$ for any $\varphi \in \mathrm{Hom}_G(G/U, S)$. Because $\varphi(*_U) = \varphi'(*_U)$ implies $\varphi(gU) = g\varphi(*_U) = g\varphi'(*_U) = \varphi'(gU)$ for any two G-maps $\varphi, \varphi': G/U \to S$, such a G-map is completely determined by $\varphi(*_U) \in S^U$. Finally for any $s \in S^U$ the map $\varphi_s = \varphi_s^U: G/U \to S: gU \mapsto gs$ is a welldefined (proof!) G-map with $\varphi_s(*_U) = s$. Thus the map $\varphi(\in \mathrm{Hom}_G(G/U, S)) \mapsto \varphi(*_U)(\in S)$ maps $\mathrm{Hom}_G(G/U, S)$ bijevtively onto S^U.

At last we have:

φ_s injective $\Leftrightarrow (\varphi_s(gU) = \varphi_s(hU) \Rightarrow gU = hU) \Leftrightarrow$
$(g\varphi_s(U) = h\varphi_s(U) \Rightarrow gU = hU) \Leftrightarrow (gs = hs \Rightarrow gU = hU) \Leftrightarrow$
$(h^{-1}gs = s \Rightarrow h^{-1}gU = U) \Leftrightarrow G_s \subseteq U.$

On the other hand $s \in S^U$ implies $U \leq G_s$ anyway and thus we have:

$$\varphi_s \text{ injective} \Leftrightarrow U = G_s.$$

Now we can state, including some results of § 2:

<u>Proposition 3.1:</u> Let S be a nonempty G-set. Then the following statements are equivalent:

(1) S is indecomposable.

(2) For any s, s' $\in$ S there exists g $\in$ G with gs = s'.

(3) S is simple.

(4) Any G-map T $\to$ S with T $\neq \emptyset$ is surjective.

(5) Any G-map G/E $\to$ S is surjective.

(6) There exists a surjective G-map G/E $\to$ S.

(7) There exists an element s $\in$ S, such that for any s' $\in$ S there exists g $\in$ G with gs = s'.

(8) The orbitspace G\S contains exactly one element.

(9) For any s $\in$ S we have a natural isomorphism $G/G_s \cong S$.

(10) There exists a subgroup $U \leqslant G$ with $G/U \cong S$.

Proof: The equivalence of (1), (2) and (3) and (3) $\Rightarrow$ (4) has been shown in § 2. (4) $\Rightarrow$ (5) is trivial, (5) $\Rightarrow$ (6) follows from $\mathrm{Hom}_G(G/E, S) \cong S^E \cong S \neq \emptyset$, (6) $\Rightarrow$ (7) follows from the explicit description of the map $\varphi_s^E: G/E \to S$, defined for any $s \in S^E = S$, given in Lemma 3.2 .
(7) $\Rightarrow$ (8) $\Rightarrow$ (2) is trivial. (4) $\Rightarrow$ (9) follows from Lemma 3.2, applied for $U = G_s$ and $s \in S^U$, because the map $\varphi_s^U: G/G_s \to S$ is surjective by (4) and injective by the last part of lemma 3.2, thus an isomorphism.
(9) $\Rightarrow$ (10) and (10) $\Rightarrow$ (7) (choose $s = *_u$) are trivial again.

<u>Corollary(P. 3.1)1:</u> For $U, V \leqslant G$ one has $G/U \cong G/V$ if and only if U and V are conjugate in G (notation: $U \sim V$ or $U \overset{G}{\sim} V$).

Proof: If $V = gUg^{-1}$, then $V = G_s$ for $s = gU \in G/U$ and thus by Prop. 3.1 : $G/V \cong G/U$.

On the other hand an isomorphism $f : G/V \cong G/U$ implies $V = G_{*_V} = G_{f(*_V)} = gUg^{-1}$, if $f(*_V) = gU$.

<u>Corollary (P. 3.1) 2:</u> Let S be a simple (indecomposable) G-set and $s \in S$. Then $|G| = |S| \cdot |G_s|$, especially $|S| \mid |G|$ for any simple G-set S ($|M|$ the number of elements in a set M, especially $|G|$ the order of G).

Proof: This follows from $S \cong G/G_s$.

<u>Corollary (P. 3.1) 3:</u> Let S and T be G-sets and X a simple G-set. Then the canonical map: $\mathrm{Hom}_G(X,S) \,\dot{\cup}\, \mathrm{Hom}_G(X,T) \to \mathrm{Hom}_G(X,S\dot{\cup}T)$ is bijective.

Proof: This follows from $X \cong G/G_x$ (x any element in X) and Lemma 3.1 and 3.2.

There are some more interesting consequences of Lemma 3.2, which should be discussed in the rest of this section. At first in the special case $U = E$ we have already used in the proof of Prop. 3.1 the fact, that G/E is a free G-set over $*_E \in G/E$, i.e. any settheoretic map $*_E \to S$ (S any G-set) can be uniquely extended to a G-map: $G/E \to S$. This can be generalized to

<u>Proposition 3.2:</u> Let S be a G-set. Then the following statements are equivalent:

(1) G acts freely on S, i.e. $G_s = E$ for all $s \in S$.

(2) S is a disjoint union of G-sets, isomorphic to G/E.

(3) There exists a subset $S_0 \subseteq S$, such that S is free over S_0.

(4) S is a projective object in $G^\wedge$, i.e. for any diagramm $\begin{array}{ccc} & & S \\ & & \downarrow \varphi \\ T' & \xrightarrow[\psi]{} & T \end{array}$ in $G^\wedge$ with surjective ψ there exists $\varphi' : S \to T'$, such that $\begin{array}{ccc} & & S \\ & \varphi' & \downarrow \varphi \\ T' & \xrightarrow[\psi]{} & T \end{array}$ commutes.

Proof: (1) $\Rightarrow$ (2) : S is a disjoint union of indecomposable G-subsets by § 2 and any such indecomposable G-subset is isomorphic to G/E by Prop. 3.1 .

(2) $\Rightarrow$ (3) : For any simple subset of S choose an isomorphism with G/E, let S_o be the union of the various images of $*_E \in G/E$ under those isomorphisms and apply Lemma 3.2 for any single simple summand of S and U = E.

(3) $\Rightarrow$ (4) : Using standardarguments one chooses for any $s \in S_o$ an element $\varphi_o'(s) \in \psi^{-1}(\varphi(s))$ and extends $\varphi_o' : S_o \to T'$ to aGmap $\varphi' : S \to T'$.

(4) $\Rightarrow$ (1) : Again using standardarguments one considers the projection $p_2 : G/E \times S \to S$.

By (4) there exists a G-map $\varphi : S \to G/E \times S$ with $p_2\varphi = Id_S$. Therefore we have for any $s \in S$: $G_s \leqslant G_{\varphi(s)} = E$, because $G_{(a,b)} = E$ for any $(a,b) \in G/E \times S$.

<u>Corollary (P. 3.2) 1:</u> If G acts freely on a G-set S, then $|G|$ divides $|S|$.

Now we want to apply Lemma 3.2 for arbitrary $U \leqslant G$ and $S = G/V$, $V \leqslant G$. Then we get

<u>Lemma 3.3:</u> For $U, V \leqslant G$ two arbitrary subgroups of G we have $Hom_G(G/U, G/V) \neq \emptyset$ if ond only if U is subconjugate to V in G, i.e. there exists $g \in G$ with $g^{-1}Ug \leqslant V$ (notation: $U \lesssim V$ or $U \underset{\sim}{\overset{G}{\leq}} V$). Especially there exist G-maps $G/U \to G/V$ and $G/V \to G/U$ if and only if U and V are conjugate in G, in which case all those G-maps are isomorphism.

Proof: By Lemma 3.2 we have

$$Hom_G(G/U, G/V) \neq \emptyset \Leftrightarrow (G/V)^U \neq \emptyset$$
$$\Leftrightarrow \exists g \in G \text{ with } UgV = gV \Leftrightarrow \exists g \in G$$

with $g^{-1}Ug \subseteq V$. The rest is obvious.

By Lemma 2.1, c) and Prop. 3.1 we know already, that any G-endomorphism of G/U ($U \leqslant G$) is a G-automorphism. Using Lemma 3.2 once more, we can easily compute the group $Aut_G(G/U)$ of all G-automorphisms (= G-endomorphisms) of G/U:

<u>Lemma 3.4:</u>

With $N_G(U) = \{n \in G \mid nUn^{-1} = U\}$ the normalizer of U in G we have

$$Aut_G(G/U) \cong N_G(U)/U,$$

more precisely: for $n \in N_G(U)$ define

$\varphi_n : G/U \to G/U : gU \mapsto gn^{-1}U = gUn^{-1}$;

then $\varphi_n \in Aut_G(G/U)$ and

$$N_G(U) \to Aut_G(G/U) : n \mapsto \varphi_n$$

is a surjective grouphomomorphism with kernel $U \trianglelefteq N_G(U)$.

Proof: One verifies easily $\varphi_n \in Aut_G(G/U)$ as well as $\varphi_{n \cdot m} = \varphi_n \cdot \varphi_m$ and $\varphi_n = Id_{G/U} \Leftrightarrow n \in U$.

The surjectivity follows from Lemma 3.2, if we can show, that for any $gU \in (G/U)^U$ there exists $n \in N_G(U)$ with $\varphi_n(*_U) = gU$. But $gU \in (G/U)^U$ implies $UgU = gU$, $g^{-1}Ug = U$, $g^{-1} \in N_G(U)$ and thus

$\varphi_n(*_U) = Un^{-1} = n^{-1}U = gU$ for $n = g^{-1} \in N_G(U)$.

<u>Corollary (L. 3.4) 1:</u> The automorphism-group of G/U acts freely on G/U and also on the set $(G/U)^V$ of V-invariant elements in G/U for any $V \leqslant G$. Especially $|N_G(U)/U| = |(G/U)^U|$ divides $|(G/U)^V|$ for any $V \leqslant G$.

Proof: At first we have to show: $\varphi_n(gU) = gU$ for some $g \in G$ implies $n \in U$. But this is obvious. By Lemma 3.1 we have $\varphi_n((G/U)^V) \subseteq (G/U)^V$ for any $V \subseteq G$ and $n \in N_G(U)$. Thus the last part follows from Cor.(P. 3.2)1.

The last remarks imply easily:

<u>Proposition 3.3:</u> If G and H are two finite groups with $G^\wedge \cong H^\wedge$, then $G \cong H$.

Proof: Any equivalence $G^\wedge \xrightarrow{\sim} H^\wedge$ must map G/E into a projective simple object in $H^\wedge$, thus into an H-set, isomorphic to H/E. Therefore

$$G \cong \mathrm{Aut}_G(G/E) \cong \mathrm{Aut}_H(H/E) \cong H.$$

Remark: U. Knauer has studied the same question for the category of arbitrary (i.e. also infinite) G-sets over an arbitrary semigroup G. He gives necessary and sufficient conditions for two semigroups G and H, to define equivalent categories of G-, resp. H-sets (i.e. to be Morita-equivalent in his terminology), and shows by constructing a counterexample that Morita-equivalence does not generally imply isomorphy (cf. [ ]).

We come back to Lemma 3.3. There we have seen: $\mathrm{Hom}_G(G/U,G/V) \neq \emptyset \Leftrightarrow U \lesssim V$. In the rest of this section we want to study more generally conditions for the existence of a G-map $\varphi : S \to T$ between arbitrary G-sets S and T. For this purpose it is convenient to define in an arbitrary category $\mathfrak{C}$ the "halfordering" $\prec$ on the class of objects of $\mathfrak{C}$ by $X \prec Y \Leftrightarrow [X,Y]_{\mathfrak{C}} \neq \emptyset$ and the equivalencerelation $X \sim Y \Leftrightarrow X \prec Y$ and $Y \prec X$.

We just state

Lemma 3.5: With X,Y,Z objects in $\mathfrak{C}$ one has:

(a) If the sum X + Y exists in $\mathfrak{C}$, then

$X < X + Y$, $Y < X + Y$ and

$X < Z,\ Y < Z \Leftrightarrow X + Y < Z$.

(b) If the product $X \times Y$ exists in $\mathfrak{C}$, then

$X \times Y < X$, $X \times Y < Y$ and

$Z < X,\ Z < Y \Leftrightarrow Z < X \times Y$.

Lemma 3.6: If S,T are simple G-sets, then

$S \sim T \Leftrightarrow S \cong T$.

Proof: Let $\varphi : S \to T$, $\psi : T \to S$ be G-maps. Then $\varphi\psi \in \mathrm{Hom}_G(T,T) = \mathrm{Aut}_G(T)$ and $\psi\varphi \in \mathrm{Hom}_G(S,S) = \mathrm{Aut}_G(S)$ are automorphisms by Lemma 2.1, which implies, that φ and ψ are isomorphisms, q.e.d.

Now assume the category $\mathfrak{C}$ to contain finite sums and products and define a class $\mathfrak{K}$ of objects in $\mathfrak{C}$ to be r-closed, resp. to be l-closed (r for right, l for left), if $\mathfrak{K}$ is nonempty and if "$X < Y,\ Y \in \mathfrak{K} \Rightarrow X \in \mathfrak{K}$" and "$X,Y \in \mathfrak{K} \Rightarrow X + Y \in \mathfrak{K}$" hold, resp. if "$X < Y,\ X \in \mathfrak{K} \Rightarrow Y \in \mathfrak{K}$" and "$X,Y \in \mathfrak{K} \Rightarrow X \times Y \in \mathfrak{K}$" hold.

Of course for any object X in $\mathfrak{C}$ the class $\mathfrak{K}_r(X) = \{Y \mid Y<X\}$ is r-closed and X maximal in $\mathfrak{K}_r(X)$ w.r.t.< and the class $\mathfrak{K}_l(X) = \{Y \mid X < Y\}$ is l-closed and X is minimal in $\mathfrak{K}_l(X)$ w.r.t.<.

Moreover we have:

Lemma 3.7:

(a) If $\mathfrak{K}$ is r-closed, resp. l-closed in $\mathfrak{C}$ and if $X \in \mathfrak{K}$

is maximal in $\mathfrak{K}$ w.r.t. $<$, resp. minimal in $\mathfrak{K}$ w.r.t. $<$, then $\mathfrak{K} = \mathfrak{K}_r(X)$, resp. $\mathfrak{K} = \mathfrak{K}_l(X)$.
Especially any r-closed, resp. l-closed class $\mathfrak{K}$ in $\mathfrak{C}$ is of the form $\mathfrak{K}_r(X)$, resp. $\mathfrak{K}_l(X)$ for some X, if any class $\mathfrak{K}$ of objects in $\mathfrak{C}$ contains maximal, resp. minimal elements w.r.t. $<$, e.g. if $\mathfrak{C}$ contains only finitely many equivalenceclasses w.r.t. to the equivalencerelation $\sim$, introduced above.

(b) One has $\mathfrak{K}_r(X) = \mathfrak{K}_r(Y) \Leftrightarrow X \sim Y \Leftrightarrow \mathfrak{K}_l(X) = \mathfrak{K}_l(Y)$.

The proof is much shorter than the statement: (b) is trivial, (a) has to be proved only for l-closed $\mathfrak{K}$ (the "r"-case is dual) and here it is enough to show: $Y \in \mathfrak{K} \Rightarrow X < Y$. But $Y \in \mathfrak{K}$ implies $X \times Y \in \mathfrak{K}$ and $X \times Y < X$ together with the minimality of X implies now $X \times Y \sim X \Rightarrow X < X \times Y < Y \Rightarrow X < Y$ q.e.d.

To apply these remarks to $\mathfrak{C} = G^\wedge$ we define for any G-set S the set $\mathfrak{U}(S) = \{U \leqslant G \mid S^U \neq \emptyset\}$ of subgroups of G. Then we can prove:

<u>Proposition 3.4:</u>

(a) For S,T G-sets one has $S < T \Leftrightarrow \mathfrak{U}(S) \subseteq \mathfrak{U}(T)$. Especially $S \sim T \Leftrightarrow \mathfrak{U}(S) = \mathfrak{U}(T)$ and there are only finitely many equivalence-classes w.r.t. $\sim$ in $G^\wedge$.

(b) Any r-closed, resp. l-closed class of G-sets is of the form $\mathfrak{K}_r(S)$, resp. $\mathfrak{K}_l(S)$ for some G-set S.

Proof: We have to prove only the first assertion.

So assume $S < T$ and choose a G-map $\varphi : S \to T$. Because $\varphi(S^U) \subseteq T^U$ by Lemma 3.1 $S^U \neq \emptyset$ implies $T^U \neq \emptyset$, i.e. we have $\mathfrak{U}(S) \subseteq \mathfrak{U}(T)$. On the other hand assume $\mathfrak{U}(S) \subseteq \mathfrak{U}(T)$ and write S in the form $\dot{\bigcup}_{i=1}^{n} G/U_i$ $(U_i \leqslant G)$. Obviously $U_i \in \mathfrak{U}(S) \subseteq \mathfrak{U}(T)$ and therefore $G/U_i < T$ by Lemma 3.2. But now Lemma 3.5(a) implies $S = \dot{\bigcup}_{i=1}^{n} G/U_i < T$, q.e.d.

Finally define for any set $\mathfrak{B}$ of subgroups of G the G-set $S(\mathfrak{B}) = \dot{\bigcup}_{V\in\mathfrak{B}} G/V$. Then $\mathfrak{U}(S(\mathfrak{B})) = \overline{\mathfrak{B}} = \{U \leqslant G \mid \exists V \in \mathfrak{B} \text{ with } U \lesssim V\}$ is the "subconjugate closure" of $\mathfrak{B}$, because $(\dot{\bigcup}_{V\in\mathfrak{B}} G/V)^U \neq \emptyset \Leftrightarrow \exists V \in \mathfrak{B}$ with $(G/V)^U \neq \emptyset \Leftrightarrow \exists\, V \in \mathfrak{B}$ with $U \lesssim V$, especially we have $\mathfrak{U}(S(\mathfrak{B})) = \mathfrak{B}$, if and only if $\mathfrak{B}$ is "subconjugately closed", i.e. contains with any $V \leqslant G$ also any $U \leqslant G$ with $U \lesssim V$.

On the other hand the set $\mathfrak{U}(T)$ obviously is subconjugately closed. This implies:

(i) $T \sim S(\mathfrak{U}(T))$,

because $\mathfrak{U}(S(\mathfrak{U}(T))) = \mathfrak{U}(T)$ by the last remarks, and

(ii) there is a 1-1 correspondence between: equivalence-classes of G-sets, r-closed classes of G-sets, l-closed classes of G-sets, subconjugately closed sets of subgroups and finally equivalence-classes of sets of subgroups, if one defines $\mathfrak{U} \sim \mathfrak{B} \Leftrightarrow \overline{\mathfrak{U}} = \overline{\mathfrak{B}}$.

§ 4 Invariants of G-sets

In this section we want to define and to study certain numerical invariants of G-sets, which - by a Theorem of Burnside - characterize G-sets up to isomorphism.

So let S,T be G-sets and let U be a subgroup of G. We define

$$\varphi_T(S) = |\mathrm{Hom}_G(T,S)|$$

and

$$\varphi_U(S) = |S^U| .$$

The following statements are more or less obvious:

Proposition 4.1:

(a) $\varphi_U(S) = \varphi_T(S)$ if $T \cong G/U$;

(b) $\varphi_U(S_1 \dot{\cup} S_2) = \varphi_U(S_1) + \varphi_U(S_2)$,
$\varphi_U(S_1 \times S_2) = \varphi_U(S_1) \cdot \varphi_U(S_2)$;

(c) $\varphi_T(S_1 \times S_2) = \varphi_T(S_1) \cdot \varphi_T(S_2)$;

(d) $\varphi_T(S_1 \dot{\cup} S_2) = \varphi_T(S_1) + \varphi_T(S_2)$, if T is simple;

(e) $\varphi_{T_1 \dot{\cup} T_2}(S) = \varphi_{T_1}(S) \cdot \varphi_{T_2}(S)$;

(f) $\varphi_T(S) \neq 0 \Leftrightarrow T < S$, $\varphi_U(S) \neq 0 \Leftrightarrow U \in \mathfrak{U}(S)$;

(g) if $U,V \leqslant G$, then $\varphi_U(S) \leqslant \varphi_V(S)$ for all G-sets S, if and only if $V \overset{G}{\lesssim} U$, especially $\varphi_U(S) = \varphi_V(S)$ for all G-sets S, if and only if $U \overset{G}{\sim} V$.

(h) If T,T' are simple, then $\varphi_T(S) = \varphi_{T'}(S)$ for all G-sets S if and only if $T \cong T'$.

Proof: (a): Lemma 3.2

(b): Lemma 3.1

(c): Definition of products in categories

(d): Cor.(P.3.1) 3.

(e): Definition of sums in categories

(f): Definition of φ, $\prec$ and $\mathfrak{U}(S)$

(g): $\varphi_U(S) \leq \varphi_V(S)$ for all G-sets S implies $\varphi_V(G/U) \neq 0$ (because $\varphi_U(G/U) > 0$) and therefore $V \lesssim U$. On the other hand $V \subseteq gUg^{-1} = W$ implies: $\varphi_U(S) = |S^U| = |S^W| \leq |S^V| = \varphi_V(S)$

(h): follows either from Prop. 3.1, Prop. 4.1, (a) and (g), and Cor.(P.3.1)1 or more directly from Prop. 4.1, f and Lemma 3.6.

Remark 1: (h) is true for arbitrary G-sets T,T', but this can be shown only with some effort.

Remark 2: Because $\varphi_U = \varphi_T$ for $T = G/U$ and $\varphi_{T_1 \dot{\cup} T_2} = \varphi_{T_1} \cdot \varphi_{T_2}$, it would be enough, to work only with the invariants φ_U, resp. the invariants φ_T. But sometimes it is more convenient to use the φ_U's, sometimes the φ_T's. Thus we will use both and pass freely from one notation to the other one.

<u>Theorem 4.1</u> (Burnside):

For S,S' G-sets we have: $S \cong S' \Leftrightarrow \varphi_U(S) = \varphi_U(S')$ for all subgroups $U \leq G$ ($\Leftrightarrow \varphi_T(S) = \varphi_T(S')$ for any simple G-set T).

Proof: Let $\mathcal{T}$ be a complete system of simple (indecomposable) G-sets. By Prop. 2.3' we have $S = \sum_{T\in\mathcal{T}} n_T T$, $S' = \sum_{T\in\mathcal{T}} n'_T T$ with certain (even unique) nonnegative integers n_T, n'_T and it is enough to show: $n_T = n'_T$ for all $T \in \mathcal{T} \Leftrightarrow \varphi_T(S) = \varphi_T(S')$ for all $T \in \mathcal{T}$.
The direction $\Rightarrow$ being trivial assume $\varphi_T(S) = \varphi_T(S')$, but $\mathcal{T}' = \{T \in \mathcal{T} | n_T \neq n'_T\} \neq \emptyset$. Then there exists an element $X \in \mathcal{T}'$, which is maximal w.r.t. $<$ in $\mathcal{T}'$, and for this $X \in \mathcal{T}'$ we get:

$$0 = \varphi_X(S) - \varphi_X(S') = \sum_{T\in\mathcal{T}} n_T\varphi_X(T) - \sum_{T\in\mathcal{T}} n'_T\varphi_X(T)$$

$$= \sum_{T\in\mathcal{T}, X<T} (n_T - n'_T)\varphi_X(T) = (n_X - n'_X)\varphi_X(X) \neq 0$$

(the last equality by the maximality of X in $\mathcal{T}'$, the one before by Prop. 4.1, (f)), a contradiction.

Remark: This proof does not use the uniqueness of the decomposition $S = \sum_{T\in\mathcal{T}} n_T T$, instead it offers another proof for this fact. One can also give a simple proof of Cor.(P.2.3)1: $S \dot{\cup} T \cong S' \dot{\cup} T \Rightarrow S \cong S'$, using Thm 4.1:
$S \dot{\cup} T \cong S' \dot{\cup} T \Rightarrow \varphi_U(S \dot{\cup} T) = \varphi_U(S' \dot{\cup} T)$ for all $U \leqslant G$
$\Rightarrow \varphi_U(S) + \varphi_U(T) = \varphi_U(S') + \varphi_U(T)$ for all
$U \leqslant G \Rightarrow \varphi_U(S) = \varphi_U(S')$ for all $U \leqslant G \Rightarrow S \cong S'$.
For later use we state:

<u>Lemma 4.1:</u>

For any two simple G-sets S and T with $S \cong G/U$ one has

$$\varphi_S(S) = |\mathrm{Aut}_G(S)| = (N_G(U) : U)$$

and

$$\varphi_S(S) \mid \varphi_T(S).$$

Proof: This is a restatement of Cor.(L.3.4)1.

Having established a complete system of invariants it seems natural to ask for a complete list of relations, which hold for such a system of invariants, so that for a given family of numbers $n_U \in \mathbb{Z}$ $(U \leqslant G)$ one can decide, whether there exists a G-set S with $\varphi_U(S) = n_U$ or not. Unfortunately such a complete list of relations seems not to be known in general, but on the other hand one can develop quite a sufficient theory, dealing with certain parts of such a complete list, e.g. it is rather easy to see, that there are no linear relations over $\mathbb{Z}$, resp. $\mathbb{Q}$ between the various different φ_U.

I want to indicate three slightly different proofs for this fact right now, to motivate the general procedure to be developped in the next section:

So let $\mathfrak{C}$ be a complete system of nonconjugate subgroups of G, so that $\varphi_U(S) = \varphi_V(S)$ for U, $V \in \mathfrak{C}$ and all G-sets S implies U = V, and assume $\sum_{U \in \mathfrak{C}} a_U \varphi_U(S) = 0$ holds for all G-sets S and some nonvanishing set of coefficients $a_U (U \in \mathfrak{S})$, $a_U \in \mathbb{Z}$ w.l.o.g.!

We have to get a contradiction.

(1) Choose $U \in \mathfrak{S}$ minimal with $a_U \neq 0$. Then we have

$\sum_{V \in \mathfrak{S}} a_V \varphi_V (G/U) = a_U \varphi_U (G/U) \neq 0$, because for

$V \lesssim U$, $V \neq U$ we have $a_V = 0$ and for $V \not\lesssim U$ we

have $\varphi_V(G/U) = 0$.

(2) We consider the $|\mathfrak{S}|$-tupels $(\varphi_U(S))_{U \in \mathfrak{S}}$ as vectors in $\prod_{U \in \mathfrak{S}} \mathbb{Q}$. If there exists a nontrivial linear relation $\sum_{U \in \mathfrak{S}} a_U \varphi_U (S) = 0$ for all S, then the subvectorspace of $\prod_{U \in \mathfrak{S}} \mathbb{Q}$, generated by all $(\varphi_U(S))_{U \in \mathfrak{S}}$ is of dimension less then $|\mathfrak{S}|$.

Thus there exists a linear relation between the $|\mathfrak{S}|$ vectors $(\varphi_U(G/V))_{U \in \mathfrak{S}}$ $(V \in \mathfrak{S})$, i.e. there are integral numbers $b_V \in \mathbb{Z}$ $(V \in \mathfrak{S})$, not all $b_V = 0$, with $\sum_{V \in \mathfrak{S}} b_V (\varphi_U(G/V))_{U \in \mathfrak{S}} = 0$, i.e. with

$\sum_{V \in \mathfrak{S}} b_V \varphi_U (G/V) = 0$ for all $U \in \mathfrak{S}$.

Let $\mathfrak{S}' = \{V | b_V > 0\}$, $\mathfrak{S}'' = \{W | b_W < 0\}$.

Then the above relation implies:

$\varphi_U(\bigcup_{V \in \mathfrak{S}'} b_V G/V) = \varphi_U(\bigcup_{W \in \mathfrak{S}''} (-b_W) G/W)$, i.e.

$\bigcup_{V \in \mathfrak{S}'} b_V \, G/V \cong \bigcup_{W \in \mathfrak{S}''} (-b_W) G/W$, a contradiction to proposition 2.1', because any G/V $(V \in \mathfrak{S}')$ is non-isomorphic to any G/W $(W \in \mathfrak{S}'')$.

(3) Now consider $\prod_{U \in \mathfrak{S}} \mathbb{Q}$ not only as a vectorspace but also as a product of fields $(\cong \mathbb{Q})$, i.e. as a

$\mathbb{Q}$-algebra. Because of Prop. 4.1, b) the $\mathbb{Q}$-span of the elements $(\varphi_U(S))_{U \in \mathfrak{C}} \in \prod_{U\in\mathfrak{C}} \mathbb{Q}$ is a subalgebra, on which the projections

$$\varphi_V: \prod_{U\in\mathfrak{C}} \mathbb{Q} \to \mathbb{Q}: (x_U)_{U\in\mathfrak{C}} \to x_V$$

define different ringhomomorphisms onto $\mathbb{Q}$. But those are necessarily linearly independent, for instance by the chinese remainder theorem.

To understand the interdependence between these three proofs, consider the following linear map: with $\mathfrak{T}$ a complete system of nonisomorphic indecomposable G-sets and $\mathfrak{C}$ a complete system of nonconjugate subgroups of G define

$$\Phi: \mathbb{Q}^{\mathfrak{T}} \to \mathbb{Q}^{\mathfrak{C}} : (f_T)_{T\in\mathfrak{T}} \mapsto (\sum_{T\in\mathfrak{T}} f_T\varphi_U(T))_{U\in\mathfrak{C}}$$

($\mathbb{Q}^{\mathfrak{S}} = \{(q_S)_{S\in\mathfrak{S}} \mid q_S \in \mathbb{Q}\}$ the set of maps $\mathfrak{S} \to \mathbb{Q}$ for any set $\mathfrak{S}$). One has to show, that Φ is surjective. With respect to the natural basis of $\mathbb{Q}^{\mathfrak{T}}$ and $\mathbb{Q}^{\mathfrak{C}}$ the map Φ is given by the matrix $(\varphi_U(T))_{U\in\mathfrak{C},\, T\in\mathfrak{T}}$ and the first proof uses the fact, that this matrix is triangular with a nowhere vanishing diagonal if one orders $\mathfrak{T} = \{T_1, T_2, \ldots\}$ and $\mathfrak{C} = \{U_1, U_2, \ldots\}$ such that $|T_i| \leqslant |T_{i+1}|$ and $G/U_i \cong T_i$, which has also been used in the proof of Thm. 4.1. The second proof uses Thm 4.1 and Prop. 2.1', to prove the injectivity of Φ, which by $|\mathfrak{T}| = |\mathfrak{C}|$ implies the surjectivity of Φ. The last proof is perhaps the most convincing one, because it uses only very basic facts, but among them one, which had been overlooked before: the multiplicativity of the maps φ_U. One can even use its idea, to give another proof of Thm. 4.1 and of Prop. 2.3 at the same time (injectivity of Φ) based only on Prop. 4.1, b) and g) (surjectivity of Φ) and the fact $|T| = |\mathfrak{C}|$.

In the next section we want to study the multiplicative structure of G-sets systematically.

§ 5 The Burnsidering and its primeideals

The last remark in § 4 shows, that the property of the maps $\varphi_U (U \leqslant G)$ to be wellbehaved with respect to sums and products - taken in $G^\wedge$ and $\mathbb{Z}$ respectively - can be surprisingly helpful. Thus in this section we want to study quite generally "wellbehaved" maps from the set of isomorphismclasses of G-sets into arbitrary (commutative) rings. It will turn out, that these maps are essentially determined by the various maps φ_U.

Later on we will be led to study congruence-relations between the φ_U modulo a prime p, thus dealing with a first basic step conserning the interesting (and principally unsolved) question of congruence-relations between the φ_U in general and associated numerical and grouptheoretical problems.

The method is to construct first a universal solution of the problem of "wellbehaved" maps, i.e. a ring $\Omega(G)$ together with a map

$\{$isomorphismclasses of G-sets$\} \to \Omega(G) : S \mapsto [S]$,

which commutes with sums and products, such that any other such map into any ring R factors uniquely through this map and a ringhomomorphism $\Omega(G) \to R$, and then to study the universal object $\Omega(G)$, the Burnsidering of G.

For this purpose we need the following well-known:

Proposition 5.1:

(a) Let A be an abelian semigroup, i.e. a set together with a map $A \times A \to A : (a,b) \mapsto a + b$, such that $a + b = b + a$ and $(a + b) + c = a + (b + c)$ holds. Then there exists an abelian group $\overline{A}$, the universal group associated to A, together with an additive map $A \to \overline{A} : a \mapsto \overline{a}$, such that any additive map $\alpha : A \to B$ into an abelian group B factors uniquely through this map and a grouphomo - morphism $\overline{A} \to B$.

(b) (Construction of $\overline{A}$): Let $\tilde{A}$ be the set of equivalence-classes $(a,b)^{\sim}$ of pairs $(a,b) \in A \times A$ with respect to the equivalence-relation: $(a,b) \sim (a',b') \Leftrightarrow \exists\, c \in A$ with $a + b' + c = a' + b + c$. Then $\tilde{A}$ is in a natural way a group, if one defines $(a,b)^{\sim} + (a',b')^{\sim} = (a+a', b+b')^{\sim}$, with $(a,a)^{\sim}$ the neutral element and the inverse of $(a,b)^{\sim}$ being given by $(b,a)^{\sim}$. Moreover $A \to \tilde{A} : a \mapsto (a+a, a)^{\sim}$ is a welldefined additive map and the thus existing unique map $\beta : \overline{A} \to \tilde{A}$ with $\beta(\overline{a}) = (a + a, a)^{\sim}$ is an isomorphism. Especially $A \to \overline{A}$ is injective, if and only if $a + c = b + c$ implies $a = b$ for all $a,b,c \in A$.

(c) If A,B,C are abelian semigroups together with a "bilinear" map $f: A \times B \to C$ (i.e. a map with $f(a+a',b) = f(a,b) + f(a',b)$, $f(a,b + b') = f(a,b) + f(a,b')$), then f extends uniquely to a bilinear map of the associated universal groups:

$$\overline{f} : \overline{A} \times \overline{B} \to \overline{C}.$$

Especially if A is a "halfring", i.e. an abelian semigroup together with an associative and distributive multiplication $A \times A \to A : (a,b) \mapsto ab$, then this multiplication extends uniquely to a multiplication $\overline{A} \times \overline{A} \to \overline{A}$, which makes $\overline{A}$ to a ring (commutative, if A has been commutative; $1 \in \overline{A}$ at least if $1 \in A$).

Moreover if B has been a "halfmodule" over the halfring A, i.e. an abelian semigroup together with a bilinear map $A \times B \to B$, $(a,b) \mapsto ab$ with $a'(ab) = (aa')b (a,a' \in A,\ b \in B)$, then the associated universal group $\overline{B}$ is in a natural way an $\overline{A}$-module.

We leave the verification of these facts to the reader, just mentioning, that of course (a) and (b) are proved together by showing, that the group $\widetilde{A}$ together with the map $A \to \widetilde{A}$, described in (b), have the universal properties stated in (a).

Now we observe that the set $\Omega^+(G)$ of isomorphismclasses of G-sets has the structure of a commutative halfring, if we define sum, resp. product of iso - morphismclasses just by the isomorphismclass of the disjoint union, resp. the cartesian product of the representing G-sets, because one has obvious natural isomorphisms:

$$S_1 \dot{\cup} S_2 \cong S_2 \dot{\cup} S_1;$$
$$(S_1 \dot{\cup} S_2) \dot{\cup} S_3 \cong S_1 \dot{\cup} (S_2 \dot{\cup} S_3);$$
$$S_1 \times S_2 \cong S_2 \times S_1;$$
$$(S_1 \times S_2) \times S_3 \cong S_1 \times (S_2 \times S_3);$$
$$S_1 \times (S_2 \dot{\cup} S_3) \cong (S_1 \times S_2) \dot{\cup} (S_1 \times S_3).$$

Furthermore $1 \in \Omega^+(G)$ exists and is represented by $*_G$, because $S \times *_G \cong S$ for any G-set S.
Thus a map from the set $\Omega^+(G)$ of isomorphismclasses of G-sets into a ring R, which commutes with sums and products, is nothing else than a homomorphism from the halfring $\Omega^+(G)$ into R and thus factors uniquely through the universal ring, associated to $\Omega^+(G)$, i.e. we have already proved part (c) of

Proposition (and Definition) 5.2:
Let $\Omega^+(G)$ be the halfring of isomorphismclasses of G-sets and let $\Omega(G)$ be the associated universal ring, - also called the "Burnsidering" of G. For a G-set S let $\lfloor S \rfloor \in \Omega(G)$ be the element in $\Omega(G)$, represented by S. Then one has:

(a) $\lfloor S \rfloor = \lfloor T \rfloor \Leftrightarrow S \cong T$;

(b) as an additive group $\Omega(G)$ is a free $\mathbb{Z}$-module with a canonical basis: the isomorphismclasses of simple or transitive G-sets, especially its rank equals the number of conjugacy-classes of subgroups of G;

(c) any homomorphism $\Omega^+(G) \to R$ into any ring R factors uniquely through $\Omega(G)$, especially for any $U \leqslant G$ one has a unique ringhomomorphism $\varphi_U : \Omega(G) \to \mathbb{Z}$ with $\varphi_U(\lfloor S \rfloor) = \varphi_U(S) = |S^U|$ and for any simple G-set T one has as well a unique ringhomomorphism $\varphi_T : \Omega(G) \to \mathbb{Z}$ with $\varphi_T(\lfloor S \rfloor) = \varphi_T(S) = |\mathrm{Hom}_G(T,S)|$.

(d) Let $\mathcal{T}$ be a complete system of nonisomorphic simple G-sets. Then the product-map

$$\prod_{T\in\mathcal{T}} \varphi_T : \Omega(G) \to \prod_{T\in\mathcal{T}} \mathbb{Z}$$

is injective.

(e) Any homomorphism $\psi : \Omega(G) \to R$ (or: $\Omega^+(G) \to R$) into an integral domain R factors (not necessarily uniquely) through some $\varphi_U : \Omega(G) \to \mathbb{Z}$ $(U \leqslant G)$, resp. some $\varphi_T : \Omega(G) \to \mathbb{Z}$ and the unique homomorphism $\mathbb{Z} \to R : n \mapsto n \cdot 1_R$.

Remark: Because $[S] = [T] \Leftrightarrow S \simeq T$ we do not loose much by considering G-sets directly as elements of $\Omega(G)$, i.e. by writing $S (\in \Omega(G))$ instead of $[S]$, $S + T$ instead of $S \dot{\cup} T$ and $S - T$ instead of $[S] - [T]$, the formal difference of G-sets, existing in $\Omega(G)$.

Especially (by Prop.5.2, (b)) any $x \in \Omega(G)$ can be written uniquely in the form $x = \sum_{T\in\mathcal{T}} n_T T$ with $\mathcal{T}$ as in Prop. 5.2, (d) a complete system of nonisomorphic simple G-sets and x is represented by a G-set, if and only if all $n_T \geqslant 0$.

Proof: (a) follows from Prop. 5.1, (b) and Cor. (P.2.3) 1; (b) follows from Prop. 2.3', which can be rephrased as saying that $\Omega^+(G)$ is a free abelian semigroup generated by the isomorphismclasses of indecomposable, i.e. simple or transitive G-sets;

(c) has been proved above;

(d) follows from Thm 4.1: Any $x \in \Omega(G)$ can be written

in the form $x = [S] - [S']$ with S and S' G-sets.
If $\varphi_T(x) = 0$ for all $T \in \mathcal{T}$, then $\varphi_T(S) = \varphi_T(S')$ for all $T \in \mathcal{T}$, thus $S \cong S'$ by Thm 4.1 and therefore $x = [S] - [S'] = 0$;

(e) we give two different proofs:

I: We prove a little more precise:

Proposition 5.2, (e)': Let $\psi : \Omega(G) \to R$ a ring-homomorphism into an integral domain. Then there exists exactly one element $T \in \mathcal{T}_\psi = \{X \in \mathcal{T} \mid \psi(X) \neq 0\}$, which is minimal w.r.t. $<$ in $\mathcal{T}_\psi$, and for this T we have $\psi(x) = \varphi_T(x) \cdot 1_R$.

<u>Proof:</u>

(a) Assume $T, T' \in \mathcal{T}_\psi$ to be minimal and consider $T \times T' = \sum_{X \in \mathcal{T}} n_X X$. Because $\psi(T \times T') = \psi(T) \cdot \psi(T') \neq 0$ in R there exists $X \in \mathcal{T}$ with $\psi(n_X X) = n_X \psi(X) \neq 0$, i.e. with $X \in \mathcal{T}_\psi$ and $X < T$, $X < T'$ $(\varphi_X(T) \cdot \varphi_X(T') = \varphi_X(T \times T') \geqslant n_X \varphi_X(X) > 0!)$, thus by the minimality of T and T' we have $T = X = T'$.

(b) The rest of the proof is based on

<u>Lemma 5.1:</u> For any G-set S and $T \in \mathcal{T}$ the decomposition of $T \times S$ into a sum of simple G-sets has the form $T \times S = \varphi_T(S) \cdot T + \sum_{X \in \mathcal{T}, X \not\geq T} n_X X$ with some nonnegative numbers $n_X \in \mathbb{Z}$.

Proof: By Prop. 2.3' one has $T \times S = \sum_{X \in \mathcal{T}} n_X X$ with $n_X \in \mathbb{Z}$, $n_X \geqslant 0$. But now $\varphi_Y(T \times S) = \varphi_Y(T) \cdot \varphi_Y(S) = 0$ if $Y \nless T$, thus $0 = \varphi_Y(\sum_{X \in \mathcal{T}} n_X X) \geqslant n_Y \varphi_Y(Y) \geqslant n_Y \geqslant 0$, i.e. $n_Y = 0$ for $Y \nless T$.

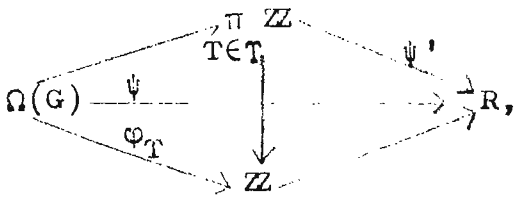
This implies $T \times S = \sum_{X<T} n_X X$ with $n_X \in \mathbb{Z}$, $n_X \geq 0$.
We have to compute n_T. But computing the value of φ_T on both sides yields:

$\varphi_T(T) \cdot \varphi_T(S) = n_T \varphi_T(T)$ which implies $\varphi_T(S) = n_T$, because $\varphi_T(T) \neq 0$.

We now use Lemma 5.1 for our minimal T and get:

$\psi(T \times S) = \psi(T) \cdot \psi(S) = \psi(\varphi_T(S) \cdot T + \sum_{X \in \mathcal{T}, X \lneq T} n_X X) = \varphi_T(S) \cdot \psi(T)$, because by the minimality of T one has $\psi(X) = 0$ for $X \lneq T$. Because R is an integral domain, we can divide both sides by $\psi(T) \neq 0$ and get $\psi(S) = \varphi_T(S) \cdot 1_R$.

II: By Prop. 5.2, (d) $\Omega(G)$ can be considered as a subring of $\prod_{T \in \mathcal{T}} \mathbb{Z}$, which is finite over $\mathbb{Z}$, thus a fortiori over $\Omega(G)$. W.l.o.g. we may assume the ring R to be an algebraically closed field. Then any map $\psi : \Omega(G) \to R$ can be extended to a map $\psi' : \prod_{T \in \mathcal{T}} \mathbb{Z} \to R$ by the going-up-Theorem (Cohen-Seidenberg).
But any such ψ' factors through a projection $\prod_{T \in \mathcal{T}} \mathbb{Z} \to \mathbb{Z}$ and the unique map $\mathbb{Z} \to R$, thus we have a commutative diagram



q.e.d.

<u>Corollary (P.5.2) 1:</u> For $U \leq G$, resp. $T \in \mathcal{T}$ and p a characteristic (i.e. p=0 or p a prime) let $\mathfrak{p}(U,p)$, resp. $\mathfrak{p}(T,p)$ be the primeideal $\{x \in \Omega(G) \mid \varphi_U(x)(\text{resp.}\varphi_T(x)) \equiv 0(p)\}$, i.e. $\mathfrak{p}(U,p) = \text{Kernel}(\Omega(G) \xrightarrow{\varphi_U} \mathbb{Z} \to \mathbb{F}_p)$, $\mathfrak{p}(T,p) = \text{Ke}(\Omega(G) \xrightarrow{\varphi_T} \mathbb{Z} \to \mathbb{F}_p$ with $\mathbb{F}_p$ the primefield of

characteristic p. Then any primeideal $\mathfrak{p}$ in $\Omega(G)$ is of the form $\mathfrak{p}(U,p)$, resp. $\mathfrak{p}(T,p)$ for some appropriate $U \leqslant G$, resp. $T \in \mathcal{T}$, and p.

Proof: Consider the natural map $\psi : \Omega(G) \to R = \Omega(G)/\mathfrak{p}$. Then $\psi(x) = \varphi_U(x) \cdot 1_R$ for some $U \leqslant G$ by Prop. 5.2, (e) and thus

$$\mathfrak{p} = \{x \in \Omega(G) \mid \psi(x) = 0\} = \{x \in \Omega(G) \mid \varphi_U(x) \cdot 1_R = 0\}$$
$$= \{x \in \Omega(G) \mid \varphi_U(x) \equiv 0(p)\} = \mathfrak{p}(U,p) \text{ with } p = \text{char } R.$$

Corollary (P. 5.2)2: For any prime ideal $\mathfrak{p} \subseteq \Omega(G)$ the quotientring $\Omega(G)/\mathfrak{p}$ is a prime ring (i.e. either isomorphic to $\mathbb{Z}$ or $\mathbb{F}_p$, $p \neq 0$), especially two homomorphisms $\psi, \psi' : \Omega(G) \to R$ into an integral domain coincide, if and only if they have the same kernel.

Corollary (P. 5.2)3: For any commutative ring S the primeidealspectrum Spec $S \otimes_{\mathbb{Z}} \Omega(G)$ equals

$$\text{Spec } S \times_{\text{Spec } \mathbb{Z}} \text{Spec } \Omega(G) = \{(\mathfrak{p},\mathfrak{q}) \mid \text{char } S/\mathfrak{p} = \text{char } \Omega(G)/\mathfrak{q}\}.$$

Proof: This is true for all rings with the property, stated in Cor.(P. 5.2)2.

By Prop. 5.2, (d) we can consider $\Omega(G)$ as a subring of a direct product of rings, isomorphic to $\mathbb{Z}$; thus it seems to have a rather simple structure. Unfor - tunately a complete and satisfying classification of isomorphism-types, e.g. a complete list of easily computable invariants of such subrings seems not to be known - and, even if known, might not be very useful for our purposes. But fortunately we can deal with certain rather useful approximations of the problem, to characterize $\Omega(G)$ as a subring of $\prod_{T\in\mathcal{T}} \mathbb{Z}$, i.e. we can decide, wether two primeideals $\mathfrak{p},\mathfrak{p}'$ in $\prod_{T\in\mathcal{T}} \mathbb{Z}$ intersect $\Omega(G)$ in the same prime-ideal (i.e. we can describe the fibers of the surjective map: Spec $\prod_{T\in\mathcal{T}} \mathbb{Z} \twoheadrightarrow$ Spec $\Omega(G)$, - injectivity of this map would imply $\Omega(G) = \prod_{T\in\mathcal{T}} \mathbb{Z}$, which easily can be seen to be wrong for $G \neq E$) and we can compute the elementary divisors of the embedding $\Omega(G) \hookrightarrow \prod_{T\in\mathcal{T}} \mathbb{Z}$, which in turn allows us to prove $\{n \in \mathbb{Z} \mid n \cdot \prod_{T\in\mathcal{T}} \mathbb{Z} \subseteq \Omega(G)\} = |G| \cdot \mathbb{Z}$.

Of course the first question is equivalent to the problem, for which $U,V \leq G$ and characteristic p one has $\mathfrak{p}(U,p) = \mathfrak{p}(V,p)$, resp. $\varphi_U(S) \equiv \varphi_V(S)$ mod p for all G-sets S, and also to the problem, through how many different φ_U a given $\psi : \Omega(G) \to R$ with R an integral domain may be factored.

Of course for $p = \text{char } R = 0$ we know already:
(I) $\mathfrak{p}(U,0) = \mathfrak{p}(V,0) \Leftrightarrow \varphi_U = \varphi_V \Leftrightarrow U \sim V$, resp. $\mathfrak{p}(T,0) = \mathfrak{p}(T',0) \Leftrightarrow \varphi_T = \varphi_{T'} \Leftrightarrow T = T'$ $(T,T' \in \mathcal{T})$, thus in this case there exists exactly one such $\varphi_T : \Omega(G) \to \mathbb{Z}$ with $\psi(x) = \varphi_T(x) \cdot 1_R$ and the corresponding subgroups $U,V,\ldots \subseteq G$ with $\psi(x) = \varphi_U(x) \cdot 1_R$ are conjugate.
So we may assume $p \neq 0$, $R = \mathbb{F}_p$.
By Prop. 5.2, (e)' any $\psi : \Omega(G) \to \mathbb{F}_p$ determines uniquely the element $T \in \mathcal{T}$, which is minimal with

respect to the condition $\psi(T) \not\equiv 0$, and for this T we have $\psi(x) = \varphi_T(x)\ 1_R$. Thus for a given $V \leqslant G$ we have to determine the minimal element $T = T(V,p) \in \mathcal{T}$ with $\varphi_V(T(V,p)) \not\equiv 0 \pmod p$ and then we have of course $\varphi_V \equiv \varphi_T \pmod p$ and

(II) $\mathfrak{p}(V,p) = \mathfrak{p}(W,p) \Leftrightarrow \varphi_V \equiv \varphi_W \pmod p \Leftrightarrow T(V,p) = T(W,p)$ for any two subgroups $V,W \leqslant G$.

Because $\varphi_V(T(V,p)) \not\equiv 0 \pmod p$ we have a fortiori $\varphi_{G/V}(T(V,p)) = \varphi_V(T(V,p)) \not\equiv 0$ and therefore

(III) $G/V < T(V,p)$, especially

(IV) $G/V \cong T(V,p) \Leftrightarrow \varphi_V(G/V) \not\equiv 0(p) \Leftrightarrow p \nmid \varphi_V(G/V) = (N_G(V):V)$, because $G/V \cong T(V,p)$ implies $\varphi_V(G/V) = \varphi_V(T(V,p)) \not\equiv 0(p)$, whereas $\varphi_V(G/V) \not\equiv 0(p)$ implies $T(V,p) < G/V$ by the minimality and uniqueness of $T(V,p)$, which together with (III) implies $G/V \cong T(V,p)$.
But - using (III) once more - (IV) is equivalent to:

(IV') $G/V \lneqq T(V,p) \Leftrightarrow p \mid (N_G(V) : V)$.
Thus $p \mid (N_G(V) : V)$ implies the existence of a subgroup W with $V \lneqq W$ and $\varphi_W \equiv \varphi_V \pmod p$, e.g. a stabilizer-group of an element in $T(V,p)$.
Because on the other hand $p \mid (N_G(V) : V)$ is equivalent to the existence of a subgroup N with $V \triangleleft N$ and $(N : V) = p$ (we write $V \overset{p}{\triangleleft} N$ in this case), the most natural guess of course is, that we may choose $W = N$, i.e. that $\varphi_V \equiv \varphi_N \pmod p$. That this is indeed the case,

states

Lemma 5.2:

Let $V \trianglelefteq N \leq G$ and $(N:V) = p^{\alpha}$ a power of p. Then

$$\varphi_V \equiv \varphi_N \pmod{p}.$$

Proof: Let S be an arbitrary G-set. We have to show $\varphi_V(S) \equiv \varphi_N(S)\ (p)$. But $\varphi_V(S) = |S^V|$ and because $V \triangleleft N$ the set S^V is N-invariant, i.e. an N-subset of S ($s \in S^V$ and $n \in N$ implies $v(ns) = n(n^{-1}vn)s = ns$ for all $v \in V$). Moreover N/V acts on S^V, leaving pointwise invariant S^N, whereas the rest $S^V - S^N$ is a disjoint union of nontrivial transitive N/V-sets, all of which have a length $\neq 1$ dividing $|N/V| = p^{\alpha}$ (Cor.(P.3.1)2). Thus $p \,\big|\, |S^V - S^N|$, i.e. $\varphi_V(S) = |S^V| \cong |S^N| = \varphi_N(S) \pmod{p}$.

Corollary (L.5.2)1:

Assume $V \overset{p}{\triangleleft} V_1 \overset{p}{\triangleleft} \dots \overset{p}{\triangleleft} V_n = U$, then $\varphi_U \equiv \varphi_V \pmod{p}$.

But now we have everything at hand, to solve our problem: For any $V \leqslant G$ consider p-chains over $V : V \overset{p}{\triangleleft} V_1 \overset{p}{\triangleleft} \dots \overset{p}{\triangleleft} V_n$, i.e. sequences of subgroups of G, starting with V and such that any group in such a sequence is normal of index p in the next one. Because G is finite, any such sequence can be continued to a maximal one, i.e. to one, which cannot be continued any further.

If $V \overset{p}{\triangleleft} V_1 \overset{p}{\triangleleft} \dots \overset{p}{\triangleleft} V_n = U$ is such a maximal p-chain over V, then we have $p \nmid (N_G(U) : U)$ - otherwise, as observed above, our sequence could be continued -

and we have $\varphi_V \equiv \varphi_U$ (mod p). The first fact implies $G/U \cong T(U,p)$ by (IV), the second fact implies $T(U,p) = T(V,p)$ by (II), thus altogether we have $T(V,p) \cong G/U$, i.e. the groups, at which a maximal p-chain over V stops, belong to the class of conjugate subgroups $H \leqslant G$ with $G/H \cong T(V,p)$, especially they are all conjugate and all maximal p-chains over V have the same length.
Moreover for two arbitrary subgroups $V,W \leqslant G$ we have $\varphi_V \equiv \varphi_W$(mod p), if and only if there exist p-chains over V and W respectively:

$$V = V_0 \overset{p}{\lhd} V_1 \overset{p}{\lhd} \dots \overset{p}{\lhd} V_n, \quad W = W_0 \overset{p}{\lhd} W_1 \overset{p}{\lhd} \dots \overset{p}{\lhd} W_m$$

(e.g. maximal ones) with $V_n \sim W_m$; i.e. the equivalence-relation $V \overset{p}{\sim} W \Leftrightarrow \mathfrak{p}(V,p) = \mathfrak{p}(W,p)$ is the finest equivalence-relation, which contains $\sim$ (G-conjugacy) and the relation $\overset{p}{\lhd}$.

Already this result seems quite satisfying, because it allows us, to describe the relation $V \overset{p}{\sim} W$ in purely grouptheoretic terms. But on the other hand, the notion of a p-chain still is a bit difficult to handle, e.g. it may be quite difficult, actually to determine maximal p-chains over a given subgroup V, especially because generally there may be many different maximal p-chains over V, even if - as we have seen above - the final groups in such chains are all conjugate, or to decide, whether $U \leqslant V \leqslant W \leqslant G$ and $\varphi_U \equiv \varphi_W$(mod p) implies $\varphi_U \equiv \varphi_V$(mod p), or whether in case $U,V \leqslant H \leqslant G$ and $\varphi_U \equiv \varphi_V$(mod p) on G-sets one may also have

$\varphi_U \equiv \varphi_V (\mathrm{mod}\ p)$ on H-sets.

But fortunately these matters can be simplified considerably, using the following

<u>Lemma (and Definition) 5.3:</u>

For a group H let $H^{(p)}$ be the (uniquely determined!) smallest normal subgroup in H with a p-power-index (i.e. the intersection of all normal subgroups in H with p-power-index). Then one has:

(a) $H^{(p)}$ is characteristic in H.

(b) If the p-part $|H|_p$ of $|H|$ (i.e. the highest power of p, dividing $|H|$) divides p^α (i.e. if $(|H|, p^\alpha) \leqslant p^\alpha$), then $H^{(p)} = \langle h^{p^\alpha} | h \in H \rangle$.

(c) If $\varphi : H \to G$ is a grouphomomorphism, then $\varphi(H^{(p)}) \subseteq G^{(p)}$, especially $U \leqslant V \Rightarrow U^{(p)} \leqslant V^{(p)}$ and $U \overset{G}{\sim} V \Rightarrow U^{(p)} \overset{G}{\sim} V^{(p)}$.

(d) $(U^{(p)})^{(p)} = U^{(p)}$.

(e) $U \lhd V$, $(V : U) = p^\alpha \Rightarrow V^{(p)} = U^{(p)}$.

Proof: Easy, left to the reader.

But now we can state:

(V) Let $U, V \leqslant G$. Then $\varphi_U \equiv \varphi_V (\mathrm{mod}\ p) \Leftrightarrow U^{(p)} \sim V^{(p)}$.

Proof: $\varphi_U \equiv \varphi_V (\mathrm{mod}\ p)$ implies the existence of p-chains $U \overset{p}{\lhd} U_1 \overset{p}{\lhd} \dots \overset{p}{\lhd} U_n$ and $V \overset{p}{\lhd} V_1 \overset{p}{\lhd} \dots \overset{p}{\lhd} V_m$ with $U_n \sim V_m$.

But then by Lemma 5.3 (e) and (c):

$$U^{(p)} = U_1^{(p)} = \dots = U_n^{(p)} \sim V_m^{(p)} = \dots = V^{(p)}.$$

On the other hand $U^{(p)} \sim V^{(p)}$ implies

$$\varphi_U \equiv \varphi_{U^{(p)}} = \varphi_{V^{(p)}} \equiv \varphi_V \bmod p, \text{ q.e.d.}$$

We sum up our results in

Theorem 5.1

(a) Let $\mathfrak{p}$ be a primeideal in $\Omega(G)$, $p = \text{char}\ \Omega(G)/\mathfrak{p}$ and $\mathfrak{f}_{\mathfrak{p}} = \{U \leqslant G \mid \mathfrak{p} = \mathfrak{p}(U,p)\}$. Then

(i) $\mathfrak{f}_{\mathfrak{p}} \neq \emptyset$.

(ii) All maximal elements in $\mathfrak{f}_{\mathfrak{p}}$ are conjugate; they are exactly the subgroups U with G/U (up to isomorphy) the unique minimal simple G-set with $G/U \notin \mathfrak{p}$, i.e. they are the minimal subgroups $U \leqslant G$ with $G/U \notin \mathfrak{p}$.

(iii) All minimal elements in $\mathfrak{f}_{\mathfrak{p}}$ are conjugate. If $U \in \mathfrak{f}_{\mathfrak{p}}$, then $U^{(p)}$ is minimal in $\mathfrak{f}_{\mathfrak{p}}$; especially U is minimal, if and only if $U = U^{(p)}$, i.e. U has no nontrivial p-factorgroup.

(iv) If $U \in \mathfrak{f}_{\mathfrak{p}}$, then $\mathfrak{f}_{\mathfrak{p}} = \{V \leqslant G \mid V^{(p)} \sim U^{(p)}\}$.

(v) If $U,V \in \mathfrak{f}_{\mathfrak{p}}$ and $U \leqslant V$, then $U^{(p)} = V^{(p)}$.

(vi) If $U \leqslant W \leqslant V$ and $U,V \in \mathfrak{f}_{\mathfrak{p}}$, then $W \in \mathfrak{f}_{\mathfrak{p}}$.

(vii) If $U,V \in \mathfrak{f}_{\mathfrak{p}}$ and $U \leqslant V \leqslant H \leqslant G$, then $\varphi_U \equiv \varphi_V (\text{mod}\ p)$ also on H-sets.

(viii) If $U,V \in \mathfrak{f}_{\mathfrak{p}}$ and $U \leqslant V$, then there exists a p-chain $U \overset{p}{\triangleleft} U_1 \overset{p}{\triangleleft} \dots \overset{p}{\triangleleft} U_n = V$. If U is minimal, then $U = V^{(p)} \triangleleft V$. Moreover V is maximal, if and only if V/U is a p-Sylow-subgroup in $N_G(U)/U$.

(b) For $U,V \leqslant G$ and p a prime the following conditions are equivalent:

(i) $\varphi_U \equiv \varphi_V \pmod p$.

(ii) $\mathfrak{p}(U,p) = \mathfrak{p}(V,p)$.

(iii) $U^{(p)} \sim V^{(p)}$.

(iv) There exist p-chains $U \overset{p}{\lhd} U_1 \overset{p}{\lhd} \dots \overset{p}{\lhd} U_n$, $V \overset{p}{\lhd} V_1 \overset{p}{\lhd} \dots \overset{p}{\lhd} V_m$ with $U_n \sim V_m$.

(v) If $U \overset{p}{\lhd} U_1 \overset{p}{\lhd} \dots \overset{p}{\lhd} U_n$, $V \overset{p}{\lhd} V_1 \overset{p}{\lhd} \dots \overset{p}{\lhd} V_m$ are maximal p-chains over U and V respectively, then $U_n \sim V_m$.

(c) For $U,V \leqslant G$ we have

$$\varphi_U = \varphi_V \Leftrightarrow \mathfrak{p}(U,0) = \mathfrak{p}(V,0) \Leftrightarrow U \sim V.$$

Proof:

(a), (i) and (ii) are restatements of results above, (iii) and (iv) follow immediately from (V) above. (v): Lemma 5.3, (c) implies $U^{(p)} \leqslant V^{(p)}$, whereas (V) implies $U^{(p)} \sim V^{(p)}$, thus we have $U^{(p)} = V^{(p)}$. (vi) follows from (v) and (iv), (vii) from (v), because the statement "$U^{(p)} = V^{(p)}$" is independent of the imbeddings $U,V \subseteq H$ or $U,V \subseteq G$. (viii) follows from (vii), applied to $V = H$, because by previous results the final group of a maximal p-chain over U in V must be conjugate in V to the final group of a maximal p-chain over V in V, i.e. must be conjugated in V to V, i.e. must be V itself, q.e.d.

The next statement in (viii) is trivial by now. But it implies, that there exists a 1-1-correspondence between groups $V \in \mathfrak{F}_{\mathfrak{p}}$ with $U \leqslant V$ and p-subgroups of $N_G(U)/U$, - thus the last statement.

(b) and (c) are trivial by now.

Finally we have to prove:

Theorem 5.2:

Identifying $\Omega(G)$ with its image in $\prod_{T\in\mathcal{T}} \mathbb{Z} = \widetilde{\Omega(G)}$ with respect to the map $\prod_{T\in\mathcal{T}} \varphi_T$ we have

$$\{n \in \mathbb{Z} \mid n\,\widetilde{\Omega(G)} \subseteq \Omega(G)\} = |G|\,\mathbb{Z}.$$

Proof: The proof is based on

Lemma 5.4:

Consider $\Omega(G)$ and $\widetilde{\Omega(G)}$ both as subrings of

$\prod_{T\in\mathcal{T}} \mathbb{Q} (\cong \mathbb{Q} \otimes \Omega(G)) = \{(x_T)_{T\in\mathcal{T}} \mid x_T \in \mathbb{Q}\}$, i.e.

$\widetilde{\Omega(G)} = \{(x_T)_{T\in\mathcal{T}} \in \prod_{T\in\mathcal{T}} \mathbb{Q} \mid x_T \in \mathbb{Z}\}$ and

$\Omega(G) = \sum_{S\in\mathcal{T}} \mathbb{Z}\cdot(\varphi_T(S))_{T\in\mathcal{T}} \subseteq \prod_{T\in\mathcal{T}} \mathbb{Q}$.

Then $\mathcal{T}' = \{\frac{1}{|\mathrm{Aut}_G(S)|} S = (\frac{\varphi_T(S)}{\varphi_S(S)})_{T\in\mathcal{T}} \mid S \in \mathcal{T}\}$

is a basis for $\widetilde{\Omega(G)}$.

Proof: By Lemma 4.1 we have $|\mathrm{Aut}_G(S)| = \varphi_S(S) \mid \varphi_T(S)$, thus $\frac{1}{|\mathrm{Aut}_G(S)|}\cdot S$ is indeed an element in $\widetilde{\Omega(G)}$ for any $S \in \mathcal{T}$. Now compare the set $\mathcal{T}'$ with the canonical basis $\mathfrak{B} = \{i_S = (\delta_T^S)_{T\in\mathcal{T}} \mid S \in \mathcal{T}\}$ of $\widetilde{\Omega(G)}$.

Because $|\mathfrak{B}| = |\mathcal{T}| = |\mathcal{T}'|$, it is enough to show, that any $i_S \in \mathfrak{B}$ is an integral linear combination of the elements of $\mathcal{T}'$. We do this by induction w.r.t. $<$:

For $S \cong G/E$ we have $i_S = \frac{1}{|\mathrm{Aut}_G(S)|} S \in \mathcal{T}'$, because $\varphi_T(\frac{1}{|\mathrm{Aut}_G(G/E)|} G/E) = \delta_T^{G/E}$. For arbitrary S we have

$$\varphi_S(\frac{1}{|\mathrm{Aut}_G(S)|} S) = 1$$

and $\varphi_T(\frac{1}{|\mathrm{Aut}_G(S)|} S) = 0$ for $T \not\leq S$, thus

$\frac{1}{|\mathrm{Aut}_G(S)|} S = i_S + \sum_{T \in \mathcal{T}, T \lneqq S} n_{T,S}\, i_T$ with

$n_{T,S} = \frac{\varphi_T(S)}{\varphi_S(S)} \in \mathbb{Z}$.

But now by induction any i_T with $T \lneqq S$ is an integral linear combination of the elements in $\mathcal{T}'$, thus the same is true for i_S, q.e.d.

Remark: In other words: If we order the elements in $\mathcal{T} = \{S_1 = G/E, S_2 \ldots\}$ in such a way, that $S_i < S_j$ implies $i \leq j$, then the matrix

$$(a_{ij})_{i,j} = \left(\frac{\varphi_{S_i}(S_j)}{\varphi_{S_j}(S_j)} \right)_{i,j},$$

which transforms $\mathfrak{B}$ into $\mathcal{T}'$ is triangular with 1's in the maindiagonal, thus unimodular and therefore $\mathcal{T}'$ is a basis of $\widetilde{\Omega(G)}$ as well as $\mathfrak{B}$.

Proof of Thm. 5.2: Obviously we have for $n \in \mathbb{Z}$: $n\widetilde{\Omega(G)} \subseteq \Omega(G) \Leftrightarrow |\mathrm{Aut}_G(S)| \,\big|\, n$ for all simple G-sets $S \Leftrightarrow |N_G(U) : U| \,\big|\, n$ for all subgroups $U \leq G \Leftrightarrow |G| \,\big|\, n$.

Remark 1:

I want to point out, that the ring $\widetilde{\Omega(G)}$ and the map $\prod_{T \in \mathcal{T}} \varphi_T : \Omega(G) \to \widetilde{\Omega(G)}$ depend only on the ring-structure of $\Omega(G)$, because for instance the maps φ_T are exactly the various different ringhomomorphisms: $\Omega(G) \to \mathbb{Z}$. Moreover their product $\prod_{T \in \mathcal{T}} \varphi_T : \Omega(G) \to \widetilde{\Omega(G)}$ can as well be considered as the canonical imbedding of $\Omega(G)$ into its integral closure $\widetilde{\Omega(G)}$ in its total quotientring, which is isomorphic to $\prod_{T\in\mathcal{T}} \mathbb{Q} \cong \mathbb{Q}\otimes\Omega(G)$.

Thus one can interpret Thm. 5.2 as a result, concerning the conductor $\mathfrak{f}_{\Omega(G)}^{\widetilde{\Omega(G)}} = \{x\in\widetilde{\Omega(G)} \mid x\cdot\widetilde{\Omega(G)} \subseteq \Omega(G)\}$ (i.e. the maximal ideal of $\widetilde{\Omega(G)}$, contained in $\Omega(G)$), stating that this ideal intersected with $\mathbb{Z}\cdot 1_{\Omega(G)}$ is generated by $|G|\cdot 1_{\Omega(G)}$. Especially it implies, that the order of G is determined by the ringstructure of $\Omega(G)$.

Remark 2:

Lemma 5.4 can of course also be used, to compute the elementary divisors of $\widetilde{\Omega(G)}$ over $\Omega(G)$; up to a reordering of prime power factors they are more or less the numbers $(N_G(U) : U)$, especially they all divide $|G|$,- which is also an obvious corollary of Thm. 5.2.

Of course there are some connections between Thm 5.1 and 5.2, as can be seen using:

<u>Lemma 5.5:</u>

Let R be a subring of a product of a finite number of copies of $\mathbb{Z}$, let $\tilde{R}$ be its integral closure in its total quotientring $\mathbb{Q} \otimes_{\mathbb{Z}} R$ and let $S \subseteq R$ be a multiplicatively closed subset with $0 \notin S$. Then the following assertions are equivalent:

(i) $R_S \to \tilde{R}_S$ is an isomorphism

(ii) $S \cap \mathfrak{f}_R^{\tilde{R}} \neq \emptyset$

(iii) If $\mathfrak{p}$ is a primeideal with $S \cap \mathfrak{p} = \emptyset$, then $\mathfrak{p}$ does not split in $\tilde{R}$, i.e. there is only one primeideal $\tilde{\mathfrak{p}}$ in $\tilde{R}$ with $\tilde{\mathfrak{p}} \cap R = \mathfrak{p}$.

Now the fact, that by Thm 5.1 a primeideal $\mathfrak{p} \subseteq \Omega(G)$ splits in $\widetilde{\Omega(G)}$ only if $0 \neq \operatorname{char} \Omega(G)/\mathfrak{p} = p$ divides $|G|$, and Lemma 5.5, applied for $R = \Omega(G)$, $S = \{|G|^n \cdot 1_{\Omega(G)} \mid n > 0\}$, imply, that at least a power of $|G|$ is contained in $\mathfrak{f}_{\Omega(G)}^{\widetilde{\Omega(G)}}$.
On the other hand, Thm 5.2 and Lemma 5.5 imply, that a primeideal $\mathfrak{p}$ in $\Omega(G)$ splits in $\widetilde{\Omega(G)}$ only if $0 \neq \operatorname{char} \Omega(G)/\mathfrak{p} = p$ divides $|G|$ and that for any such $p \mid |G|$ there exists at least one primeideal $\mathfrak{p}$ in $\Omega(G)$, which splits in $\widetilde{\Omega(G)}$.
We are now prepared, to develop the rather general theory of Mackey-functors in the following chapter. Of course one can also use the above results as a starting point for a more thorough treatment of the ringtheoretic properties of $\Omega(G)$ and their relations to grouptheoretic properties of G. But here only partial results are known - some indeed rather peculiar - and we will deal with them only after the general theory is developed.

Chapter II.

Mackey - functors

In this chapter we will develop a rather general axiomatic theory, using the language of functors and categories. This theory will be applied later on to the study of integral representation rings of finite groups, especially to induction-theory and related topics.
To justify a new axiomatic theory one has to show, that it covers various interesting examples and still allows to draw sufficiently strong conclusions, which applied to the various examples lead to admittedly important (known, unknown or conjectured) results, thus offering a unified treatment of various, sometimes even rather different concrete problems.

Unfortunately one cannot do all three things (giving examples, developing the theory, applying the results to the examples) at the same time. Moreover the main point of this lecture is to apply our theory to integral representationtheory, where things get anyway rather complicated and lengthy, whereas the best psychological justification for the theory might be the fact, that applied to classical examples it offers a rather easy, quick and unified approach to many wellknown basic results (which of course also will turn out later as special cases of considerably more general results).

Therefore we will first develop the axiomatic theory - giving only few examples and asking the reader just to believe, that this general abstract nonsense really has some useful consequences - then construct some rather general examples, mostly related to integral representationtheory, which can be discussed thoroughly only in the following chapters, but finally include a section, where the relation to classical examples and results is explained briefly and independent of the rest of these lectures.

§ 6 Some basic definitions

We start with the definition of bifunctors between two categories $\mathfrak{C}$ and $\mathfrak{D}$. Later on $\mathfrak{C}$ will be the category $G^\wedge$ of G-sets (G a finite group) and $\mathfrak{D}$ the category $\underline{\underline{\mathfrak{Ab}}}$ of abelian groups. But right now let $\mathfrak{C}$ and $\mathfrak{D}$ be arbitrary. Then a bifunctor $F : \mathfrak{C} \to \mathfrak{D}$ from $\mathfrak{C}$ into $\mathfrak{D}$ is a pair (F^*, F_*) of functors $F^*: \mathfrak{C} \to \mathfrak{D}$ and $F_*: \mathfrak{C} \to \mathfrak{D}$ with F^* covariant and F_* contravariant, which are identical on the objects of $\mathfrak{C}$, i.e. F associates to any object X in $\mathfrak{C}$ an object F(X) in $\mathfrak{D}$ and to any map $\varphi : X \to Y$ in $\mathfrak{C}$ two maps $F^*(\varphi) = \varphi^*: F(X) \to F(Y)$ and $F_*(\varphi) = \varphi_*: F(Y) \to F(X)$, such that for any composition $\varphi\psi$ of maps in $\mathfrak{C}$ one has $(\varphi\psi)^* = \varphi^*\psi^*$ and $(\varphi\psi)_* = \psi_*\varphi_*$ in $\mathfrak{D}$ and $(\mathrm{Id}_X)^* = (\mathrm{Id}_X)_* = \mathrm{Id}_{F(X)}$ for any object X in $\mathfrak{C}$.

In other words: One may define for any category $\mathfrak{D}$ the category $\mathrm{Bi}(\mathfrak{D})$, which has the same objects as $\mathfrak{D}$ but morphisms $[A,B]_{\mathrm{Bi}(\mathfrak{D})} = [A,B]_{\mathfrak{D}} \times [B,A]_{\mathfrak{D}}$ with obvious compositions and identities. Then a bifunctor $F : \mathfrak{C} \to \mathfrak{D}$ is nothing else than a functor $F : \mathfrak{C} \to \mathrm{Bi}(\mathfrak{D})$. But this last description is unsuitable for defining natural transformations between bifunctors in a useful manner. Because if one wants to define - at least for a small category $\mathfrak{C}$ - the category $\mathfrak{B}(\mathfrak{C},\mathfrak{D})$ of bifunctors from $\mathfrak{C}$ into $\mathfrak{D}$ in such a way, that $\mathfrak{B}(\mathfrak{C},\mathfrak{D})$

preserves all those good properties of $\mathfrak{D}$, which are generally known to be preserved by functor-categories over $\mathfrak{D}$, then one has to define a natural transformation $\Theta : E \to F$ between two bi-functors $E, F : \mathfrak{C} \to \mathfrak{D}$ (i.e. a morphism in $\mathfrak{B}(\mathfrak{C},\mathfrak{D})$) as a family of maps:

$\Theta(X) : E(X) \to F(X)$ (X ranges over the objects of $\mathfrak{C}$), such that for any map $\varphi : X \to Y$ in $\mathfrak{C}$ one has two commutative diagrams:

$$\begin{array}{ccc} E(X) & \xrightarrow{E^*(\varphi)} & E(Y) \\ \downarrow{\scriptstyle \Theta(X)} & & \downarrow{\scriptstyle \Theta(Y)} \\ F(X) & \xrightarrow{F^*(\varphi)} & F(Y) \end{array} \quad \text{and} \quad \begin{array}{ccc} E(Y) & \xrightarrow{E_*(\varphi)} & E(X) \\ \downarrow{\scriptstyle \Theta(Y)} & & \downarrow{\scriptstyle \Theta(X)} \\ F(Y) & \xrightarrow{F_*(\varphi)} & F(X), \end{array}$$

i.e. such that Θ is as well a natural transformation from E^* into F^* as from E_* into F_*.

For our purpose we have to consider special bifunctors: We define a bifunctor $\mathfrak{M} : \mathfrak{C} \to \mathfrak{D}$ to be a Mackey-functor, if the following two conditions are fullfilled by $\mathfrak{M}$:

(M1) If
$$\begin{array}{ccc} X & \xrightarrow{\bar{\varphi}} & X_2 \\ \downarrow{\scriptstyle \bar{\psi}} & & \downarrow{\scriptstyle \psi} \\ X_1 & \xrightarrow[\varphi]{} & Y \end{array}$$
is a pull back diagram in $\mathfrak{C}$,

then the diagram
$$\begin{array}{ccc} \mathfrak{M}(X) & \xrightarrow{\bar{\varphi}^*} & \mathfrak{M}(X_2) \\ \uparrow{\scriptstyle \bar{\psi}_*} & & \uparrow{\scriptstyle \psi^*} \\ \mathfrak{M}(X_1) & \xrightarrow[\varphi^*]{} & \mathfrak{M}(Y) \end{array}$$
commutes.

(M2) If $X + Y = Z$ is the sum of X and Y in $\mathfrak{C}$ with $i : X \to X+Y = Z$, $j : Y \to X+Y = Z$ the canonical maps of the summands into the sum, then

$$\mathfrak{M}(Z) \xrightarrow{i_* \times j_*} \mathfrak{M}(X) \times \mathfrak{M}(Y)$$

is an isomorphism, i.e. $\mathfrak{M}(Z)$ together with the two maps $i_*: \mathfrak{M}(Z) \to \mathfrak{M}(X)$, $j_*: \mathfrak{M}(Z) \to \mathfrak{M}(Y)$ is the product of $\mathfrak{M}(X)$ and $\mathfrak{M}(Y)$ in $\mathfrak{D}$. Or in other words: $\mathfrak{M}_*$ trans - forms sums in $\mathfrak{C}$ into products in $\mathfrak{D}$. We want to state a few immediate consequences:

<u>Lemma 6.1:</u>

If $\mathfrak{M} : \mathfrak{C} \to \mathfrak{D}$ is a Mackey-functor and $\alpha : X \to Y$ an isomorphism, then $\alpha^*: \mathfrak{M}(X) \to \mathfrak{M}(Y)$ is the inverse of $\alpha_*: \mathfrak{M}(Y) \to \mathfrak{M}(X)$.

Proof: Obviously

$$\begin{array}{ccc} X & \xrightarrow{Id} & X \\ \downarrow{Id} & & \downarrow{\alpha} \\ X & \xrightarrow[\alpha]{} & Y \end{array}$$

is a pull back diagram in $\mathfrak{C}$, thus using (M1) we get a commutative diagram

$$\begin{array}{ccc} \mathfrak{M}(X) & \xrightarrow{Id} & \mathfrak{M}(X) \\ \uparrow{Id} & & \uparrow{\alpha_*} \\ \mathfrak{M}(X) & \xrightarrow[\alpha^*]{} & \mathfrak{M}(Y) \end{array} ,$$

i.e. $\alpha_*\alpha^* = Id_{\mathfrak{M}(X)}$. Because $\alpha_*$and α^* are isomorphisms, we have also $\alpha^*\alpha_* = Id_{\mathfrak{M}(Y)}$, q.e.d.

<u>Lemma 6.2:</u>

Let $\mathfrak{C}$ be the category $G^{\wedge}$ of G-sets (G a finite group), $\mathfrak{D}$ an abelian category and $\mathfrak{M} : \mathfrak{C} \to \mathfrak{D}$ a Mackey-functor. Then

(a) $\mathfrak{M}(\emptyset) = 0$ and

(b) for any sum $S \dot{\cup} T$ of two G-sets S and T with the canonical imbeddings $i : S \to S \dot{\cup} T$, $j : T \to S \dot{\cup} T$ the composite

$$\mathfrak{M}(S) \oplus \mathfrak{M}(T) \xrightarrow{i^* \oplus j^*} \mathfrak{M}(S \dot\cup T) \xrightarrow{i_* \times j_*} \mathfrak{M}(S) \times \mathfrak{M}(T)$$

is the canonical isomorphism

$$\mathfrak{M}(S) \oplus \mathfrak{M}(T) \to \mathfrak{M}(S) \times \mathfrak{M}(T),$$

existing in abelian categories.

Especially $i^* \oplus j^*$ is as well an isomorphism as $i_* \times j_*$ and one has $Id_{\mathfrak{M}(S \dot\cup T)} = i^* i_* + j^* j_*$.

Proof:

(a) We consider $\emptyset$ as the sum of $\emptyset$ and $\emptyset$ with the identities $\begin{array}{l} \emptyset \xrightarrow{Id} \\ \emptyset \xrightarrow[Id]{} \end{array} \emptyset$ as the canonical maps of the summands into the sum. Then (M2) implies, that the diagonal $\mathfrak{M}(\emptyset) \xrightarrow{Id_* \times Id_*} \mathfrak{M}(\emptyset) \times \mathfrak{M}(\emptyset)$ is an isomorphism, which can hold only for $\mathfrak{M}(\emptyset) = 0$.

(b) We have to show: $i_* i^* : \mathfrak{M}(X) \to \mathfrak{M}(X \dot\cup Y) \to \mathfrak{M}(X)$ is the identity and $j_* i^* : \mathfrak{M}(X) \to \mathfrak{M}(X \dot\cup Y) \to \mathfrak{M}(Y)$ the zeromap. But this follows from (M1), applied to the pull back diagrams $\begin{array}{ccc} X & \xrightarrow{Id} & X \\ \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle i} \\ X & \xrightarrow[i]{} & X \dot\cup Y \end{array}$ and $\begin{array}{ccc} \emptyset & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle j} \\ X & \xrightarrow[i]{} & X \dot\cup Y \end{array}$, and $\mathfrak{M}(\emptyset) = 0$.

Remark:

Of course Lemma 6.2 is true for much more general categories than $G^\wedge$, e.g. categories of functors into the category of sets. More precisely the proof shows, that (a) holds for any initial object in an arbitrary category, whereas (b) holds, whenever the above diagrams are pull backs.

Finally we want to give two examples of Mackey-functors, defined over $\mathfrak{C} = G^{\wedge}$, G a finite group and with $\mathfrak{D}$ the category of abelian groups:

(1) We consider G-sets as compact (discret) topological G-spaces and use complex G-vector bundles over a G-set S (with not necessarily constant, but finite fiberdimensions), to construct the Grothendieckgroup $K_G(S)$. A complete treatment of this and related examples without any reference to K_G-theory will be given in § 9. For any G-map: $\varphi: S \to T$ the pullback of bundles over T to bundles over S defines a map $\varphi_*: K_G(T) \to K_G(S)$. Moreover because $\varphi: S \to T$ is finite, one has for any bundle $\begin{smallmatrix}\xi\\ \downarrow\\ S\end{smallmatrix}$ the direct image $\begin{smallmatrix}\varphi^*(\xi)\\ \downarrow\\ T\end{smallmatrix}$ with fibers $\varphi^*(\xi)_t = \bigoplus_{s\in\varphi^{-1}(t)} \xi_s$ and G-action $g\cdot(\sum_{s\in\varphi^{-1}(t)} x_s) = \sum_{s\in\varphi^{-1}(t)} gx_s$ $(x_s \in \xi_s)$. Because $\varphi^*(\xi \oplus \xi') = \varphi^*(\xi) \oplus \varphi^*(\xi')$ one thus gets an additive map

$$\varphi^*: K_G(S) \to K_G(T).$$

(M1) and (M2) are easily verified. Moreover there is a wellknown canonical isomorphism

$$K_G(G/U) \xrightarrow{\sim} X(U),$$

where X(H) is the characterring of U, especially $K_G(*_G) \xrightarrow{\sim} X(G)$. Thus the map $\eta = \eta_{G/U} : G/U \to *_G$ defines maps

$$\eta_*: X(G) \to X(U),$$
$$\eta^*: X(U) \to X(G).$$

These maps are easily identified with the wellknown restriction and induction of characters.

Now consider the pullback

$$\begin{array}{ccc} G/U \times G/V & \to & G/V \\ \downarrow & & \downarrow \\ G/U & \to & G/G \end{array}$$

and apply (M1) for $\mathfrak{M} = K_G$. Using Prop. 1.1 (G-orbits in $G/U \times G/V$ are in 1-1 correspondence with the double cosets UgV in G), (M2) and the identifications $K_G(G/H) \xrightarrow{\sim} X(H)$ one sees easily, that (M1) is equivalent with the Mackey-subgroup-theorem (see CR, p.324). This of course was the reason for the choice of the name "Mackey"-functor.

Finally let $\mathfrak{g}$ be set of all cyclic subgroups $C \leqslant G$ and consider $\eta = \eta_S: S = \dot{\bigcup}_{C \in \mathfrak{g}} G/C \to *_G$. Then

$$\eta_*: K_G(*) \to K_G(S)$$

is injective, because $K_G(S) \cong \prod_{C \in \mathfrak{g}} X(C)$ and the product of the restrictions $X(G) \to \prod_{C \in \mathfrak{g}} X(C)$ is obviously injective. Moreover

$$\eta^*(K_G(S)) \supseteq |G| \cdot K_G(*_G)$$

by Artin's inductiontheorem (with $|G| \cdot \mathfrak{A} = \{|G| \cdot a \mid a \in \mathfrak{A}\}$ for any abelian group $\mathfrak{A}$), using again the above identification.

It will be one of the first applications of the theorie of Mackey-functors, to show, that the injectivity of η_* together with the fact, that K_G is a Mackey-functor, already implies Artin's theorem. Moreover with $\mathfrak{H} = \{H \leqslant G \mid \exists C \trianglelefteq H,\ C \text{ cyclic},\ H/C \text{ a p-group}\}$ the set of hyperelementary subgroups of G our theory implies the surjectivity of the inductionmap

$$\sum_{H \in \mathfrak{H}} X(H) \to X(G).$$

Of course Brauer has shown, that in this case $\mathfrak{H}$ may be replaced by the even smaller set of elementary subgroups, so our theory does not lead to the best possible results for K_G.
But on the one hand the techniques of Mackey-functors apply as well to rational representation-rings, where $\mathfrak{H}$ is known to be the smallest possible set, on the other hand it is possible to refine our theory by considering monomial and not only permutation representations, to get the Theorem of Brauer as well as the Berman-Witt-Theorem as applications.

(2) For any G-set S let $G^\wedge/S$ be the category of G-sets over S. (For a category $\mathfrak{C}$ and an object X in $\mathfrak{C}$ the category $\mathfrak{C}/X$ has objects the pairs (Y,φ) with Y an object in $\mathfrak{C}$ and $\varphi : Y \to X$ a morphism in $\mathfrak{C}$ and morphisms $[(Y,\varphi),(Y',\varphi')]_{\mathfrak{C}/Y} = \{\psi \in [Y,Y']_{\mathfrak{C}} \mid \varphi = \varphi'\psi\}$, i.e. a morphism from $\varphi : Y \to X$ into $\varphi' : Y' \to X$ is a commutative triangel $\begin{array}{ccc} Y & \xrightarrow{\psi} & Y' \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\varphi'} & \\ & X & \end{array}$ in $\mathfrak{C}$.).

Because $*$ is a final object in $G^\wedge$, one has obviously $G^\wedge/* \cong G^\wedge$.
In $G^\wedge/S$ one has sums: $(T,\varphi) + (T',\varphi') = (T\dot{\cup}T', \varphi\dot{\cup}\varphi' : T\dot{\cup}T \to S)$.
Thus the set of isomorphismclasses $\Omega^+(S)$ in $G^\wedge/S$ has naturally the structure of an abelian semigroup. Let $\Omega(S)$ be the associated group. Of course $\Omega(*_G)$ is just the additive group of the ring $\Omega(G)$, considered in § 5.

For any G-map : $\varphi : S \to S'$ I want to define additive maps:

$$\Omega^*(\varphi) = \varphi^*: \Omega(S) \to \Omega(S')$$

$$\Omega_*(\varphi) = \varphi_*: \Omega(S') \to \Omega(S),$$

such that Ω becomes a Mackey-functor.

It is enough to define additive maps

$$\varphi^*: \Omega^+(S) \to \Omega^+(S'),$$

$$\varphi_*: \Omega^+(S') \to \Omega^+(S),$$

which is rather easy: for $(T,\alpha) \in \Omega^+(S)$ we define $\varphi^*(T,\alpha) = (T, \varphi\alpha : T \to S') \in \Omega^+(S)$, for $(T,\beta) \in \Omega^+(S')$ we define $\varphi_*(T,\beta)$ to be the pull back $\overline{\beta} = \beta|_\varphi$ of β with respect to φ, i.e. we construct the pull back

$$\begin{array}{ccc} T \times_{S'} S & \xrightarrow{\overline{\varphi}} & T \\ {\scriptstyle \overline{\beta}=\beta|_\varphi}\downarrow & & \downarrow{\scriptstyle \beta} \\ S & \xrightarrow[\varphi]{} & S' \end{array}$$

and define

$$\varphi_*(T,\beta) = (T \times_{S'} S,\ \overline{\beta}: T \times_{S'} S \to S) \in \Omega^+(S).$$

One verifies easily, that both maps are additive and thus extend from Ω^+ to Ω. (M1) holds for Ω, because by general category theory one has

<u>Lemma 6.3:</u>

In a diagram

$$\begin{array}{ccccc} \overline{T} & \to & S_1 \times_S S_2 & \xrightarrow{\overline{\varphi}} & S_2 \\ \downarrow & & \downarrow{\scriptstyle \overline{\psi}} & & \downarrow{\scriptstyle \psi} \\ T & \xrightarrow[\alpha]{} & S_1 & \xrightarrow[\varphi]{} & S \end{array}$$

with the second square a pull back the first square is a pull back if and only if the rectangle is a pull back.

This shows that for (T,α) - or just α - in $\Omega(S_1)$ one has

$$\psi_*\varphi^*(\alpha) = \psi_*(\varphi\alpha) = (\varphi\alpha)|_{\psi} = \bar{\varphi}(\alpha|_{\bar{\psi}}) = \bar{\varphi}^*\bar{\psi}_*(\alpha).$$

(M2) follows from the fact, that in $G^\wedge$ a map into a disjoint union of G-sets is always the disjoint union of the maps of the preimages of the single summands into these summands, i.e. from

Lemma 6.4:

Let $S = S_1 \dot{\cup} S_2$ be the disjoint union of S_1 and S_2 with imbeddings: $\alpha_i : S_i \to S (i=1,2)$. Then in a commutative diagram

$$\begin{array}{ccccc} T_1 & \xrightarrow{\beta_1} & T & \xleftarrow{\beta_2} & T_2 \\ \downarrow & & \downarrow & & \downarrow \\ S_1 & \xrightarrow{\alpha_1} & S & \xleftarrow{\alpha_2} & S_2 \end{array}$$

the two squares are pull backs, if and only if T is the sum of T_1 and T_2 with respect to the two maps: $\beta_i : T_i \to T (i=1,2)$

Remark: The property of $G^\wedge$, stated in Lemma 6.4, of course holds in all functorcategories over the category of sets and might be an interesting axiom, to characterize (of course together with other axioms) "set like" categories.

By Lemma 6.4 the maps $\Omega_*(\alpha_i) : \Omega^+(S) \to \Omega^+(S_i)$ induce a bijection $\Omega_*(\alpha_1) \times \Omega_*(\alpha_2) : \Omega^+(S) \to \Omega^+(S_1) \times \Omega^+(S_2)$, the inverse map being given by

$(T_1, \gamma_1 : T_1 \to S_1) \times (T_2, \gamma_2 : T_2 \to S_2) \mapsto (T_1 \dot{\cup} T_2, \gamma_1 \dot{\cup} \gamma_2 : T_1 \dot{\cup} T_2 \to S)$,

i.e. Lemma 6.4 verifies (M2) for Ω.

As may be motivated by considering example (1) we are interested in computing for any G-set S the kernel

$K_\Omega(S)$ of $(\eta_S)_* : \Omega(G) = \Omega(*_G) \to \Omega(S)$
and the image $I_\Omega(S)$ of
$(\eta_S)^* : \Omega(S) \to \Omega(*_G) = \Omega(G)$.

So let $\mathcal{T}$ be a complete system of nonisomorphic simple G-sets. We claim

<u>Proposition 6.1:</u>
(a) $K_\Omega(S) = \{x \in \Omega(G) | \varphi_T(x) = 0 \text{ for all } T \in \mathcal{T},\ T < S\}$
(b) $I_\Omega(S) = \{x \in \Omega(G) | \varphi_T(x) = 0 \text{ for all } T \in \mathcal{T},\ T \not< S\}$

Proof:
Because $x = X - X' \in \Omega(G)$ is contained in $K_\Omega(S)$, if and only if the projections $\psi_S : X \times S \to S$ and $\psi'_S : X' \times S \to S$ represent the same element in $\Omega(S)$, i.e. get isomorphic after eventually adding another object $\alpha : S' \to S$ in $G^\wedge/S$, we need criterions for the isomorphy of G-sets over S, similar to Burnside's criterion for the isomorphy of G-sets, as stated in Thm 4.1.
We can get such criterions either by first reducing the problem to simple G-sets S, using the canonical equivalence $G^\wedge/S_1 \dot\cup S_2 \equiv G^\wedge/S_1 \times G^\wedge/S_2$, then identifying the category $G^\wedge/S$ for $S \cong G/U$ with the category $U^\wedge$ of U-sets (see §9) and finally applying Thm 4.1 for U-sets or more directly by generalizing the statement and the proof of Thm 4.1 to more general situations.

We will follow the second path: So let $\mathfrak{C}$ be a category, which contains finite sums (especially an initial object, being identified with the sum over an empty set of objects) and let $\mathfrak{T}$ be a finite set of objects of $\mathfrak{C}$, such that

(I) any object in $\mathfrak{C}$ is isomorphic to a finite sum of copies of elements in $\mathfrak{T}$,

(II) any endomorphism of an element in $\mathfrak{T}$ is an automorphism,

(III) for any $T \in \mathfrak{T}$ and any two objects S_1, S_2 in $\mathfrak{C}$ the canonical map

$$[T, S_1]_{\mathfrak{C}} \dot{\cup} [T, S_2]_{\mathfrak{C}} \to [T, S_1 + S_2]_{\mathfrak{C}}$$

is a bijection,

(IV) for any $T, T' \in \mathfrak{T}$ the set of morphisms $[T, T']_{\mathfrak{C}}$ is finite.

We call such a pair $(\mathfrak{C}, \mathfrak{T})$ or just the category $\mathfrak{C}$ a based category and $\mathfrak{T}$ the basis of $\mathfrak{C}$. The objects in $\mathfrak{T}$ are uniquely determined up to isomorphism: They are exactly the indecomposable objects in $\mathfrak{C}$, especially the initial object is not contained in $\mathfrak{T}$. Moreover w.l.o.g. we may and will assume $\mathfrak{T}$ to contain only pairwise nonisomorphic objects. (I), (III) and (IV) together imply $\varphi_S(S') = |[S, S']_{\mathfrak{C}}| < \infty$ for any two objects S, S' in $\mathfrak{C}$; (II) implies for $T, T' \in \mathfrak{T}$: $T \sim T' \Leftrightarrow T \cong T'$, i.e. $T = T'$ using the above assumption. Thus the relation $<$ defines a strict ordering on $\mathfrak{T}$ in this case.

Of course $G^\wedge$ is a based category with basis a complete set of nonisomorphic simple G-sets. We want to show, that for any G-set S the category $G^\wedge/S$ is based, too. More generally let S be an arbitrary object in $\mathfrak{C}$.

Because a sum of objects $(S_i, \varphi_i : S_i \to S)_{i\in I}$ in $\mathfrak{C}/S$ is just the sum $\sum_{i\in I} S_i$ of the objects S_i in $\mathfrak{C}$ together with the canonical map: $\sum_{i\in I} \varphi_i : \sum_{i\in I} S_i \to S$, one verifies easily:

<u>Lemma 6.5:</u>

If $(\mathfrak{C}, \mathcal{T})$ is a based category and S an object in $\mathfrak{C}$, then $(\mathfrak{C}/S, \mathcal{T}/S)$ with $\mathcal{T}/S = \{(T,\varphi) | T \in \mathcal{T}, \varphi \in [T,S]\}$ is a based category, too.

We can generalize Thm 4.1 to based categories and claim (even generalizing the statement for $\mathfrak{C} = G^\wedge$ slightly):

<u>Lemma 6.6:</u>

Let $(\mathfrak{C}, \mathcal{T})$ be a besed category and $X = \sum_{T\in\mathcal{T}} x_T T$, $X' = \sum_{T\in\mathcal{T}} x'_T T$ and S objects in $\mathfrak{C}$.

Then $x_T = x'_T$ for all $T \nless S$, $T \in \mathcal{T}$, if and only if $\varphi_T(X) = \varphi_T(X')$ for all $T \nless S$, $T \in \mathcal{T}$. Especially $X \cong X' \Leftrightarrow \varphi_T(X) = \varphi_T(X')$ for all $T \in \mathcal{T} \Leftrightarrow x_T = x'_T$ for all $T \in \mathcal{T}$.

Proof:

Of course $x_T = x'_T$ for $T \nless S$, $T \in \mathcal{T}$ implies

$$\varphi_Y(X) = \sum_{T\in\mathcal{T}} x_T \varphi_Y(T) = \sum_{T\in\mathcal{T}, Y<T} x_T \varphi_Y(T) = \sum_{T\in\mathcal{T}, Y<T} x'_T \varphi_Y(T) =$$

$= \varphi_Y(X')$ for all $Y \in \mathcal{T}$, $Y \nless S$, because $Y < T$ and $Y \nless S$ implies $T \nless S$, thus $x_T = x'_T$.

On the other hand assume $\varphi_T(X) = \varphi_T(X')$ for all $T \nless S$, $T \in \mathcal{T}$. Then we have to show that

$T' = \{T \in \mathcal{T} \mid T \nless S,\ x_T \neq x'_T\} = \emptyset$. Otherwise choose $Y \in T'$ maximal in T' w.r.t $<$. Then

$$\varphi_Y(X) = \sum_{T \in \mathcal{T}} x_T \varphi_Y(T) = \sum_{T \in \mathcal{T}, Y<T} x_T \varphi_Y(T) = x_Y \varphi_Y(Y) + \sum_{T \in \mathcal{T}, Y \lneq T} x_T \varphi_Y(T)$$

$$\neq x'_Y\ \varphi_Y(Y) + \sum_{T \in \mathcal{T}, Y \lneq T} x'_T\ \varphi_Y(T) = \varphi_Y(X'),$$

because $x_T = x'_T$ for $Y \lneq T$ by the maximality of Y in T'.

<u>Corollary(L.6.6) 1:</u> The abelian semigroup $\Omega^+(\mathfrak{C})$ of isomorphism-classes of objects in $\mathfrak{C}$ (with composition the "categorical" sum in $\mathfrak{C}$) is freely generated by the isomorphism-classes of the elements in $\mathcal{T}$, especially $\Omega(\mathfrak{C})$ is a free abelian group with the same basis and $\Omega^+(\mathfrak{C}) \to \Omega(\mathfrak{C})$ is injective.

Now we can prove Prop. 6.1 easily:

(a) For $x = X - X' \in \Omega(G)$ we have:

$x \in K_\Omega(S) \Leftrightarrow$ the projections $\psi_S\colon X \times S \to S$ and $\psi'_S\colon X' \times S \to S$ are isomorphic in $G^\wedge/S \Leftrightarrow \varphi_{T\to S}(X \times S \to S) = \varphi_{T\to S}(X' \times S \to S)$ for all maps $\alpha : T \to S$ $(T \in \mathcal{T})$.

But $\varphi_{T\to S}(X \times S \to S) = |\{\psi : T \to X \times S \mid T \xrightarrow{\psi} X \times S$ comm. $\}|$ (with $\alpha\colon T \to S$, $\psi_S\colon X \times S \to S$)

and a map $\psi : T \to X \times S$ is nothing else than a pair (ψ_1, ψ_2) of maps $\psi_1\colon T \to X$, $\psi_2\colon T \to S$, thus

$\varphi_{T\to S}(X \times S \to S) = |\{(\psi_1,\psi_2)\colon T \to X \times S \mid \psi_2 = \alpha\}|$

$= |\{\psi_1\colon T \to X\}| = \varphi_T(X)$. Thus:

$x = X - X' \in K_\Omega(S) \Leftrightarrow \varphi_T(X) = \varphi_T(X')$ for all $T < S$, $T \in \mathcal{T} \Leftrightarrow \varphi_T(x) = 0$ for all $T < S$, $T \in \mathcal{T}$, q.e.d.

(b) By the definition of $(\eta_S)^*: \Omega(S) \to \Omega(*) = \Omega(G)$ we have obviously: $I_\Omega(S) = \sum_{T\in\mathcal{T}, T<S} \mathbb{Z}\cdot T$, thus

$$I_\Omega(S) = \{x = \sum_{T\in T} x_T T \in \Omega(G) \mid x_T = 0 \text{ for } T\in T,\ T\nless S\}$$

$$= \{x \in \Omega(G) \mid \varphi_T(x) = 0 \text{ for } T \in \mathcal{T},\ T \nless S\},$$

q.e.d.

§ 7 An Inductiontheorem for Mackeyfunctors

In this section we consider only Mackeyfunctors from the category $G^\wedge$ of G-sets (G a finite group) into the category $\underline{\underline{\mathfrak{Ab}}}$ of abelian groups. (The results generalize immediately to arbitrary abelian image categories instead of $\underline{\underline{\mathfrak{Ab}}}$).
As can be motivated for example by considering Ex.(1) in § 6 we are interested to study for a Mackeyfunctor $\mathfrak{M}$ and a G-set S the kernel $K_{\mathfrak{M}}(S)$ (or just K(S)) of the map $(\eta_S)_* : \mathfrak{M}(*) \to \mathfrak{M}(S)$ and the image $I_{\mathfrak{M}}(S) = I(S)$ of the map $(\eta_S)^* : \mathfrak{M}(S) \to \mathfrak{M}(*)$ or occasionally more general for an arbitrary G-map $\varphi : S \to T$ the kernel $K_{\mathfrak{M}}(\varphi) = K(\varphi)$ of $\varphi_* : \mathfrak{M}(T) \to \mathfrak{M}(S)$ and the image $I_{\mathfrak{M}}(\varphi) = I(\varphi)$ of $\varphi^* : \mathfrak{M}(S) \to \mathfrak{M}(T)$.

A special, but important case of our main result is the relation

$$(1) \quad K(S) + I(S) \supseteq |G|\ \mathfrak{M}(*)$$

for any G-set S and any Mackeyfunctor $\mathfrak{M}$. Already this relation shows that for $\mathfrak{M} = K_G$ and $S = S(\mathcal{G}) = \dot{\bigcup}_{C \in \mathcal{G}} G/C$ with $\mathcal{G} = \{C \leqslant G \mid C \text{ cyclic}\}$ the injectivity of $(\eta_S)_* : K_G(*) \to K_G(S)$ implies $|G| \cdot K_G(*) \subseteq \eta_S^*(K_G(S))$, i.e. Artin's inductiontheorem.

Before stating the main result just a few remarks and definitions:

At first let us observe, that $S < T$ implies $K(S) \supseteq K(T)$, $I(S) \subseteq I(T)$, especially $S \sim T \Rightarrow K(S) = K(T)$, $I(S) = I(T)$, because $S < T$ implies the existence of a commutative triangle: $\begin{matrix} S & \xrightarrow{\eta_S} & \\ \downarrow\alpha & & * \\ T & \xrightarrow{\eta_T} & \end{matrix}$. Thus $K(S)$ and $I(S)$ depend only on the equivalence class of S, i.e. on $\mathfrak{U}(S)$ (see § 3). Therefore it makes sense (and also is closer to conventional notations as can be seen again by considering example 1 in § 6) to introduce the notation $K(\mathfrak{U}) = K(S(\mathfrak{U}))$, $I(\mathfrak{U}) = I(S(\mathfrak{U}))$ $(S(\mathfrak{U}) = \dot{\bigcup}_{U \in \mathfrak{U}} G/U$, see § 3) for any set $\mathfrak{U}$ of subgroups of G and $K(U)$, resp. $I(U)$ in case $\mathfrak{U} = \{U\}$ contains exactly one subgroup. Of course $K(\mathfrak{U}) = K(\overline{\mathfrak{U}})$, $I(\mathfrak{U}) = I(\overline{\mathfrak{U}})$ with $\overline{\mathfrak{U}}$ the sub-conjugate closure of $\mathfrak{U}$. Moreover Lemma 6.2 implies $K(S \dot{\cup} T) = K(S) \cap K(T)$, $I(S \dot{\cup} T) = I(S) + I(T)$, especially $K(\mathfrak{U}) = \bigcap_{U \in \mathfrak{U}} K(U)$, $I(\mathfrak{U}) = \sum_{U \in \mathfrak{U}} I(U)$.

Now let π be a (possibly empty) set of prime numbers and let π' be its complement (in the set of all prime numbers). Thus any natural number n can be written uniquely as the product of its π-part n_π and its π'-part $n_{\pi'}$, with n_π for $n = \prod p^{\alpha_p}$ defined by $n_\pi = \prod_{p \in \pi} p^{\alpha_p}$.

For $\mathfrak{U}$ a set of subgroups of G define

$$\mathfrak{S}_\pi \mathfrak{U} = \mathfrak{S}_\pi \overline{\mathfrak{U}} = \overline{\mathfrak{U}} \cup \{V \leq G \mid \exists p \in \pi, N \trianglelefteq V \text{ with } V/N \text{ a } p\text{-group and } N \in \overline{\mathfrak{U}}\}$$

and $\mathfrak{S}\mathfrak{U} = \mathfrak{S}_\pi \mathfrak{U}$ for π the set of all primes, i.e. $\pi' = \emptyset$. Thus using the results of § 5 $\mathfrak{S}_\pi \mathfrak{U}$ contains with any

$U \in \overline{\mathfrak{U}}$ all $V \leq G$ with $V \overset{p}{\sim} U$ (i.e. $\mathfrak{p}(V,p) = \mathfrak{p}(U,p)$) for some $p \in \pi$.

Finally for a G-set T we define

$T_\pi = S(\mathfrak{H}_\pi(\mathfrak{U}(T))) = \dot{\bigcup}_{V \in \mathfrak{H}_\pi(\mathfrak{U}(T))} G/V$. Thus $T \sim T_\emptyset < T_\pi$,

$T \sim S$ implies $T_\pi = S_\pi$, one has $\mathfrak{H}_\pi(\mathfrak{U}(T)) = \mathfrak{U}(T_\pi)$ and the above remark can be translated into the following form: For two simple G-sets X,Y with $X < T$ and $X \overset{p}{\sim} Y$ for some $p \in \pi$ (i.e. $\mathfrak{p}(X,p) = \mathfrak{p}(Y,p)$) one has $Y < T_\pi$.

Now we can state:

<u>Theorem 7.1</u> (general inductionlemma for Mackeyfunctors): Let G be a finite group, $\mathfrak{M}: G^\wedge \to \underline{\underline{\mathfrak{Ab}}}$ a Mackeyfunctor and π a set of prime-numbers. Then one has for any G-set S

$$K(S) + I(S_\pi) \supseteq |G|_{\pi'} \cdot \mathfrak{M}(*)$$

and

$$|G|_{\pi'} \cdot (I(S) \cap K(S_\pi)) = 0$$

resp. for any set $\mathfrak{U}$ of subgroups of G:

$$K(\mathfrak{U}) + I(\mathfrak{H}_\pi \mathfrak{U}) \supseteq |G|_{\pi'} \mathfrak{M}(*),$$

especially $K(\mathfrak{U}) + I(\mathfrak{H}\mathfrak{U}) = \mathfrak{M}(*)$, and

$$|G|_{\pi'} \cdot (I(\mathfrak{U}) \cap K(\mathfrak{H}_\pi \mathfrak{U})) = 0.$$

Proof:

(1) Outline: The idea of the proof is to reduce the general statement to the case $\mathfrak{M} = \Omega$ and then to apply the results of §§ 5 and 6. The reduction is based on the following fact: Any G-map $\varphi : S \to T$ defines an endomorphism $\varphi^* \varphi_* : \mathfrak{M}(T) \to \mathfrak{M}(T)$.

Thus we get a pairing:

$$\Omega^{+}(T) \times \mathfrak{M}(T) \to \mathfrak{M}(T):$$
$$(\varphi, x) \mapsto \varphi^{*}\varphi_{*}(x),$$

which by bilinearity extends to a pairing:

$$\Omega(T) \times \mathfrak{M}(T) \to \mathfrak{M}(T).$$

The study of the behaviour of this map w.r.t. induction (i.e. the maps $\Omega^{*}(\varphi)$, $\mathfrak{M}^{*}(\varphi)$) and restriction (i.e. the maps $\Omega_{*}(\varphi)$, $\mathfrak{M}_{*}(\varphi)$) leads to the abstract definition of exterior composition (or pairings) of Mackey-functors or more generally bifunctors, which specializes (e.g. in the case $\mathfrak{M} = \Omega$) to the definition of interior composition of bifunctors, Frobenius-functors and Frobeniusmodules. In our case the composition $\Omega \times \mathfrak{M} \to \mathfrak{M}$ defines for $\Omega = \mathfrak{M}$ on Ω the structure of a Frobeniusfunctor and on any Mackey-functor $\mathfrak{M}$ the structure of an Ω-module. Thus we can apply T.Y. Lam's theory of Frobeniusfunctors, which finishes the reduction to the case $\Omega = \mathfrak{M}$.
We split up the proof into a sequence of Lemmata.

(2) <u>Lemma 7.1:</u>

Let $\mathfrak{M}$ be a Mackeyfunctor from $G^\wedge$ into $\underline{\underline{\mathfrak{Ab}}}$. For any G-map $\varphi : S \to T$ in $G^\wedge$ and $x \in \mathfrak{M}(T)$ define:

$$\langle \varphi, x \rangle = \varphi^{*}(\varphi_{*}(x)).$$

Then one has for any two G-maps $\varphi : S \to T$, $\varphi' : S' \to T$:

(a) $\langle \varphi \mathbin{\dot\cup} \varphi', x \rangle = \langle \varphi, x \rangle + \langle \varphi', x \rangle$

with $\varphi \mathbin{\dot\cup} \varphi' : S \mathbin{\dot\cup} S' \to T$ the sum of φ and φ'.

(b) $\langle \varphi \underset{T}{\times} \varphi', x \rangle = \langle \varphi \langle \varphi', x \rangle \rangle = \langle \varphi' \langle \varphi, x \rangle \rangle$.

(c) $\langle Id_T, x \rangle = x$.

Proof:

(a) Let $i : S \to S \dot{\cup} S'$ and $j : S' \to S \dot{\cup} S'$ be the canonical imbeddings, thus $\varphi = (\varphi \dot{\cup} \varphi')i$ and $\varphi' = (\varphi \dot{\cup} \varphi')j$ and therefore

$\langle\varphi,x\rangle + \langle\varphi',x\rangle = \varphi^*\varphi_*(x) + \varphi'^*\varphi'_*(x) =$

$= (\varphi \dot{\cup} \varphi')^* i^* i_* \ (\varphi \dot{\cup} \varphi')_* x + (\varphi \dot{\cup} \varphi')^* j^* j_*(\varphi \dot{\cup} \varphi')_* x =$

$= (\varphi \dot{\cup} \varphi')^*(i^* i_* + j^* j_*)(\varphi \dot{\cup} \varphi')_* x =$

$=$ (by Lemma 6.2, (b)) $(\varphi \dot{\cup} \varphi')^*(\varphi \dot{\cup} \varphi')_* x =$

$\langle\varphi \dot{\cup} \varphi',x\rangle$, q.e.d.

(b) Using (M1) for the diagram

$$\begin{array}{ccc} S \underset{T}{\times} S' & \xrightarrow{\ \bar{\varphi}\ } & S' \\ {\scriptstyle \bar{\varphi}'}\downarrow & \searrow^{\varphi \underset{T}{\times} \varphi'} & \downarrow{\scriptstyle \varphi'} \\ S & \xrightarrow[\ \varphi\ ]{} & T \end{array}$$

we have:

$\langle\varphi \underset{T}{\times} \varphi',x\rangle = (\varphi \underset{T}{\times} \varphi')^* \ (\varphi \underset{T}{\times} \varphi')_* x =$

$= (\varphi\bar{\varphi}')^* \ (\varphi'\bar{\varphi})_* x = \varphi^*(\bar{\varphi}'^*\bar{\varphi}_*)\varphi'_* x =$

$= \varphi^*(\varphi_*\varphi'^*)\varphi'_* x = \varphi^*\varphi_*\langle\varphi',x\rangle = \langle\varphi,\langle\varphi',x\rangle\rangle$

and because $\varphi \underset{T}{\times} \varphi' \cong \varphi' \underset{T}{\times} \varphi$ as well $\langle\varphi \underset{T}{\times} \varphi',x\rangle = \langle\varphi',\langle\varphi,x\rangle\rangle$.

(c) is trivial.

By Lemma 7.1, (a) the pairing

$$\Omega^+(T) \times \mathfrak{M}(T) \to \mathfrak{M}(T) \ : \ (\varphi,x) \mapsto \langle\varphi,x\rangle$$

is bilinear and thus extends to a bilinear map:

$$\Omega(T) \times \mathfrak{M}(T) \to \mathfrak{M}(T).$$

We have further, using the above notations:

Lemma 7.2:

Let $\alpha : T' \to T$ be a G-map. Then the following three diagrams commute

(a) $$\begin{array}{ccc} \Omega(T)\times\mathfrak{M}(T) & \longrightarrow & \mathfrak{M}(T)\\ \big\downarrow{\scriptstyle \Omega_*(\alpha)\times\mathfrak{M}_*(\alpha)} & & \big\downarrow{\scriptstyle \mathfrak{M}_*(\alpha)}\\ \Omega(T')\times\mathfrak{M}(T') & \longrightarrow & \mathfrak{M}(T'), \end{array}$$

(b) $$\begin{array}{ccccc} \Omega(T)\times\mathfrak{M}(T') & \xrightarrow{\mathrm{Id}\times\mathfrak{M}^*(\alpha)} & \Omega(T)\times\mathfrak{M}(T) & & \\ & & & \searrow & \\ \big\downarrow{\scriptstyle \Omega_*(\alpha)\times \mathrm{Id}} & & & & \mathfrak{M}(T)\\ & & & \nearrow{\scriptstyle \mathfrak{M}^*(\alpha)} & \\ \Omega(T')\times\mathfrak{M}(T') & \longrightarrow & \mathfrak{M}(T') & & \end{array}$$

(c) $$\begin{array}{ccccc} \Omega(T')\times\mathfrak{M}(T) & \xrightarrow{\Omega^*(\alpha)\times\mathrm{Id}} & \Omega(T)\times\mathfrak{M}(T) & & \\ & & & \searrow & \\ \big\downarrow{\scriptstyle \mathrm{Id}\times\mathfrak{M}_*(\alpha)} & & & & \mathfrak{M}(T)\\ & & & \nearrow{\scriptstyle \mathfrak{M}^*(\alpha)} & \\ \Omega(T')\times\mathfrak{M}(T') & \to & \mathfrak{M}(T') & & \end{array},$$

i.e. (a) for $y \in \Omega(T)$, $x \in \mathfrak{M}(T)$ one has

$$\alpha_*(\langle y,x\rangle) = \langle \alpha_* y, \alpha_* x\rangle,$$

(b) for $y \in \Omega(T)$, $x \in \mathfrak{M}(T')$ one has

$$\langle y, \alpha^*(x)\rangle = \alpha^*(\langle \alpha_* y, x\rangle),$$

(c) for $y \in \Omega(T')$, $x \in \mathfrak{M}(T)$ one has

$$\langle \alpha^* y, x\rangle = \alpha^*\langle y, \alpha_* x\rangle.$$

Proof:

It is enough, to prove all formulas for elements in Ω^+, i.e. for maps $\varphi : S \to T$, resp. $\varphi' : S' \to T'$.

(a) and (b): Let $\varphi : S \to T$ represent an element in $\Omega^+(T)$ and apply (M2) to the diagram:

$$\begin{array}{ccc} T'\times_T S & \xrightarrow{\bar{\alpha}} & S\\ \big\downarrow{\scriptstyle \bar{\varphi}} & & \big\downarrow{\scriptstyle \varphi}\\ T' & \xrightarrow{\alpha} & T \end{array}.$$

Then we get for $x \in \mathfrak{M}(T)$:

$$\alpha_*\langle\varphi,x\rangle = \alpha_*\varphi^*\varphi_*x = \bar{\varphi}^*\bar{\alpha}_*\varphi_*x = \bar{\varphi}^*\bar{\varphi}_*\alpha_*x = \langle\bar{\varphi},\alpha_*x\rangle =$$
$$= \langle\alpha_*(\varphi),\alpha_*x\rangle$$

and for $x \in \mathfrak{M}(T')$:

$$\langle\varphi,\alpha^*x\rangle = \varphi^*\varphi_*\alpha^*x = \varphi^*\bar{\alpha}^*\bar{\varphi}_*x = \alpha^*\bar{\varphi}^*\bar{\varphi}_*x = \alpha^*\langle\bar{\varphi},x\rangle =$$
$$= \alpha^*\langle\alpha_*\varphi,x\rangle, \text{ q.e.d.}$$

(c) Functoriality alone (already used above together with (M1)) implies for $\varphi' : S' \to T' \in \Omega^+(T')$:
$\langle\alpha^*(\varphi'),x\rangle = \langle\alpha\varphi',x\rangle = \alpha^*\varphi'^*\varphi'_*\alpha_*x = \alpha^*\langle\varphi',\alpha_*x\rangle$,
q.e.d.

Following ideas of T.Y. Lam (see [ ]), slightly varied with respect to our situation as stated in Lemma 7.2, we now define (a bit more circumstantial than necessary in the moment because of possible further applications):

Let $\mathfrak{C}$ be an arbitrary category and $D,E,F : \mathfrak{C} \to \underline{\underline{Ab}}$ (or any abelian category) bifunctors. A pairing or exterior composition $\Gamma : D \times E \to F$ is a family of bilinear maps $\Gamma_X : D(X) \times E(X) \to F(X)$, indexed by the objects X in $\mathfrak{C}$, such that for any map $\varphi : Y \to X$ in $\mathfrak{C}$ one has three commutative diagrams:

(C1)
$$\begin{array}{ccc} D(X) \times E(X) & \xrightarrow{\Gamma_X} & F(X) \\ \downarrow{\scriptstyle D_*(\varphi)\times E_*(\varphi)} & & \downarrow{\scriptstyle F_*(\varphi)} \\ D(Y) \times E(Y) & \xrightarrow{\Gamma_Y} & F(Y), \end{array}$$

i.e. $\varphi_*(\Gamma_X(d,e)) = \Gamma_Y(\varphi_*d,\ \varphi_*e)$ for $d \in D(X)$, $e \in E(X)$,

(C2)

$$\begin{array}{ccccc} D(X)\times E(Y) & \xrightarrow{\mathrm{Id}\times E^*(\varphi)} & D(X)\times E(X) & & \\ & & & \searrow^{\Gamma_X} & \\ \Big\downarrow{\scriptstyle D_*(\varphi)\times \mathrm{Id}} & & & & F(X) \\ & & & \nearrow_{F^*(\varphi)} & \\ D(Y)\times E(Y) & \xrightarrow{\Gamma_Y} & F(Y) & & \end{array},$$

i.e. $\Gamma_X(d,\varphi^*(e)) = \varphi^*(\Gamma_Y(\varphi_* d,e))$, $d \in D(X)$, $e \in E(Y)$,

(C3)

$$\begin{array}{ccccc} D(Y)\times E(X) & \xrightarrow{D^*(\varphi)\times \mathrm{Id}} & D(X)\times E(X) & & \\ & & & \searrow^{\Gamma_X} & \\ \Big\downarrow{\scriptstyle \mathrm{Id}\times E_*(\varphi)} & & & & F(X) \\ & & & \nearrow_{F^*(\varphi)} & \\ D(Y)\times E(Y) & \xrightarrow{\Gamma_Y} & F(Y) & & \end{array},$$

i.e. $\Gamma_X(\varphi^* d,e) = \varphi^*(\Gamma_Y(d,\varphi_* e))$, $d \in D(Y)$, $e \in E(X)$.

If $D = E = F$, we say, that Γ defines an inner composition of F, and if $E = F$, we say, that Γ defines an action of D on F.

Remark: The G-functors, considered by J.A. Green in [ ], can be identified with Mackey-functors $\mathfrak{M} : G^\wedge \to$ k-modules (k the base-ring considered in Green's paper) together with an inner composition, which is supposed to be k-bilinear.

The most interesting inner compositions of course are the associative ones. For our purpose we specialize even further and define a bifunctor $F : \mathfrak{C} \to \mathfrak{Ab}$ together with an inner composition $\Gamma : F\times F \to F$ to be a (commutative) Frobenius-functor, if for any object X in $\mathfrak{C}$ the multiplication $F(X)\times F(X) \to F(X) : (x,y) \mapsto \Gamma_X(x,y) = x\cdot y$ defines on $F(X)$ the structure of a commutative ring with a unit $1_{F(X)} \in F(X)$, such that for any $\varphi : Y \to X$ in $\mathfrak{C}$ one has $\varphi_*(1_{F(X)}) = 1_{F(Y)}$ (i.e. φ_* is a ringhomomorphism).

And an F-module (or more precisely a Frobenius-module over F) is defined to be a bifunctor $M : \mathfrak{C} \to \underline{\underline{\mathfrak{Ab}}}$ together with an action $\Gamma : F \times M \to M$, such that for any object X in $\mathfrak{C}$ Γ_X defines on $M(X)$ the structure of an unitary $F(X)$-module, i.e. an $F(X)$-module with $\Gamma_X(1_{F(X)}, m) = 1_{F(X)} \cdot m = m$ for all $m \in M(X)$.

To state the next Lemma, we define for a bifunctor $F : \mathfrak{C} \to \underline{\underline{\mathfrak{Ab}}}$ and a map $\varphi : Y \to X$ in $\mathfrak{C}$ the subgroups $K_F(\varphi) = \mathrm{Ke}(\varphi_* : F(X) \to F(Y))$ and $I_F(\varphi) = \mathrm{Im}(\varphi^* : F(Y) \to F(X))$.

One sees easily, that for a Frobenius-module $M : \mathfrak{C} \to \underline{\underline{\mathfrak{Ab}}}$ over an Frobenius-functor $F : \mathfrak{C} \to \underline{\underline{\mathfrak{Ab}}}$ the subgroups $K_M(\varphi)$ and $I_M(\varphi)$ are indeed $F(X)$-submodules for any $\varphi : Y \to X$ in $\mathfrak{C}$, especially $K_F(\varphi)$ and $I_F(\varphi)$ are ideals in $F(X)$. Moreover we have:

<u>Lemma 7.3 (T.Y. Lam):</u>

(a) If $D,E,F : \mathfrak{C} \to \underline{\underline{\mathfrak{Ab}}}$ are bifunctors and if $\Gamma : D \times E \to F$ is an exterior composition, then for any map $\varphi : Y \to X$ the composition Γ defines maps:

$$K_D(\varphi) \times E(X) \to K_F(\varphi),$$
$$I_D(\varphi) \times E(X) \to I_F(\varphi),$$
$$K_D(\varphi) \times I_E(\varphi) \to 0,$$
$$I_D(\varphi) \times K_E(\varphi) \to 0.$$

(b) Especially for $F : \mathfrak{C} \to \underline{\underline{\mathfrak{Ab}}}$ a Frobenius-functor and M an F-module one has $K_F(\varphi) \cdot M(X) \subseteq K_M(\varphi)$, $I_F(\varphi) \cdot M(X) \subseteq I_M(\varphi)$, $K_F(\varphi) \cdot I_M(\varphi) = I_F(\varphi) \cdot K_M(\varphi) = 0$ for any $\varphi : Y \to X$, and if $\psi : Z \to X$ is another map in $\mathfrak{C}$, such that $n \cdot 1_{F(X)} \in K_F(\varphi) + I_F(\psi)$ for some natural number n, then $K_M(\varphi) + I_M(\psi) \supseteq n \cdot M(X)$,

$$n \cdot (K_M(\psi) \cap I_M(\varphi)) = 0.$$

Proof:

(a) The first formula follows from (C1), the second and third from (C2) and the last from (C3).

(b) The first part is an immediate consequence of (a), applied for D = F, E = F = M, the second part follows easily from the first part:

$nM(X) = n \cdot 1_{F(X)} \cdot M(X) \subseteq (K_F(\varphi) + I_F(\psi)) \cdot M(X) =$
$= K_F(\varphi) \cdot M(X) + I_F(\psi) \cdot M(X) \subseteq K_M(\varphi) + I_M(\psi),$
$n(K_M(\psi) \cap I_M(\varphi)) = n \cdot 1_{F(X)} \cdot (K_M(\psi) \cap I_M(\varphi)) \subseteq$
$\subseteq (K_F(\varphi) + I_F(\psi))(K_M(\psi) \cap I_M(\varphi)) \subseteq$
$\subseteq K_F(\varphi) \cdot I_M(\varphi) + I_F(\psi) \cdot K_M(\psi) = 0.$

By Lemma 7.3,(b) our Theorem follows now from

Lemma 7.4:

The canonical pairing $\Omega \times \mathfrak{M} \to \mathfrak{M}$, defined for any Mackey-functor $\mathfrak{M} : G^\wedge \to \underline{\underline{\mathfrak{Ab}}}$, defines for $\mathfrak{M} = \Omega$ on Ω the structure of a Frobenius-functor with a canonical ringisomorphism $\Omega(*_G) \xrightarrow{\sim} \Omega(G)$ and on any Mackey-functor $\mathfrak{M} : G^\wedge \to \underline{\underline{\mathfrak{Ab}}}$ the structure of a Frobenius-module over Ω;

and from

Lemma 7.5:

The first formula in Theorem 7.1 is true for $\mathfrak{M} = \Omega$.

Proof of Lemma 7.4:

For any G-set S we have to show at first, that the multiplication $\Omega(S) \times \Omega(S) \to \Omega(S)$ defined by

$\Omega^+(S) \times \Omega^+(S) \to \Omega^+(S) \; : \; (\varphi, \psi) \mapsto \varphi^*(\varphi_*(\psi)).$

$(\varphi : T \to S,\ \psi : Y \to S)$ is associative and commutative and that there exists a unit $1_{\Omega(S)} \in \Omega(S)$. But

$$\varphi^*(\varphi_*(\psi)) = \varphi^*(\psi|_\varphi) = \varphi\ \psi|_\varphi = \varphi \underset{S}{\times} \psi : T\underset{S}{\times}Y \longrightarrow Y$$

$$\begin{array}{ccc} T\underset{S}{\times}Y & \longrightarrow & Y \\ {\scriptstyle \psi|_\varphi}\downarrow & \overset{\varphi\underset{S}{\times}\psi}{\searrow} & \downarrow{\scriptstyle \psi} \\ T & \underset{\varphi}{\longrightarrow} & S, \end{array}$$

i.e. $\varphi^*(\varphi_*(\psi))$ is the product of φ and ψ in the category $G\hat{}/S$ and therefore associative and commutative. Moreover the final object $Id_S : S \to S$ in $G^\wedge/S$ represents the wanted unit in $\Omega(S)$, for $\varphi : T \to S$ one has of course $\varphi_*(Id_S) = Id_S|_\varphi = Id_T$ and because $G^\wedge/*_G \cong G^\wedge$ one has the canonical ringisomorphism: $\Omega(*_G) \cong \Omega(G)$.

Thus the first part of Lemma 7.4 is proved. For the second part we have to show for any G-set S, Mackey-functor $\mathfrak{M} : G^\wedge \to \underline{\underline{\mathfrak{Ab}}}$ and elements $x,y \in \Omega(S)$, $m \in \mathfrak{M}(S) : \langle Id_S, m\rangle = m$, $\langle x,\langle y,m\rangle\rangle = \langle\langle x,y\rangle,m\rangle$. But because w.l.o.g. $x = \varphi : T \to S$, $y = \psi : Y \to S \in \Omega^+(S)$ and $\langle\varphi,\psi\rangle = \varphi \underset{S}{\times} \psi$ this follows from Lemma 7.1, (b) and (c).

Proof of Lemma 7.5:

(1) At first we want to show: $|G|\cdot 1_{\Omega(*)} \in K_\Omega(S) + I_\Omega(S)$ for any G-set S. For this purpose we choose a complete system $\mathcal{T}$ of nonisomorphic simple G-sets and consider the imbedding $\prod_{T\in\mathcal{T}} \varphi_T : \Omega(*_G) = \Omega(G) \to \prod_{T\in\mathcal{T}} \mathbb{Z} = \widetilde{\Omega(G)}$.

Define $e_S, f_S \in \widetilde{\Omega(G)}$ by

$$\varphi_T(e_S) = \begin{cases} 1 & T < S \\ 0 & T \not< S, \end{cases}$$

$$\varphi_T(f_S) = \begin{cases} 0 & T < S \\ 1 & T \not< S. \end{cases}$$

Thus $e_S + f_S = 1_{\Omega(G)} = 1_{\Omega(G)}$ for any G-set S. Moreover $|G| \cdot 1_{\Omega(G)} = |G| \cdot e_S + |G| \cdot f_S$ and $|G| e_S$, $|G| f_S \in \Omega(G)$ by Thm 5.2. But $\varphi_T(|G| e_S) = 0$ for $T \nless S$, $\varphi_T(|G| \cdot f_S) = 0$ for $T < S$, thus by Prop. 6.1:

$|G| \cdot 1_{\Omega(G)} = |G| \cdot e_S + |G| \cdot f_S \in I_\Omega(S) + K_\Omega(S)$.

(2) Now we want to show:

<u>Lemma 7.6:</u>

If $I(S_\pi) + K(S) \subseteq \mathfrak{p}$, $\mathfrak{p}$ a prime ideal in $\Omega(G)$, then $\mathfrak{p} = \mathfrak{p}(T,p)$ for some $T \in \mathcal{T}$, $T < S$ and some $p \neq 0$ with $p \mid |G|$, $p \in \pi'$. Especially $I(S_\pi) + K(S) = \Omega(G)$, if $\{p \mid p \mid |G| \text{ and } p \in \pi'\} = \emptyset$.

Proof:

By Prop. 6.1 we have $K(S) = \bigcap_{T \in \mathcal{T}, T < S} \mathfrak{p}(T,0)$ and $I(S_\pi) = \bigcap_{T \in \mathcal{T}, T \nless S_\pi} \mathfrak{p}(T,0)$. Thus the assumption implies $\bigcap_{T < S, T \in \mathcal{T}} \mathfrak{p}(T,0) \subseteq \mathfrak{p}$, and $\bigcap_{T \nless S_\pi, T \in \mathcal{T}} \mathfrak{p}(T,0) \subseteq \mathfrak{p}$. But an intersection of a finite number of ideals is contained in a primeideal $\mathfrak{p}$, if and only if at least one of the ideals is contained in $\mathfrak{p}$ (otherwise there exists for any ideal $\mathfrak{a}_i$ $(i = 1,\ldots,n)$ an element $x_i \in \mathfrak{a}_i - \mathfrak{p}$ and for the product $y = \prod_{i=1}^{n} x_i$ one would have: $y \in \prod_{i=1}^{n} \mathfrak{a}_i \subseteq \bigcap_{i=1}^{n} \mathfrak{a}_i$, but $y \notin \mathfrak{p}$, a contradiction).

Thus we have a $T < S$, $T \in \mathcal{T}$ with $\mathfrak{p}(T,0) \subseteq \mathfrak{p}$ and we have a $T' \nless S_\pi$ with $\mathfrak{p}(T',0) \subseteq \mathfrak{p}$.

Because $\Omega(G)/\mathfrak{p}(T,0) \cong \mathbb{Z}$ any primeideal $\mathfrak{p}$ in $\Omega(G)$, containing $\mathfrak{p}(T,0)$ is necessarily of the form $\mathfrak{p}(T,p)$ with $p = \operatorname{char} \Omega(G)/\mathfrak{p}$. Thus we have $\mathfrak{p}(T,p) = \mathfrak{p} = \mathfrak{p}(T',p)$, i.e. $T \overset{p}{\sim} T'$ for $p = \operatorname{char} \Omega(G)/\mathfrak{p}$. Because $T < S < S_\pi$ and $T' \nless S_\pi$, especially $T \nsim T'$ the relation $T \overset{p}{\sim} T'$ implies $0 \neq p \mid |G|$. Moreover $p \in \pi$ and $T' \overset{p}{\sim} T < S$ would imply $T' < S_\pi$, thus we get $p \in \pi'$, q.e.d.

(3) Lemma 7.5 now follows immediately from (1) and (2) and the purely ringtheoretic:

<u>Lemma 7.7:</u>

Let R be a commutative ring with $1 \in R$ and let $\mathfrak{a}$ be an ideal in R. Let π' be a (possibly empty) set of primenumbers. Then the following statements are equivalent:

(i) for any primeideal $\mathfrak{p} \subseteq R$ with $\mathfrak{a} \subseteq \mathfrak{p}$ one has $0 \neq \operatorname{char} R/\mathfrak{p} = p \in \pi'$.

(ii) There exists a natural π'-number k with $k \cdot 1_R \in \mathfrak{a}$.

(iii) There exists a natural number k with $k \cdot 1_R \in \mathfrak{a}$ and for any natural number n the relation $n \cdot 1_R \in \mathfrak{a}$ implies $n_{\pi'} \cdot 1_R \in \mathfrak{a}$.

Proof:

(i) $\Rightarrow$ (ii) Consider the multiplicatively closed set $S = \{n \cdot 1_R \mid n \text{ a natural } \pi'\text{-number}\}$. If $\mathfrak{a} \cap S = \emptyset$ one can find a primeideal $\mathfrak{p}$ with $\mathfrak{a} \subseteq \mathfrak{p}$, $\mathfrak{p} \cap S = \emptyset$

(e.g. a maximal ideal with these conditions) and for this $\mathfrak{p}$ we have $p = \text{char } R/\mathfrak{p} \notin \pi'$, because otherwise $p \cdot 1_R \in \mathfrak{p} \cap S \neq \emptyset$. But this is a contradiction to (i), thus we have a natural π'-number k with $k \cdot 1_R \in \mathfrak{a}$.

(ii) → (iii): The first part is trivial. Now assume $n \cdot 1_R \in \mathfrak{a}$. Because there exists already a π'-number k with $k \cdot 1_R \in \mathfrak{a}$ we get $(n,k) \cdot 1_R \in \mathfrak{a}$ and because $(n,k) | n_{\pi'}$ we get also $n_{\pi'} \cdot 1_R \in \mathfrak{a}$.

(iii) → (i): Assume $\mathfrak{a} \subseteq \mathfrak{p}$ and $p = \text{char } R/\mathfrak{p}$. $p = 0$ contradicts $k \cdot 1_R \in \mathfrak{a} \subseteq \mathfrak{p}$ and $0 \neq p \notin \pi'$ implies $p \cdot 1_R \in \mathfrak{p}$, thus $p_{\pi'} \cdot 1_R = 1_R \in \mathfrak{p}$, also a contradiction. Thus we have $0 \neq p \in \pi'$.

<u>Remark:</u>

In this proof we have made use of Thm 5.2, to prove (1), and of Thm 5.1, to prove (2). Already (1) implies $|G| \cdot \mathfrak{M}(*) \subseteq K(S) + I(S)$, i.e. "Artin's Inductiontheorem" for any Mackey-functor $\mathfrak{M}$ and G-set S, thus for this part of the theory we do not need the study of the primeideal structure of $\Omega(G)$. On the other hand (2) together with (3) implies, that at least for a certain power $|G|^n$ of $|G|$ one has $|G|^n \cdot \mathfrak{M}(*) \subseteq K(S) + I(S)$ and then $|G|^n_{\pi'} \mathfrak{M}(*) \subseteq K(S) + I(S_\pi)$ for any set π of primes. Especially one gets $\mathfrak{M}(*) = K(S) + I(S_\pi)$, whenever π contains all primedivisors of $|G|$, without using Thm 5.2.

Of course, studying a given Mackey-functor $\mathfrak{M}$, one of the basic problems is to find "small" (w.r.t. $<$) G-sets S with $I_{\mathfrak{M}}(S) = \mathfrak{M}(*)$. This problem is not solved by Thm 7.1. But it is easy to imagine, that Thm 7.1 can be used very well, to deal with this question. For the special case of Mackey-functors, which are at the same time Frobenius-functors, we will discuss this problem in the next section.

§ 8 Green - functors

Let us start with some more abstract definitions concerning bifunctors. So let $\mathfrak{C}$ be an arbitrary category, $F : \mathfrak{C} \to \underline{\underline{\mathfrak{Ab}}}$ a Frobenius-functor and $M_i : \mathfrak{C} \to \underline{\underline{\mathfrak{Ab}}}$ $(i=1,2,3)$ three Frobenius-modules over F. A pairing $\Gamma: M_1 \times M_2 \to M_3$ is called F-bilinear, if for any object X in $\mathfrak{C}$ the pairing $\Gamma_X : M_1(X) \times M_2(X) \to M_3(X)$ is F(X)-bilinear.

We claim at first:

<u>Lemma 8.1:</u>

Let G be a finite group and $\mathfrak{M}_i : G^\wedge \to \underline{\underline{\mathfrak{Ab}}}$ $(i=1,2,3)$ Mackey-functors (thus Ω-modules). Then any pairing $\Gamma : \mathfrak{M}_1 \times \mathfrak{M}_2 \to \mathfrak{M}_3$ is Ω-bilinear.

Proof: For a G-set S, a G-map $\varphi : T \to S$ and elements $x_i \in \mathfrak{M}_i(S)$ $(i=1,2)$ we have to show:

$\Gamma_S(\langle\varphi,x_1\rangle,x_2) = \Gamma_S(x_1,\langle\varphi,x_2\rangle) = \langle\varphi,\Gamma_S(x_1,x_2)\rangle$.

But $\langle\varphi,\cdot\rangle = \varphi^*\varphi_*$ and thus we get (using § 7, (C1), (C2) and (C3)):

$\Gamma_S(\langle\varphi,x_1\rangle,x_2) = \Gamma_S(\varphi^*\varphi_* x_1,x_2) = \varphi^*\Gamma_T(\varphi_* x_1,\varphi_* x_2) =$
$= \varphi^*\varphi_*\Gamma_S(x_1,x_2) = \langle\varphi,\Gamma_S(x_1,x_2)\rangle$ and similarly
$\Gamma_S(x_1,\langle\varphi,x_2\rangle) = \langle\varphi,\Gamma_S(x_1,x_2)\rangle$.

Now let $F' : \mathfrak{C} \to \underline{\underline{\mathfrak{Ab}}}$ be another Frobenius-functor. A natural transformation $\Theta : F \to F'$ is called a (Frobenius-)homomorphism from F into F', if for any object X in $\mathfrak{C}$ the map $\Theta_X : F(X) \to F'(X)$ is a ringhomomorphism. Any such homomorphism $\Theta_X : F(X) \to F'(X)$ can be used, to define an F-module-structure on

$F' : \Theta^o : F \times F' \to F' : \Theta^o_X : F(X) \times F'(X) \to F'(X) :$

$(x,x') \mapsto \Theta_X(x) \cdot x'$, such that the multiplication becomes F-bilinear.

On the other hand, given an F-module-structure $\Gamma : F \times F' \to F'$ on F', such that the multiplication $F' \times F' \to F'$ is F-bilinear, then the map:

$\Gamma^\bullet : F \to F' : \Gamma^\bullet_X : F(X) \to F'(X) : x \mapsto \Gamma(x, 1_{F'(X)}) = x \cdot 1_{F'(X)}$

is a Frobenius-homomorphism from F into F'.

Moreover $(\Theta^o)^\bullet = \Theta$ and $(\Gamma^\bullet)^o$, thus we have a 1-1-correspondence between "bilinear" F-module-structures on F' and homomorphisms $F \to F'$. We also call a Frobeniusfunctor F' together with an homomorphism $F \to F'$ a Frobenius-algebra over F or an F-algebra.

We want to apply this to Mackey-functors. Therefore let us introduce for a bifunctor $\mathfrak{G} : G^\wedge \to \underline{\underline{\mathfrak{Ab}}}$ with an inner composition, which is at the same time a Mackey-functor and a Frobenius-functor, the name Green-functor and similarly the name Green-modules, resp. Green-algebras over $\mathfrak{G}$ for Mackey-functors, which are Frobenius-modules, resp. Frobenius-algebras over $\mathfrak{G}$.

Of course Ω is a Green-functor and one can also make K_G to a Green-functor, using the tensorproduct of vectorbundles.

Moreover Lemma 8.1 implies:

Lemma 8.2:

Let G be a finite group. Then any Green-functor $\mathfrak{G} : G^\wedge \to \underline{\underline{\mathfrak{Ab}}}$ is in a natural (and unique) way an Ω-algebra, i.e. in the category of Green-functors: $G^\wedge \to \underline{\underline{\mathfrak{Ab}}}$ the functor Ω is an initial object.

In this section we want to consider special properties of Green-functors, especially we want to consider G-sets S with $I_{\mathfrak{G}}(S) = \mathfrak{G}(*)$ for a given Green-functor $\mathfrak{G}$. Of course $I_{\mathfrak{G}}(S) = \mathfrak{G}(*) \Leftrightarrow 1_{\mathfrak{G}(*)} \in I_{\mathfrak{G}}(S)$, because $I_{\mathfrak{G}}(S)$ is an ideal in $\mathfrak{G}(*)$.

At first we have the basic

Theorem 8.1 (J.A.Green):

Let $\mathfrak{G} : G^\wedge \to \underline{\underline{\mathfrak{Ab}}}$ be a Green-functor. Then there exists a G-set $S = S_{\mathfrak{G}}$, such that for any G-set T we have

$$I_{\mathfrak{G}}(T) = \mathfrak{G}(*) \Leftrightarrow S < T.$$

Remark:

Of course S is determined only up to equivalence $\sim$ by this property. But this implies, that $\mathfrak{U}(S)$ is uniquely determined by $\mathfrak{G}$. $\mathfrak{U}(S)$ is called the defect basis $\mathfrak{D}_{\mathfrak{G}}$ of $\mathfrak{G}$. Obviously one has:

$$I_{\mathfrak{G}}(T) = \mathfrak{G}(*) \Leftrightarrow \mathfrak{D}_{\mathfrak{G}} \subseteq \mathfrak{U}(T),$$
$$I_{\mathfrak{G}}(\mathfrak{U}) = \mathfrak{G}(*) \Leftrightarrow \mathfrak{D}_{\mathfrak{G}} \subseteq \overline{\mathfrak{U}}.$$

We also call S a defect-set of $\mathfrak{G}$.

Proof:

By **Prop.3.4(b)** we have to show, that $\mathfrak{K}_{\mathfrak{G}} = \{T \mid I_{\mathfrak{G}}(T) = \mathfrak{G}(*)\}$ is 1-closed. But $T < T'$, $T \in \mathfrak{K}_{\mathfrak{G}}$ obviously implies $T' \in \mathfrak{K}_{\mathfrak{G}}$,

thus it remains to show: $T, T' \in \mathfrak{R}_{\mathfrak{G}} \Rightarrow T \times T' \in \mathfrak{R}_{\mathfrak{G}}$.
But this follows immediately from

<u>Lemma 8.3:</u>

Let $\mathfrak{G} : G^{\wedge} \to \underline{\underline{\mathfrak{Ab}}}$ be a Green-functor and let

$$\begin{array}{ccc} X & \overset{\bar{\alpha}}{\dashrightarrow} & T' \\ {\scriptstyle\bar{\beta}}\downarrow & & \downarrow{\scriptstyle\beta} \\ T & \underset{\alpha}{\longrightarrow} & S \end{array}$$

be a pull-back diagram in $G^{\wedge}$. Then the surjectivity of α^* (i.e. $1_{\mathfrak{G}(S)} \in I_{\mathfrak{G}}(\alpha)$) implies the surjectivity of $\bar{\alpha}^*$ (i.e. $1_{\mathfrak{G}(T')} \in I_{\mathfrak{G}}(\bar{\alpha})$).

Proof:

Assume $1_{\mathfrak{G}(S)} = \alpha^*(x)$ for some $x \in \mathfrak{G}(T)$. Then
$1_{\mathfrak{G}(T')} = \beta_*(1_{\mathfrak{G}(S)}) = \beta_*(\alpha^*(x)) = \bar{\alpha}^*(\bar{\beta}_*(x)) \in I_{\mathfrak{G}}(\bar{\alpha})$,
q.e.d.

Another way, to prove the above statement, is to use the following Lemma, which generalizes Mackey's Tensorproduct-Theorem (cf. CR, p.325) to arbitrary Mackey-functors:

<u>Lemma 8.4:</u>

Let $\mathfrak{M}_i : G^{\wedge} \to \underline{\underline{\mathfrak{Ab}}}$ $(i = 1,2,3)$ be Mackey-functors with a pairing $\Gamma : \mathfrak{M}_1 \times \mathfrak{M}_2 \to \mathfrak{M}_3$ and let

$$\begin{array}{ccc} X & \overset{\bar{\alpha}}{\dashrightarrow} & T' \\ {\scriptstyle\bar{\beta}}\downarrow & \searrow^{\alpha \times_S \beta} & \downarrow{\scriptstyle\beta} \\ T & \underset{\alpha}{\longrightarrow} & S \end{array}$$

be a pull-back diagram.

Then for $x \in \mathfrak{M}_1(T)$, $y \in \mathfrak{M}_2(T')$ one has:

$$\Gamma_S(\alpha^*(x), \beta^*(y)) = (\alpha \underset{S}{\times} \beta)^* \Gamma_X(\bar{\beta}_*(x), \bar{\alpha}_*(y)).$$

Proof:

We write $\langle \cdot, \cdot \rangle_S$ instead of $\Gamma_S(\cdot, \cdot)$. Then we have:

$$\langle \alpha^*(x), \beta^*(y)\rangle_S \overset{(C3)}{=} \alpha^*(\langle x, \alpha_*\beta^* y\rangle_T) \overset{(M1)}{=} \alpha^*(\langle x, \bar{\beta}^*\bar{\alpha}_* y\rangle_T)$$
$$\overset{(C2)}{=} \alpha^*\bar{\beta}^*(\langle \bar{\beta}_* x, \bar{\alpha}_* y\rangle_X) = (\alpha \underset{S}{\times} \beta)^*(\langle \bar{\beta}_* x, \bar{\alpha}_* y\rangle_X).$$

Remark: Using Lemma 8.4 one sees easily, that $\mathfrak{K}_{\mathfrak{G}}$ is 1-closed for any Mackey-functor $\mathfrak{G}$ with an inner composition, such that $\mathfrak{G}(*) \times \mathfrak{G}(*) \to \mathfrak{G}(*)$ **is surjective.**

By Thm 8.1 the problem to determine for a given Green-functor $\mathfrak{G}$ all G-sets S with $I_{\mathfrak{G}}(S) = \mathfrak{G}(*)$ has been transformed into the problem, to determine the defect basis $\mathfrak{D}_{\mathfrak{G}}$ of $\mathfrak{G}$. Unfortunately there are many cases, especially in integral representation-theory, in which only lower and upper bounds for $\mathfrak{D}_{\mathfrak{G}}$ can be given. But already the knowledge of sufficiently strong upper bounds for $\mathfrak{D}_{\mathfrak{G}}$ can be quite useful. Indeed in many of the following results the surjectivity of the inductionmap $\mathfrak{G}(S) \to \mathfrak{G}(*)$ is much more important than the minimality of S. Therefore they may be stated sometimes for any surjective inductionmap.

At first let us remark, that the proof of Thm 7.1 immediately yields:

<u>Theorem 7.1':</u>

Let $\mathfrak{G} : G^\wedge \to \underline{\underline{\mathfrak{Ab}}}$ be a Green-functor and X an arbitrary G-set. Then there exists a G-map $\alpha : S \to X$, such that for any G-map $\beta : T \to X$ the inductionmap $\beta^* : \mathfrak{M}(T) \to \mathfrak{M}(X)$ is surjective if and only if α factors through β, i.e. if and only if there exists a morphism from α into β in $G^\wedge/X$.

Such a map $\alpha : S \to X$ is called a defect-map over X and of course it is uniquely determined by $\mathfrak{G}$ up to equivalence $\sim$ in $G^\wedge/X$.

Especially we define X to be without defect (w.r.t. $\mathfrak{G}$) if the identity $Id_X : X \to X$ is a defect-map over X, i.e. if for $\alpha : S \to X$ the surjectivity of $\mathfrak{G}^*(\alpha) : \mathfrak{G}(S) \to \mathfrak{G}(X)$ implies the existence of a section $\beta : X \to S$, i.e. a G-map with $\alpha\beta = Id_X$.

In a way a Green-functor $\mathfrak{G}$ is determined by its behaviour on G-sets without defect and the theory of defects can be considered as a way to reduce the study of $\mathfrak{G}$ over arbitrary G-sets to the case of G-sets without defect. More precisely one has the following two propositions:

Proposition 8.1:

(a) Two G-maps $\alpha_i : S_i \to X_i$ $(i=1,2)$ are defect-maps over X_i $(i=1,2)$ resp., if and only if their sum $\alpha_1 \dot{\cup} \alpha_2 : S_1 \dot{\cup} S_2 \to X_1 \dot{\cup} X_2$ is a defect-map over $X_1 \dot{\cup} X_2$. Especially $X_1 \dot{\cup} X_2$ is without defect if and only if X_1 and X_2 are without defect.

(b) If $\mathfrak{T}$ is (as usual) a complete set of nonisomorphic simple G-sets and $\mathfrak{T}_{\mathfrak{G}} = \{T \in \mathfrak{T} | T$ without defect w.r.t. $\mathfrak{G}\}$, then $S = \dot{\bigcup}_{T \in \mathfrak{T}_{\mathfrak{G}}} T$ is a defect set w.r.t. $\mathfrak{G}$.

Proof:

(a) This follows immediately from Lemma 6.2 and the definitions.

(b) We have to show:

(i) If $I_{\mathfrak{G}}(S') = \mathfrak{G}(*)$, then $S < S'$,

(ii) $I_{\mathfrak{G}}(S) = \mathfrak{G}(*)$.

(i): If $I_{\mathfrak{G}}(S') = \mathfrak{G}(*)$, then by Lemma 8.3 the induction map $\mathfrak{G}^*(\psi) : \mathfrak{G}(S \times S') \to \mathfrak{G}(S)$ with $\psi : S \times S' \to S$ the projection onto the first factor is surjective. But by (a) S is without defect, thus $S < S \times S' < S'$, q.e.d.

(ii) The proof is based on the following

<u>Lemma 8.5:</u>

Let $\mathfrak{K}$ be an equivalence-class of G-sets w.r.t. $\sim$. Then one has

(a) If $\alpha : X \to Y$ is a G-map with $X, Y \in \mathfrak{K}$, then the image $\alpha(X)$, considered as a G-subset of Y (cf. § 2), is in $\mathfrak{K}$.

(b) If X_o is a G-set in $\mathfrak{K}$ with a minimal number of elements, i.e. $|X_o| \leqslant |Y|$ for all $Y \in \mathfrak{K}$, then:

(i) Any G-map $\alpha : X_o \to Y$ is injective and has a left inverse $\alpha' : Y \to X_o$, i.e. $\alpha' \cdot \alpha = Id_{X_o}$.

(ii) Any G-map $\beta : Y \to X_o$ is surjective and has a section, i.e. a right inverse $\beta' : X_o \to Y$, $\beta \cdot \beta' = Id_{X_o}$.

(iii) If $X_1 \in \mathfrak{K}$ with $|X_1| = |X_0|$, then any G-map $\alpha : X_o \to X_1$ is an isomorphism, esp. X_o is determined by $\mathfrak{K}$ up to isomorphism and $End_G(X_o) = Aut_G(X_o)$.

We call X_o a smallest G-set in $\mathfrak{K}$, resp. a smallest G-set with $X_o \sim Y$, if Y is any G-set in $\mathfrak{K}$.

Proof:

(a) We have $X < \alpha(X) < Y < X$, thus $\alpha(X) \sim X$, i.e. $\alpha(X) \in \mathfrak{K}$.

(b) The injectivity of α and the surjectivity of β follow immediately from (a). But this already implies (iii), which in turn implies, that any G-map $\alpha' : Y \to X_o$, resp. $\beta' : X_o \to Y$ is a left-, resp. a right-inverse up to an automorphism of X_o, i.e. the rest of (i) and (ii).

<u>Remark:</u>

If $T_{\mathfrak{K}} = \{T \in T \mid T < Y$ for some (all) $Y \in \mathfrak{K}\}$ and if $T'_{\mathfrak{K}} = \{T \in T_{\mathfrak{K}} \mid T$ max. in $T_{\mathfrak{K}}$ w.r.t. $<\}$, then one can show: $X_o \cong \bigcup_{T \in T'_{\mathfrak{K}}} T$ is a smallest G-set in $\mathfrak{K}$.

This way one may construct "smallest objects" in a $\sim$-equivalence-class $\mathfrak{K}$ of objects in any based category $\mathfrak{C}$, which then have the properties (i), (ii), (iii).

End of the proof of Prop. 8.1, (b): Let X_o be a smallest defect-set w.r.t. $\mathfrak{G}$. If $\alpha : Y \to X_o$ is a G-map with $\alpha^* : \mathfrak{G}^*(Y) \to \mathfrak{G}^*(X_o)$ surjective, then $\eta_Y^* = (\eta_{X_o}\alpha)^* = \eta_{X_o}^*\alpha^*$ is surjective as well, thus $X_o < Y$ and therefore $X_o \sim Y$. But by Lemma 8.5 this implies, that $\alpha : Y \to X_o$ has a section, i.e. X_o is without defect. By Prop. 8.1, (a) this implies, that any indecomposable (i.e. simple) subset of X_o is without defect, thus isomorphic to some $T \in T_{\mathfrak{G}}$ and therefore $X_o < \bigcup_{T \in T_{\mathfrak{G}}} T = S$, $I_{\mathfrak{G}}(S) = \mathfrak{G}(*)$.

Remark:

Unfortunately it is not true in general, that any defect-set X is without defect w.r.t. $\mathfrak{G}$, or equivalently: $T' < T$ and T without defect does not generally imply T' without defect. Indeed it will be one of the maindifficulties to show, that for certain Green-functors $\mathfrak{G}$, associated with relative integral representation theory, the above implication does hold. We will come back to this problem.

The following Proposition is a formal version of the idea of R. Brauer, to use the surjectivity of the inductionmap $\sum_{H\in\mathfrak{E}} X(H) \to X(G)$ (with $\mathfrak{E}$ the set of elementary subgroups of G) to characterize generalized character (i.e. elements in X(G)) among class functions (i.e. elements in $\mathbb{C} \otimes_{\mathbb{Z}} X(G)$) by their restrictions to elementary subgroups.

Proposition 8.2:

Let $\mathfrak{G} : G^{\wedge} \to \underline{\underline{\mathfrak{Ab}}}$ be a Green-functor and $\mathfrak{M} : G^{\wedge} \to \underline{\underline{\mathfrak{Ab}}}$ a Green-module over $\mathfrak{G}$. Let $\varphi : S \to X$ be a G-map with $\mathfrak{G}^*(\varphi) : \mathfrak{G}(S) \to \mathfrak{G}(X)$ surjective. Then

(a) $\mathfrak{M}^*(\varphi) : \mathfrak{M}(S) \to \mathfrak{M}(X)$ is surjective,

(b) $\mathfrak{M}_*(\varphi) : \mathfrak{M}(X) \to \mathfrak{M}(S)$ is injective and maps $\mathfrak{M}(X)$ isomorphically onto $\{x \in \mathfrak{M}(S) \mid \alpha_*(x) = \beta_*(x)\}$ with α,β the two "projections" $S \times_X S \to S$ in the pull-back diagram

$$\begin{array}{ccc} S\times_X S & \xrightarrow{\beta} & S \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\varphi} \\ S & \xrightarrow[\varphi]{} & X \end{array} .$$

Proof:

We have by Lemma 7.3:

(a): $\mathfrak{M}(X) = \mathfrak{G}(X) \cdot \mathfrak{M}(X) = I_{\mathfrak{G}}(\varphi) \cdot \mathfrak{M}(X) \subseteq I_{\mathfrak{M}}(\varphi)$; and

(b): $K_{\mathfrak{M}}(\varphi) = \mathfrak{G}(X) \cdot K_{\mathfrak{M}}(\varphi) = I_{\mathfrak{G}}(\varphi) \cdot K_{\mathfrak{M}}(\varphi) = 0$.

Because $\varphi\alpha = \varphi\beta : S \underset{X}{\times} S \to X$ we have moreover

$\alpha_*\varphi_* = \beta_*\varphi_*$, i.e. $\varphi_*(\mathfrak{M}(X)) \subseteq \mathfrak{M}(S)' = \{x \in \mathfrak{M}(S) \mid \alpha_*(x) = \beta_*(x)\}$.

On the other hand assume $x \in \mathfrak{M}(S)'$ and choose $y \in \mathfrak{G}(S)$ with $\varphi^*(y) = 1_{\mathfrak{G}(X)}$.

We claim: $\varphi_*(\varphi^*(yx)) = x$, which proves, that $\mathfrak{M}_*(\varphi)$ maps $\mathfrak{M}(X)$ onto $\mathfrak{M}(S)'$: $\varphi_*(\varphi^*(yx)) = \alpha^*(\beta_*(yx)) =$

$= \alpha^*(\beta_*(y)\beta_*(x)) = \alpha^*(\beta_*(y)\alpha_*(x)) = \alpha^*(\beta_*(y)) \cdot x =$

$= \varphi_*(\varphi^*(y)) \cdot x = \varphi_*(1_{\mathfrak{G}(X)}) \cdot x = 1_{\mathfrak{G}(S)} \cdot x = x$.

Remark:

Sometimes (for example see [3], §8) the last result has been interpreted in the following way (with $X = *$ for the sake of simplicity): Let $\{\{S\}\}$ be the full subcategory of $G^\wedge$, whose objects are the G-sets T with $T < S$. Then the various restrictions $(\eta_T)_* : \mathfrak{M}(*) \to \mathfrak{M}(T)$ define a map $\mathfrak{M}(*) \to \varprojlim_{\{\{S\}\}} \mathfrak{M}_*$, which - after identifying $\varprojlim_{\{\{S\}\}} \mathfrak{M}_*$ with $\mathfrak{M}(S)'$ - turns out to be an isomorphism by Prop. 8.2 .

Another - and probably nicer - interpretation is the following: For any G-map $\varphi : S \to X$ one can consider the Amitsur-complex

$$\mathfrak{A}(\varphi): X \leftarrow S \overset{\leftarrow}{\underset{\leftarrow}{}} S \underset{X}{\times} S \overset{\leftarrow}{\underset{\leftarrow}{\leftarrow}} S \underset{X}{\times} S \underset{X}{\times} S \overset{\leftarrow}{\underset{\leftarrow}{\leftleftarrows}} \ldots \quad .$$

Applying $\mathfrak{M}_*$ and taking cohomology yields $\mathfrak{M}'(S)/\varphi_*(\mathfrak{M}(X)) \cong H^1(\varphi, \mathfrak{M}_*)$, the first cohomology-group of $\mathfrak{M}_*$ w.r.t. φ. Thus we have by Prop. 8.2:

$H^1(\varphi,\mathfrak{M}_*) = 0$, whenever $\mathfrak{M} : G^\wedge \to \underline{\underline{\mathfrak{Ab}}}$ is a Green-module over a Green-functor $\mathfrak{G}$ with $I_{\mathfrak{G}}(\varphi) = \mathfrak{G}(X)$. Amitsur-cohomology is a generalization of group-cohomology: If $\varphi = \eta_{G/E} : G/E \to *$, then $H^i(\varphi,\mathfrak{M}_*) \cong H^i(G,\mathfrak{M}(G/E))$ for the natural action of $G = \mathrm{Aut}(G/E)$ on $\mathfrak{M}(G/E)$. We will give some further results concerning the Amitsur-cohomology of Mackey-functors and generalizing Prop. 8.2 in an appendix to this section.

<u>Corollary (P.8.2) 1:</u>

If $\mathfrak{G},\mathfrak{G}' : G^\wedge \to \underline{\underline{\mathfrak{Ab}}}$ are Green-functors and if there exists a homomorphism $\Theta : \mathfrak{G} \to \mathfrak{G}'$, then $\mathfrak{D}_{\mathfrak{G}'} \subseteq \mathfrak{D}_{\mathfrak{G}}$.

Proof:

This follows immediately from Prop. 8.2,(a).

Now we want to apply the results of § 7 to the study of defect-basis. Using the "Artin-part" of Thm 7.1 (i.e. the case $\pi = \emptyset$) we get already:

<u>Proposition 8.3:</u>

Let $\mathfrak{G} : G^\wedge \to \underline{\underline{\mathfrak{Ab}}}$ be a Green-functor, S a G-set, $n \in \mathbb{N}$ a fixed natural number and let $\Theta : \Omega \to \mathfrak{G}$ be the unique homomorphism from Ω into $\mathfrak{G}$. Consider the following statements:

(i) $n \cdot 1_{\mathfrak{G}(*)} \in I_{\mathfrak{G}}(S)$

(ii) $n \cdot \mathfrak{G}(*) \subseteq I_{\mathfrak{G}}(S)$

(iii) $n \cdot K_{\mathfrak{G}}(S) = 0$

(iv) $\Theta(n \cdot K_{\Omega}(S)) = 0$

(v) $|G| \cdot n \cdot 1_{\mathfrak{G}(*)} \in n \cdot I_{\mathfrak{G}}(S) \subseteq I_{\mathfrak{G}}(S)$.

Then we have the following implications:

(i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v), especially for $|G|\cdot\mathfrak{G}(*) = \mathfrak{G}(*)$ they are all equivalent.

Proof:

(i) $\Leftrightarrow$ (ii) is trivial ($I_{\mathfrak{G}}(S)$ is an ideal in $\mathfrak{G}(*)$);

(ii) $\Rightarrow$ (iii) : $n\cdot K_{\mathfrak{G}}(S) = n\cdot\mathfrak{G}(*)\cdot K_{\mathfrak{G}}(S) \subseteq I_{\mathfrak{G}}(S)\cdot K_{\mathfrak{G}}(S) = 0$ (Lemma 7.3);

(iii) $\Rightarrow$ (iv): This follows from the commutativity of the diagram:

$$\begin{array}{ccc} \Omega(*) & \xrightarrow{\ \Theta_*\ } & \mathfrak{G}(*) \\ \big\downarrow{\scriptstyle \Omega_*(\eta_S)} & & \big\downarrow{\scriptstyle \mathfrak{G}_*(\eta_S)} \\ \Omega(S) & \xrightarrow{\ \Theta_S\ } & \mathfrak{G}(S) \end{array}$$

$\Omega_*(\eta_S)(x) = 0$ implies $\mathfrak{G}_*(\eta_S)(\Theta_*(x)) = 0$, thus $n \cdot \Theta_*(x) = \Theta_*(nx) = 0$.

(iv) $\Rightarrow$ (v): By Thm 7.1 we have $|G|\cdot 1_{\Omega(*)} \in K_\Omega(S) + I_\Omega(S)$. Multiplying with n and applying Θ we get:

$$|G|\cdot n\cdot 1_{\mathfrak{G}(*)} \in \Theta(n\cdot K_\Omega(S)) + \Theta(n\cdot I_\Omega(S)) =$$

$$= 0 + n\cdot\Theta(I_\Omega(S)) \subseteq nI_{\mathfrak{G}}(S) \subseteq I_{\mathfrak{G}}(S).$$

<u>Corollary (P. 8.3) 1:</u>

If $|G|\cdot\mathfrak{G}(*) = \mathfrak{G}(*)$, then $D_{\mathfrak{G}} \subseteq \mathfrak{U}(S) \Leftrightarrow K_{\mathfrak{G}}(S) = 0$ for any $\mathfrak{G}$-set S.

Proof:

Choose n = 1.

<u>Corollary (P. 8.3) 2:</u>

Let $\mathfrak{G}' \subseteq \mathfrak{G}$ be a sub-Green-functor of $\mathfrak{G}$ and assume $|G|\cdot\mathfrak{G}'(*) = \mathfrak{G}'(*)$. Then $D_{\mathfrak{G}} = D_{\mathfrak{G}'}$.

Proof:

By Cor. (8.5.2) 1 we have $D_{\mathfrak{G}} \subseteq D_{\mathfrak{G}'}$. On the other hand we have $K_{\mathfrak{G}'}(D_{\mathfrak{G}}) \subseteq K_{\mathfrak{G}}(D_{\mathfrak{G}}) = 0$, thus $D_{\mathfrak{G}'} \subseteq D_{\mathfrak{G}}$, q.e.d.

Corollary (8.5.3) 3:

Let $\mathfrak{G}(*)$ be torsionfree. Then $n \cdot 1_{\mathfrak{G}(*)} \in I_{\mathfrak{G}}(S)$ implies $(n,|G|) \cdot 1_{\mathfrak{G}(*)} \in I_{\mathfrak{G}}(S)$.

Proof:

$n \cdot K_{\mathfrak{G}}(S) = 0$ implies now $K_{\mathfrak{G}}(S) = 1 \cdot K_{\mathfrak{G}}(S) = 0$, thus $|G| \cdot 1_{\mathfrak{G}(*)} \in I_{\mathfrak{G}}(S)$ and therefore $(n,|G|) \cdot 1_{\mathfrak{G}(*)} \in I_{\mathfrak{G}}(S)$.

To state further results we introduce for any commutative ring R and any Green-functor $\mathfrak{G}$ the notation $D_{\mathfrak{G}}^{R}$ for the defect basis of $R \otimes_{\mathbb{Z}} \mathfrak{G}$ (which obviously is a Green-functor in a natural way) and also $D_{\mathfrak{G}}^{\pi}$ instead of $D_{\mathfrak{G}}^{R}$ in the special case $R \subseteq \mathbb{Q}$ and $\pi = \pi_R = \{p \mid p \text{ a prime with } pR \neq R\}$ (and thus $R = \mathbb{Z}_\pi = \mathbb{Z}[\frac{1}{q} \mid q \in \pi']$).

We collect a few facts concerning these notations in

Lemma 8.6:

Let $\mathfrak{G} : G^\wedge \to \underline{\underline{\mathfrak{Ab}}}$ be a Green-functor and $\pi, \pi_1, \pi_2, \ldots$ sets of primes. Then

(a) For a G-set S we have $D_{\mathfrak{G}}^{\pi} \subseteq \mathfrak{U}(S) \Leftrightarrow$ there exists a π'-number $n \in \mathbb{N}$ with $n \cdot 1_{\mathfrak{G}(*)} \in I_{\mathfrak{G}}(S)$.

(b) $\pi \subseteq \pi_1 \Rightarrow D_{\mathfrak{G}}^{\pi} \subseteq D_{\mathfrak{G}}^{\pi_1}$, especially $D_{\mathfrak{G}}^{\pi_1 \cap \pi_2} \subseteq D_{\mathfrak{G}}^{\pi_1} \cap D_{\mathfrak{G}}^{\pi_2}$ (equality does not necessarily hold!).

(c) $D_{\mathfrak{G}}^{\pi_1 \cup \pi_2} = D_{\mathfrak{G}}^{\pi_1} \cup D_{\mathfrak{G}}^{\pi_2}$.

(d) If $\pi_1 = \{p \in \pi \mid p\mathfrak{G}(*) \neq \mathfrak{G}(*)\} = \pi \cap \pi_{\mathfrak{G}(*)}$, then $D_{\mathfrak{G}}^{\pi} = D_{\mathfrak{G}}^{\pi_1}$.

(e) If $\mathfrak{G}(*)$ is torsionfree and $\pi_2 = \{p \in \pi \,\big|\, p \,\big|\, |G|\}$, then $D_{\mathfrak{G}}^{\pi} = D_{\mathfrak{G}}^{\pi_2}$.

Proof:

(a) This follows from the fact, that tensoring with $\mathbb{Z}_{\pi}$ is the same as localizing w.r.t. $\{n \cdot 1_{\mathfrak{G}(*)} \mid n \text{ a natural } \pi'\text{-number}\}$.

(b) We have $\mathbb{Z}_{\pi_1} \subseteq \mathbb{Z}_{\pi}$, thus a homomorphism: $\mathbb{Z}_{\pi_1} \otimes \mathfrak{G} \to \mathbb{Z}_{\pi} \otimes \mathfrak{G}$ and therefore $D_{\mathfrak{G}}^{\pi} \subseteq D_{\mathfrak{G}}^{\pi_1}$ by Cor(P.8.2)1. An example with $D_{\mathfrak{G}}^{\pi_1 \cap \pi_2} \neq D_{\mathfrak{G}}^{\pi_1} \cap D_{\mathfrak{G}}^{\pi_2}$ will be given later.

(c) $D_{\mathfrak{G}}^{\pi_1} \cup D_{\mathfrak{G}}^{\pi_2} \subseteq D_{\mathfrak{G}}^{\pi_1 \cup \pi_2}$ follows from (b). Now let S_i $(i=1,2)$ be defect sets for $\mathbb{Z}_{\pi_i} \otimes \mathfrak{G}$, i.e. $\mathfrak{U}(S_i) = D_{\mathfrak{G}}^{\pi_i}$ $(i=1,2)$. Then there exists a natural π_i'-number n_i with $n_i \cdot 1_{\mathfrak{G}(*)} \in I_{\mathfrak{G}}(S_i)$, thus $(n_1,n_2) \cdot 1_{\mathfrak{G}(*)} \in I_{\mathfrak{G}}(S_1) + I_{\mathfrak{G}}(S_2) = I_{\mathfrak{G}}(S_1 \dot{\cup} S_2)$ and because (n_1,n_2) is a $(\pi_1 \cup \pi_2)'$-number: $D_{\mathfrak{G}}^{\pi_1 \cup \pi_2} \subseteq \mathfrak{U}(S_1 \dot{\cup} S_2) = \mathfrak{U}(S_1) \cup \mathfrak{U}(S_2) = D_{\mathfrak{G}}^{\pi_1} \cup D_{\mathfrak{G}}^{\pi_2}$.

(d) $D_{\mathfrak{G}}^{\pi_1} \subseteq D_{\mathfrak{G}}^{\pi}$ follows from $\pi_1 \subseteq \pi$ and (b).

On the other hand let S be a defect-set for $\mathbb{Z}_{\pi_1} \otimes \mathfrak{G}$, i.e. $\mathfrak{U}(S) = D_{\mathfrak{G}}^{\pi_1}$. Then there exists a natural π_1'-number n with $n \cdot 1_{\mathfrak{G}(*)} \in I_{\mathfrak{G}}(S)$.

But $n = n_{\pi} \cdot n_{\pi'}$ and n_{π} is a $\pi \cap \pi_1' \subseteq \pi_{\mathfrak{G}(*)}'$-number, i.e. $n_{\pi} \cdot \mathfrak{G}(*) = \mathfrak{G}(*)$, and thus

$n_{\pi'} \cdot 1_{\mathfrak{G}(*)} \subseteq n_{\pi'}(n_{\pi}\mathfrak{G}(*)) = n\mathfrak{G}(*) \subseteq I_{\mathfrak{G}}(S)$,

i.e. $D_{\mathfrak{G}}^{\pi} \subseteq \mathfrak{U}(S) = D_{\mathfrak{G}}^{\pi_1}$.

(e) Again $D_{\mathfrak{G}}^{\pi_2} \subseteq D_{\mathfrak{G}}^{\pi}$ by (b). So let S be a defect-set for $\mathbb{Z}_{\pi_2} \otimes \mathfrak{G}$. Then there exists a natural π_2'-number n with $n \cdot 1_{\mathfrak{G}(*)} \in I_{\mathfrak{G}}(S)$. But then $(n,|G|) \cdot 1_{\mathfrak{G}(*)} \in I_{\mathfrak{G}}(S)$ by Cor.(P.8.3)2 and $(n,|G|)$ is a π'-number, because $\pi = \pi_2 \cup \{p \in \pi \mid p \nmid |G|\}$.

Thus $D_{\mathfrak{G}}^{\pi} \subseteq \mathfrak{U}(S) = D_{\mathfrak{G}}^{\pi_2}$, q.e.d.

Now we want to use Thm 7.1, to study the relations between the defect sets $D_{\mathfrak{G}}^{\pi}$ for various π. At first we have

<u>Lemma 8.7:</u>

If $\mathfrak{G} : G^{\wedge} \to \underline{\underline{\mathfrak{Ab}}}$ is a Green-functor and S a G-set, such that any element in $K_{\mathfrak{G}}(S)$ is nilpotent, then:

(a) for any set π of primes one has $D_{\mathfrak{G}}^{\pi} \subseteq \mathfrak{H}_{\pi}(\mathfrak{U}(S))$, especially $D_{\mathfrak{G}} \subseteq \mathfrak{H}(\mathfrak{U}(S))$.

(b) $|G|^n \cdot K_{\mathfrak{G}}(S) = 0$ for a certain power $|G|^n$ of $|G|$.

Proof:

(a) For any set π of primes we have

$|G|_{\pi'} \cdot 1_{\mathfrak{G}(*)} = x + y$ with $x \in K_{\mathfrak{G}}(S)$,

$y \in I_{\mathfrak{G}}(S_\pi)$ by Thm 7.1. Assume $x^n = 0$.

Then $|G|_{\pi'}^n \cdot 1_{\mathfrak{G}(*)} = (x+y)^n = x^n + y(n \cdot x^{n-1} + \dots + y^{n-1}) =$

$= 0 + y \cdot z \in I_{\mathfrak{G}}(S_\pi)$, thus $D_{\mathfrak{G}}^\pi \subseteq \mathfrak{U}(S_\pi) = \mathfrak{H}_\pi(\mathfrak{U}(S))$ by

Lemma 8.6, (a).

(b) Especially for $\pi = \emptyset$ we get $|G|^n \cdot 1_{\mathfrak{G}(*)} \in I_{\mathfrak{G}}(S)$

for some $n \in \mathbb{N}$ and thus $|G|^n \cdot K_{\mathfrak{G}}(S) = 0$.

The last Lemmata together imply now:

Theorem 8.2:

Let $\mathfrak{G} : G^\wedge \to \underline{\underline{\mathfrak{Ab}}}$ be a Green-functor and assume, that all torsion-elements in $\mathfrak{G}(*)$ are nilpotent (e.g. $\mathfrak{G}(*)$ torsionfree. More precisely our assumption is equivalent to char $\mathfrak{G}(*)/\mathfrak{p} = 0$ for any minimal prime-ideal $\mathfrak{p}$ in $\mathfrak{G}(*)$). Then we have for any set π of primes:

$D_{\mathfrak{G}}^\pi \subseteq \mathfrak{H}_\pi D_{\mathfrak{G}}^{\mathbb{Q}}$, especially $D_{\mathfrak{G}} \subseteq \mathfrak{H} D_{\mathfrak{G}}^{\mathbb{Q}}$.

Proof:

Let S be a defect-set of $\mathbb{Q} \otimes \mathfrak{G}$, i.e. $D_{\mathfrak{G}}^{\mathbb{Q}} = \mathfrak{U}(S)$.

By Lemma 8.6, (a) and Prop. 8.3 this implies

$n \cdot K_{\mathfrak{G}}(S) = 0$ for some $n \in \mathbb{N}$, thus $K_{\mathfrak{G}}(S)$ is nilpotent by our assumption and therefore

$D_{\mathfrak{G}}^\pi \subseteq \mathfrak{H}_\pi(\mathfrak{U}(S)) = \mathfrak{H}_\pi D_{\mathfrak{G}}^{\mathbb{Q}}$ by Lemma 8.7, (a).

<u>Remark 1:</u>

Using Lemma 8.7, (b) one sees easily, that also in Cor.(P.8.3)2 and in Lemma 8.6, (e) the assumption "$\mathfrak{G}(*)$ torsionfree" can be replaced by the more general condition "all torsion-elements in $\mathfrak{G}(*)$ are nilpotent" (replacing $(n,|G|)$ by $(n,|G|^k)$ for a certain power $|G|^k$ of $|G|$ in Cor.(P.8.3)2 as well, which does not afflict the argument used in the proof of Lemma 8.6, (e)).

<u>Remark 2:</u>

For all results, concerning the defect-basis $D_{\mathfrak{G}}^{\pi}$, it would be enough, to know only the qualitative description of primeideals in $\Omega(G)$, as stated in Thm 5.1 (sometimes replacing $|G|$ or $|G|_{\pi'}$ by certain powers $|G|^k$ or $|G|_{\pi'}^k = (|G|^k)_{\pi'}$) and not the quantitative result Thm 5.2. Especially Thm 8.2 can be based completely on Thm 5.1.

<u>Remark 3:</u>

In many special cases one gets the best results by combining Thm 8.2 with Lemma 8.6, (d) and (e). But it doesn't seem to be worth while, to state all these possible implications as extra corollaries.

<u>Remark 4:</u>

Let $\mathfrak{G} : G^\wedge \to \underline{\underline{\mathfrak{Ab}}}$ be a Green-functor and $\Theta : \Omega \to \mathfrak{G}$ the canonical homomorphism. Then obviously the image $\mathfrak{G}' = \Theta(\Omega) \subseteq \mathfrak{G}$ is a sub-Green-functor of $\mathfrak{G}$, thus $D_{\mathfrak{G}'}^{\mathbb{Q}} = D_{\mathfrak{G}}^{\mathbb{Q}}$ by Cor.(P.8.3)2 and under the assumptions of Thm 8.2 one can get useful upper bounds for $D_{\mathfrak{G}}$ by considering only the image of Ω in $\mathfrak{G}$, tensored with $\mathbb{Q}$. This is one of the reasons for the importance of permutation-representations in the theory of induced representations.

By Thm 8.2 the problem of the determination of the defect-basis of a Green-functor is partly reduced to the study of those Green-functors, whose images are $\mathbb{Q}$-vectorspaces, i.e. to Green-functors $\mathfrak{G} : G^{\wedge} \to \underline{\underline{\mathbb{Q}\text{-mod}}}$. By Prop. 8.1 this in turn is reduced to the study of simple G-sets without defect. Here we can state:

<u>Proposition 8.4:</u>

Let $\mathfrak{G} : G^{\wedge} \to \underline{\underline{\mathbb{Q}\text{-mod}}}$ be a Green-functor and S a simple G-set. Then

(a) S is without defect w.r.t. $\mathfrak{G}$, if and only if there exists a linear map $\varepsilon : \mathfrak{G}(S) \to \mathbb{Q}$ (or $\mathbb{C}$), such that the diagram

$$\begin{array}{ccc} \Omega(S) & \xrightarrow{\Theta_S} & \mathfrak{G}(S) \\ \downarrow{\scriptstyle \varepsilon_S} & & \downarrow{\scriptstyle \varepsilon} \\ \mathbb{Z} & \hookrightarrow & \mathbb{C} \end{array}$$

commutes, where the "augmentation" $\varepsilon_S = \varphi_{\mathrm{Id}_S} : \Omega(S) \to \mathbb{Z}$ maps an object $\alpha : S' \to S$ in $G^{\wedge}/S$ onto the number of sections $|\{\beta : S \to S' \in \mathrm{Hom}_G(S,S') \mid \alpha\beta = \mathrm{Id}_S\}|$, which equals the number of simple components of S', isomorphic to S;

and

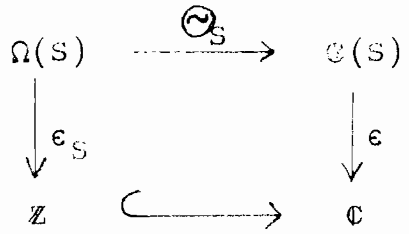
(b) S has a defect w.r.t. $\mathfrak{G}$ (i.e. is not without defect!), if and only if there exists an element $x \in \Omega(S)$ with $\varepsilon_S(x) \neq 0$, but $\Theta_S(x) = 0$, resp. if and only if $\Theta_S(x) = 0$ for one (all) element(s) $x \in \Omega(S)$ with $\varepsilon_S(x) = \varphi_{Id_S}(x) \neq 0$, but $\varphi_\alpha(x) = 0$ for all $\alpha \in T_S$, $\alpha \neq Id_S$, $(T_S = \{\alpha : T \to S | T \in \mathfrak{T}\}$, $\varphi_\alpha(\beta : S' \to S) = $
$= |\{\gamma \in Hom_G(T,S') | \beta\gamma = \alpha\}|)$.

Remark:

These statements have mainly practical interest. If for a given Greenfunctor $\mathfrak{G} : G^\wedge \to \underline{\underline{Q\text{-mod}}}$ and a simple G-set S one has reason to believe that S is without defect w.r.t. $\mathfrak{G}$, one has try to construct a map $\varepsilon : \mathfrak{G}(S) \to \mathbb{C}$ as in part (a). If one conjectures the opposite, one has to find an element x in $\Omega(S)$ with $\varepsilon_S(x) \neq 0$, but $\Theta_S(x) = 0$ and in case there is no other information at hand, one may first construct an element $x \in \Omega(S)$ with $\varepsilon_S(x) \neq 0$, but $\varphi_\alpha(x) = 0$ for all maps $\alpha : T \to S \in T_S$ with $\alpha \neq Id_S$ (such an element is uniquely determined up to a scalar factor, because if x and y are of this type, then $\varphi_\alpha(\varepsilon_S(y)x - \varepsilon_S(x)y) = 0$ for all $\alpha \in T_S$ and thus $\varepsilon_S(y)x = \varepsilon_S(x)y$!) and then has to verify: $\Theta_S(x) = 0$.

Proof of Prop. 6.4:

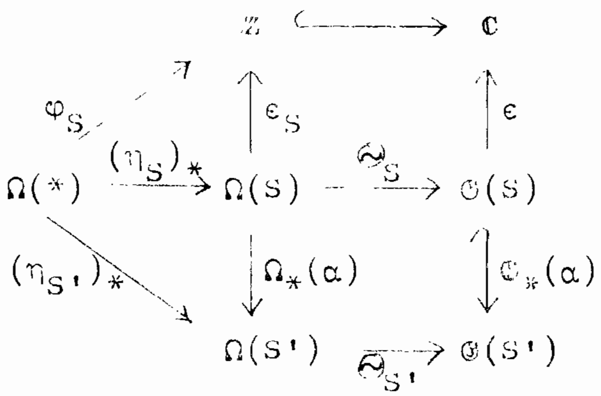
(a) Assume there exists a commutative diagram



and let $\alpha : S' \to S$ be a G-map with $\alpha^* : \mathcal{C}(S') \to \mathcal{C}(S)$ surjective. We have to show, that α has a section. But, because S - being simple - is a smallest G-set in its equivalence class, it is enough to show $S \sim S'$, i.e. $S < S'$.

By **Thm 5.2** there exists an element $x \in \Omega(*)$ with $\varphi_S(x) \neq 0$ (e.g. $= |G|$) and $\varphi_T(x) = 0$ for any simple T with $T \neq S$. By **Prop. 6.1 (a)** we have for any G-set Y : $(\eta_Y)_*(x) \neq 0 \Leftrightarrow S < Y$, thus we have to show $(\eta_{S'})_*(x) \neq 0$.

Now consider the commutative diagram:



The commutativity of the upper left triangel has been observed already in the proof of **Prop. 6.1, (a)** the injectivity of $\mathcal{C}_*(\alpha)$ follows from Prop. 6.2, (b). Considering the various images of x in this diagram

we get $\epsilon(\Phi_S((\eta_S)_*(x))) = \varphi_S(x) \neq 0 \Rightarrow \Phi_S((\eta_S)_*(x)) \neq 0 \Rightarrow$
$0 \neq \mathfrak{C}_*(\alpha)(\Phi_S((\eta_S)_*(x))) = \Phi_{S'}((\eta_{S'})_*(x)) \Rightarrow$
$(\eta_{S'})_*(x) \neq 0$, q.e.d.

Now assume S to be without defect. In $G^\wedge/S$ we have a unique maximal $\sim$-equivalence-class just below the final class, which is represented for instance by the sum σ_S of all maps $\alpha : T \to S (T \in \mathfrak{T})$ with $\alpha \neq Id_S$: $\sigma_S = \dot{\bigcup}_{\substack{(\alpha:T\to S)\in \mathfrak{T}_S \\ \alpha \neq Id_S}} \alpha : S^o = \dot{\bigcup}_{\substack{(\alpha:T\to S)\in \mathfrak{T}_S \\ \alpha\neq Id_S}} T \to S.$

By its definition σ_S has no section. We have moreover

$$\Omega(S) = \sum_{\alpha\in\mathfrak{T}_S} \mathbb{Z}\alpha = \mathbb{Z}\cdot Id_S \oplus I_\Omega(\sigma_S),\ Ke(\epsilon_S) = I_\Omega(\sigma_S).$$

Consider now the diagram:

$$\begin{array}{ccccccc}
\Omega(S^o) & \xrightarrow{\Omega^*(\sigma_S)} & \Omega(S) & \xrightarrow{\epsilon_S} & \mathbb{Z} & \hookrightarrow & \\
\downarrow \Phi_{S^o} & & \downarrow \Phi_S & & \downarrow \bar{\Phi}_S & \searrow & \mathbb{Q} \\
\mathfrak{C}(S^o) & \xrightarrow{\mathfrak{C}^*(\sigma_S)} & \mathfrak{C}(S) & \longrightarrow & \mathfrak{C}(S)/I_{\mathfrak{C}}(\sigma_S) & \nearrow &
\end{array}.$$

The exactness of the two rows and the surjectivity of ϵ_S implies the existence of $\bar{\Phi}_S$.
Because $1_{\mathfrak{C}(S)} \in \Phi_S(\Omega(S))$, but $1_{\mathfrak{C}(S)} \notin I_{\mathfrak{C}}(\sigma_S)$ - S has no defect and σ_S no section - the induced map $\mathbb{Z} \xrightarrow{\bar{\Phi}_S} \mathfrak{C}(S)/I_{\mathfrak{C}}(\sigma_S)$ is nonzero, thus injective, because $\mathfrak{C}(S)/I_{\mathfrak{C}}(\sigma_S)$ is a $\mathbb{Q}$-vectorspace. Thus there exists the wanted map $\mathfrak{C}(S)/I_{\mathfrak{C}}(\sigma_S) \to \mathbb{Q}$, which makes the whole diagram commutative.

A perhaps less direct, but more instructive version of the last part is the following: A map $\epsilon : \mathfrak{G}(S) \to \mathbb{Q}$ with the above properties exists, if and only if $\mathrm{Ke}(\Theta_S : \Omega(S) \to \mathfrak{G}(S)) \subseteq \mathrm{Ke}(\epsilon_S : \Omega(S) \to \mathbb{Z})$. Thus assume there exists $x \in \Omega(S)$ with $\Theta_S(x) = 0$, but $\epsilon_S(x) \neq 0$. We can write x in the form $y + n \cdot \mathrm{Id}_S$ with $y \in I_\Omega(\sigma_S)$ and $n = \epsilon_S(y + n\mathrm{Id}_S) = \epsilon_S(x) \neq 0$. But now $\Theta_S(x) = 0$ implies $0 = \Theta_S(y + n\ \mathrm{Id}_S) = \Theta_S(y) + n \cdot 1_{\mathfrak{G}(S)}$, thus $n \cdot 1_{\mathfrak{G}(S)} = -\Theta_S(y) \in I_{\mathfrak{G}}(\sigma_S) \Rightarrow 1_{\mathfrak{G}(S)} \in I_{\mathfrak{G}}(\sigma_S) \Rightarrow I_{\mathfrak{G}}(\sigma_S) = \mathfrak{G}(S)$, a contradiction to S being without defect.

(b) The first part is a direct consequence of (a) and has been stated just above. For the second part let $y \in \Omega(S)$ be an element with $\epsilon_S(y) \neq 0$, $\varphi_\alpha(y) = 0$ for all $\alpha \in T_S$, $\alpha \neq \mathrm{Id}_S$ (e.g. $y = (\eta_S)_*(x)$ with $x \in \Omega(*)$ as in the proof of the first part of (a)). If $\Theta_S(y) = 0$, then - as has been shown above - S has a defect w.r.t. $\mathfrak{G}$, on the other hand, if S has a defect w.r.t. $\mathfrak{G}$, then there exists $x \in \Omega(S)$ with $\epsilon_S(x) \neq 0$ and $\Theta_S(x) = 0$ and then $\epsilon_S(xy) = \epsilon_S(x) \cdot \epsilon_S(y) \neq 0$, $\Theta_S(xy) = \Theta_S(x) \cdot \Theta_S(y) = 0$, $\varphi_\alpha(xy) = \varphi_\alpha(x) \cdot \varphi_\alpha(y) = 0$ for all $\alpha \in T_S$, $\alpha \neq \mathrm{Id}_S$. Thus there exists at least one such element (i.e. xy) in $\Omega(S)$ and because any other element z with $\varphi_\alpha(z) = 0$ for all $\alpha \in T_S$, $\alpha \neq \mathrm{Id}_S$ differs from xy only by a scalar, we have $\Theta_S(z) = 0$ for any such element.

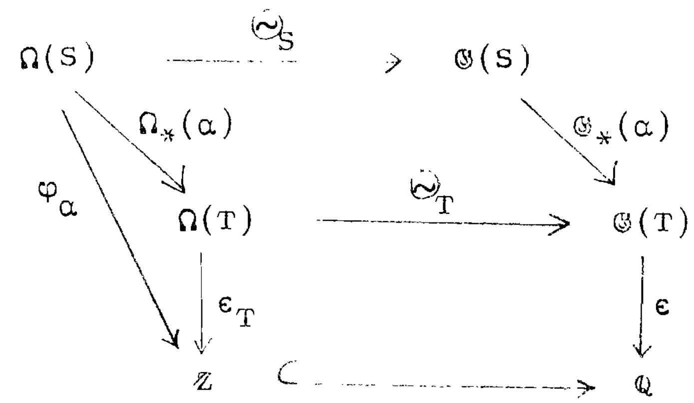
Now we want to characterize those G-sets S, for which any G-set S' with S' < S is without defect. As has been remarked above, this is not necessarily the case for an arbitrary Green-functor, and the fact, that it is true for all Green-functors, which occur naturally in integral representationtheory, seems to be an important feature of this theory. Generally one can state:

<u>Proposition 8.5:</u>

Let $\mathfrak{G} : G^{\wedge} \to \mathbb{Q}\text{-mod}$ be a Green-functor and S a G-set. Then all G-sets $S' < S$ have no defect w.r.t. $\mathfrak{G}$, if and only if the canonical map: $\Theta_S : \Omega(S) \to \mathfrak{G}(S)$ is injective.
In this case S is called faithful w.r.t. $\mathfrak{G}$.

Proof:

(i) Assume that any $S' < S$ has no defect w.r.t. $\mathfrak{G}$ and let $x \in \Omega(S)$ be in the kernel of $\Theta_S : \Omega(S) \to \mathfrak{G}(S)$.
We have to show, that $x = 0$, i.e. $\varphi_\alpha(x) = 0$ for all $\alpha : T \to S \in T_S$. But by our assumption and Prop. 8.4 we have a commutative diagram



($\varphi_\alpha = \epsilon_T\, \Omega_*(\alpha)$ is easily verified: For $\beta : S' \to S$ we have $\varphi_\alpha(\beta) = |\{\gamma : T \to S' \mid \beta\gamma = \alpha\}| = |\{\gamma' : T \to S' \times_S T \mid p_T \gamma' = Id_T\}| =$

$= \epsilon_T(p_T) = \epsilon_T(\Omega_*(\alpha)(\beta))$, where $p_T = \Omega_*(\alpha)(\beta)$ is the projection of $S' \underset{S}{\times} T$ onto T, in other words: the functors $\Omega_*(\alpha) : G^\wedge/S \to G^\wedge/T : (\beta : S' \to S) \mapsto (S' \underset{S}{\times} T \to T)$ and $\Omega^*(\alpha) : G^\wedge/T \to G^\wedge/S : (\beta' : T' \to T) \mapsto (\alpha\beta' : T' \to S)$ are adjoint to each other and this is used for $\beta' = Id_T : T \to T$, cf. also the proof of **Prop.6.1 (a)** and § **2**). Now $\Theta_S(x) = 0$ implies $\epsilon(\mathfrak{G}_*(\alpha)(\Theta_S(x))) = \varphi_\alpha(x) = 0$, q.e.d.

(ii) The proof of the converse is based on

<u>Lemma 8.8:</u>

If $\beta : S' \to S$ is a G-map, then the only ideal, contained in $Ke(\Omega^*(\beta) : \Omega(S') \to \Omega(S))$, is the zero-ideal.

Proof:

Assume $0 \neq x \in \Omega(S')$ and $\Omega(S') \cdot x \subseteq Ke(\Omega^*(\beta))$. Because $0 \neq x$ there exists $(\alpha : T \to S') \in T_{S'}$ with $\varphi_\alpha(x) \neq 0$. Similarly as in § 5 , **Lemma 5.1** ($S' = *!$) we have $x \cdot \alpha = \varphi_\alpha(x)\alpha + \sum_{\gamma \in T_{S'}} n_\gamma \gamma$ with $n_\gamma \neq 0$ only for such $(\gamma : Y \to S') \in T_{S'}$ with $\gamma < \alpha$, $\gamma \neq \alpha$ in $G^\wedge/S'$, especially $|Y| \gneq |T|$. But $\alpha \cdot x \in \Omega(S')x \subseteq Ke(\Omega^*(\beta))$ now implies $0 = \Omega^*(\beta)(\alpha x) = \varphi_\alpha(x)\beta\alpha + \sum_{\gamma \in T_{S'}} n_\gamma \beta\gamma$ in $\Omega(S)$ and, because $\beta\alpha$, $\beta\gamma$ represent elements of the canonical basis T_S of $\Omega(S)$ and $\varphi_\alpha(x) \neq 0$, such an equation can hold only if $\beta\alpha = \beta\gamma$ in $\Omega(S)$, i.e. $(\beta\alpha : T \to S) \cong (\beta\gamma : Y \to S)$ in $G^{\wedge}/S$ for some $\gamma : Y \to S'$ with $n_\gamma \neq 0$, a contradiction to $|Y| \neq |T|$.

Now we can prove: If $\mathfrak{G} : G^{\wedge} \to \underline{\underline{\mathfrak{Ab}}}$ is any Green-functor (not necessarily with $\mathbb{Q}$-vectorspaces as images), then the injectivity of the canonical map $\Theta_S : \Omega(S) \to \mathfrak{G}(S)$ implies the injectivity of $\Theta_{S'} : \Omega(S') \to \mathfrak{G}(S')$ for any $S' < S$: choose a map $\beta : S' \to S$ and consider the diagram:

$$\begin{array}{ccc} \Omega(S') & \xrightarrow{\quad \Omega^*(\beta) \quad} & \Omega(S) \\ \Big\downarrow \Theta_{S'} & & \Big\downarrow \Theta_S \\ \mathfrak{G}(S') & \xrightarrow{\quad \mathfrak{G}^*(\beta) \quad} & \mathfrak{G}(S) \end{array} .$$

Because Θ_S is injective we have $\mathrm{Ke}(\Theta_{S'}) \subseteq \mathrm{Ke}(\Omega^*(\beta))$. But $\mathrm{Ke}(\Theta_{S'})$ is an ideal, thus by Lemma 8.8: $\mathrm{Ke}\,\Theta_{S'} = 0$, q.e.d.

Now, if $\mathfrak{G} : G^{\wedge} \to \underline{\underline{\mathbb{Q}\text{-mod}}}$ is a Green-functor, Prop. 8.4 implies easily, that a G-set S' is without defect w.r.t. $\mathfrak{G}$, if $\Theta_{S'} : \Omega(S') \to \mathfrak{G}(S')$ is injective.

<u>Remark:</u>

It seems sensible to define for any Green-functor $\mathfrak{G} : G^{\wedge} \to \underline{\underline{\mathfrak{Ab}}}$ (not necessarily $\underline{\underline{\mathbb{Q}\text{-mod}}}$) a G-set S to be faithful w.r.t. $\mathfrak{G}$, if the canonical map $\Theta_S : \Omega(S) \to \mathfrak{G}(S)$ is injective. Using the above remarks and the fact, that with S_1 and S_2 also $S_1 \dot{\cup} S_2$ is faithful w.r.t. $\mathfrak{G}$, one sees, that the class $\mathfrak{K}^{\mathfrak{G}}$ of G-sets, which are faithful w.r.t. $\mathfrak{G}$, is r-closed, thus there exists a G-set S, unique up to equivalence, such that a G-set S' is faithful w.r.t. $\mathfrak{G}$, if and only if $S' < S$. Fortunately for all Greenfunctors $\mathfrak{G}$, occuring in integral representationtheory, this G-set S equals the defect base of $\mathbb{Q} \otimes \mathfrak{G}$, thus does not offer new problems.

Finally we want to give some examples of Green-functors, to show, that any subconjugately closed family of subgroups can occur as a defect base of a Green-functor, that a G-set without defect is not necessarily faithful and that $D_{\mathfrak{G}}^{\pi_1 \cap \pi_2}$ is not necessarily equal to $D_{\mathfrak{G}}^{\pi_1} \cap D_{\mathfrak{G}}^{\pi_2}$.

At first let us observe, that for any subfunctor $\mathfrak{M}'$ of a Mackey-functor $\mathfrak{M} : G^{\wedge} \to \underline{\underline{\mathfrak{Ab}}}$ the quotient $\mathfrak{M}/\mathfrak{M}'$: $G^{\wedge} \to \underline{\underline{\mathfrak{Ab}}}$: $S \mapsto \mathfrak{M}(S)/\mathfrak{M}'(S)$ is again a Mackey-functor in a natural way (i.e. such that the map: $\mathfrak{M} \to \mathfrak{M}/\mathfrak{M}'$: $\mathfrak{M}(S) \to \mathfrak{M}(S)/\mathfrak{M}'(S)$ becomes a natural transformation). Especially if $\mathfrak{M}$ is a Green-functor $\mathfrak{G}$ and $\mathfrak{M}' = \mathfrak{J} \subset \mathfrak{G}$ an ideal, i.e. $\mathfrak{M}'(S) = \mathfrak{J}(S) \subseteq \mathfrak{G}(S)$ an ideal in $\mathfrak{G}(S)$ for any G-set S, then the quotient $\mathfrak{G}/\mathfrak{J}$ is a Green-functor as well in a natural way.

Moreover for any Mackey-functor $\mathfrak{M}$ and any G-set S one has two natural subfunctors $\mathfrak{M}_S^K$ and $\mathfrak{M}_S^I$ of $\mathfrak{M}$, defined by $\mathfrak{M}_S^K(T) = K_{\mathfrak{M}}(p_T)$, $\mathfrak{M}_S^I(T) = I_{\mathfrak{M}}(p_T)$ with $p_T : T \times S \to T$ the projection onto the first factor. Again for a Green-functor $\mathfrak{G}$ the subfunctors $\mathfrak{G}_S^K$ and $\mathfrak{G}_S^I$ are ideals in $\mathfrak{G}$, thus the quotients $\mathfrak{G}_K^S = \mathfrak{G}/\mathfrak{G}_S^K$, $\mathfrak{G}_I^S = \mathfrak{G}/\mathfrak{G}_S^I$ are again Green-functors.

We apply this to $\mathfrak{G} = \Omega$ and claim:

Proposition 8.6:

(a) If S is a G-set and π a set of primes, then $\mathfrak{D}_{\mathfrak{G}}^{\pi} = \mathfrak{H}_{\pi}(\mathfrak{U}(S))$ for $\mathfrak{G} = \Omega_K^S$, especially $\mathfrak{D}_{\mathfrak{G}}^{\mathbb{Q}} = \mathfrak{U}(S)$ - thus any subconjugately closed family of subgroups of G can occur as a defect base of a Green-functor - and $\mathfrak{D}_{\mathfrak{G}}^{\pi} = \mathfrak{H}_{\pi}\mathfrak{D}_{\mathfrak{G}}^{\mathbb{Q}}$ - thus Thm 8.2 cannot be sharpened without additional assumptions.

(b) If $S \not\to *$, then $*$ is a defect-set of $\mathbb{Q} \otimes \Omega_I^S$, but $*$ is not faithful w.r.t. $\mathbb{Q} \otimes \Omega_I^S$ (If one is troubled by the fact, that $\mathbb{Q} \otimes \Omega_I^S(T) = 0$ for $T < S$, one may consider the direct product $(\mathbb{Q} \otimes \Omega_I^S) \times \Omega_K^{G/E}$ and get the same statement if $S \not\leftarrow G/E$, $S \not\to *$).

The proof is left to the reader. As a corollary we have for $G = U \times V$ a cyclic group of order $p \cdot q$ ($p \neq q$ two primes) and say $|U| = p$, $|V| = q$, $S = G/U \,\dot\cup\, G/V$: $\mathfrak{D}_{\mathfrak{G}}^{\emptyset} = \mathfrak{D}_{\mathfrak{G}}^{\mathbb{Q}} = \{U, V, E\} \not\ni G$ for $\mathfrak{G} = \Omega_K^S$, but $\mathfrak{D}_{\mathfrak{G}}^{p} = \mathfrak{H}_p\mathfrak{D}_{\mathfrak{G}}^{\mathbb{Q}} \ni G$ and $\mathfrak{D}_{\mathfrak{G}}^{q} = \mathfrak{H}_q\mathfrak{D}_{\mathfrak{G}}^{\mathbb{Q}} \ni G$, thus $G \notin \mathfrak{D}_{\mathfrak{G}}^{\emptyset} = \mathfrak{D}_{\mathfrak{G}}^{\{p\}\cap\{q\}} \subsetneqq \mathfrak{D}_{\mathfrak{G}}^{\{p\}} \cap \mathfrak{D}_{\mathfrak{G}}^{\{q\}} \ni G$.

Appendix to § 8:

Mackey-functors and Amitsur cohomology.

Let $\mathfrak{C}$ be an arbitrary category with finite pullbacks. Then for any map $\varphi : X \to Y$ in $\mathfrak{C}$ one has the semisimplicial Amitsur complex : in $\mathfrak{C}$:

$$\mathfrak{A}(\varphi):\quad Y \overset{\varphi=\varphi_o^o}{\longleftarrow} X \underset{\varphi_1^1}{\overset{\varphi_o^1}{\leftleftarrows}} X \underset{Y}{\times} X \underset{\varphi_2^2}{\overset{\varphi_o^2}{\Lleftarrow}} X \underset{Y}{\times} X \underset{Y}{\times} X \Lleftarrow \ldots ,$$

where $\varphi_m^n : X^{[n+1]} = \underbrace{X \underset{Y}{\times} X \underset{Y}{\times} \ldots \underset{Y}{\times} X}_{n+1 \text{ times}} \to X^{[n]}$

is the product (over Y) of the projections onto the first m factors, i.e. $X^{[m]}$,**resp. the last n-m factors, i.e. $X^{[n-m]}$ together with the natural identification**

$$X^{[m]} \underset{Y}{\times} X^{[n-m]} \overset{\sim}{\to} X^{[n]}.$$

Now if $\mathfrak{M} : \mathfrak{C} \to \underline{\underline{\mathfrak{Ab}}}$ is a contravariant functor, one gets a semisimplicial complex $\mathfrak{C}(\mathfrak{M},\varphi)$ in $\underline{\underline{\mathfrak{Ab}}}$ by applying $\mathfrak{M}$ onto $\mathfrak{A}(\varphi)$. The cohomology-groups $H^i(\mathfrak{M},\varphi)$ of this complex-with $H^o(\mathfrak{M},\varphi) = \mathrm{Ke}(\mathfrak{M}(Y) \overset{\mathfrak{M}(\varphi)}{\to} \mathfrak{M}(X))$ - are the Amitsur-cohomology-groups of $\mathfrak{M}$ w.r.t. φ *).

If $\alpha : Y' \to Y$ is another map in $\mathfrak{C}$, one can take the pullback of $\mathfrak{A}(\varphi)$ w.r.t. α and gets the complex:

$$Y' \underset{Y}{\times} \mathfrak{A}(\varphi):\; Y' \overset{\varphi'}{\leftarrow} Y' \underset{Y}{\times} X = X' \leftleftarrows Y' \underset{Y}{\times} X^{[2]} \Lleftarrow \ldots$$

*) In our context, it is sensible, to numerate the H^i as indicated, in other connections it might be better, to cut of the first step of $\mathfrak{A}(M,\varphi)$, thus starting with $\mathfrak{M}(X) \rightrightarrows \mathfrak{M}(X\times X) \Rrightarrow \ldots$ and to define $H^o(\mathfrak{M},\varphi)=\mathrm{Ke}(\mathfrak{M}(\varphi_o^1)-\mathfrak{M}(\varphi_1^1))$.

which is easily identified with

$$\mathfrak{A}(\varphi') \; : \; Y' \overset{\varphi'}{\leftarrow} X' \leftleftarrows X'^{[2]} \leftleftarrows \ldots \quad .$$

Especially one has a natural transformation
$\mathfrak{A}(\varphi') \cong Y' \underset{Y}{\times} \mathfrak{A}(\varphi) \to \mathfrak{A}(\varphi)$ and thus for any contravariant $\mathfrak{M} : \mathfrak{C} \to \underline{\underline{\mathfrak{Ab}}}$ homomorphisms:
$\mathfrak{A}(\mathfrak{M},\varphi) \to \mathfrak{A}(\mathfrak{M},\varphi')$; $H^i(\mathfrak{M},\varphi) \xrightarrow{H^i(\alpha,\mathfrak{M}) = H^i(\alpha)} H^i(\mathfrak{M},\varphi')$.

Moreover if $Y' = X$ and $\alpha : Y' \to Y$ equals $\varphi : X \to Y$, then one can show, that $H^i(\alpha) = H^i(\varphi)$ are zero-maps, thus statements concerning the kernel of the $H^i(\alpha) = H^i(\alpha,\mathfrak{M})$ can always be specialized to statements, concerning the cohomology-groups $H^i(\mathfrak{M},\varphi)$ themselves.
We want to make such a statement in case $\mathfrak{M}$ is not an arbitrary functor, but the contravariant part of a Mackey-functor $\mathfrak{M} : \mathfrak{C} \to \underline{\underline{\mathfrak{Ab}}}$. More precisely let $\mathfrak{M},\mathfrak{N},\mathfrak{L} : \mathfrak{C} \to \underline{\underline{\mathfrak{Ab}}}$ be Mackey-functors and let $\Gamma : \mathfrak{M} \times \mathfrak{N} \to \mathfrak{L}$ be a pairing. It is easy to see, that for any $\varphi^n_m : X^{[n+1]} \to X^{[n]}$ one has a commutative diagram:

$$\begin{array}{ccc} \mathfrak{M}(Y) \times \mathfrak{N}(X^{[n]}) & \xrightarrow{\Gamma_n} & \mathfrak{L}(X^{[n]}) \\ \downarrow{\scriptstyle Id_{\mathfrak{M}(Y)} \times \mathfrak{N}_*(\varphi^n_m)} & & \downarrow{\scriptstyle \mathfrak{L}_*(\varphi^n_m)} \\ \mathfrak{M}(Y) \times \mathfrak{N}(X^{[n+1]}) & \xrightarrow{\Gamma_{n+1}} & \mathfrak{L}(X^{[n+1]}), \end{array}$$

where Γ_n is the composition of
$\mathfrak{M}_*(\varphi^n) \times Id_{\mathfrak{N}(X^{[n]})} : \mathfrak{M}(Y) \times \mathfrak{N}(X^{[n]}) \to \mathfrak{M}(X^{[n]}) \times \mathfrak{N}(X^{[n]})$
($\varphi^n : X^{[n]} \to Y$ <u>the</u> canonical map) with the given pairing, evaluated at $X^{[n]}$. Thus one has an induced pairing

$$\mathfrak{M}(Y) \times H^i(\mathfrak{N},\varphi) \to H^i(\mathfrak{L},\varphi).$$

We claim:

<u>Proposition 8.A.1:</u>

Let $\alpha : Y' \to Y$ be a morphism in $\mathfrak{C}$. Then - using the above notations - the pairing $\mathfrak{M}(Y) \times H^i(\mathfrak{N},\varphi) \to H^i(\Omega,\varphi)$ vanishes on $I_{\mathfrak{M}}(\alpha) \times Ke(H^i(\alpha,\mathfrak{N}))$.

Proof:

Let $B^i(\mathfrak{N},\varphi)$ be image of $\sum_{j=0}^{i-1} (-1)^j \mathfrak{N}_*(\varphi_j^{i-1}) : \mathfrak{N}(X^{\lfloor i-1 \rfloor}) \to \mathfrak{N}(X^{\lfloor i \rfloor})$

and $C^i(\mathfrak{N},\varphi)$ the kernel of $\sum_{j=0}^{i} (-1)^j \mathfrak{N}_*(\varphi_j^i)$, thus

$H^i(\mathfrak{N},\varphi) = C^i(\mathfrak{N},\varphi)/B^i(\mathfrak{N},\varphi)$.

Choose $x \in C^i(\mathfrak{N},\varphi)$ and assume $\mathfrak{N}_*(\alpha^i)(x) \in B^i(\mathfrak{N},\varphi')$, where $\mathfrak{N}_*(\alpha^i) : \mathfrak{N}(X^{\lfloor i \rfloor}) \to \mathfrak{N}(Y' \times_Y X^{\lfloor i \rfloor}) = \mathfrak{N}(X'^{\lfloor i \rfloor})$ comes from the projection $\alpha^i : Y' \times_Y X^{\lfloor i \rfloor} \to X^{\lfloor i \rfloor}$ onto the second factor. Let

$y = \mathfrak{M}_*(\alpha)(z) \in I_{\mathfrak{M}}(\alpha) \subseteq \mathfrak{M}(Y)$, $z \in \mathfrak{M}(Y')$.

We have to show: $\langle y,x \rangle \in B^i(\Omega,\varphi)$. ($\langle y,x \rangle$ the image of $y \in \mathfrak{M}(Y)$ and $x \in \mathfrak{N}(X^{\lfloor i \rfloor})$ w.r.t. the pairing into $\Omega(X^{\lfloor i \rfloor})$).

Because $\mathfrak{N}_*(\alpha^i)(x) \in B^i(\mathfrak{N},\varphi')$, we have

$u \in \mathfrak{N}(Y' \times_Y X^{\lfloor i-1 \rfloor}) = \mathfrak{N}(X'^{\lfloor i-1 \rfloor})$ with

$$\mathfrak{N}_*(\alpha^i)(x) = \sum_{j=0}^{i-1} (-1)^j \mathfrak{N}_*(\varphi'^{i-1}_j)(u).$$

We claim: $\langle y,x \rangle = \sum_{j=0}^{i-1} (-1)^j \Omega_*(\varphi_j^{i-1})(\Omega^*(\alpha^{i-1})(\langle z,u \rangle))$:

consider the diagram

$$\begin{array}{ccccc}
Y & \longleftarrow & X^{\lfloor i-1 \rfloor} & \xleftarrow{\ \varphi_j^{i-1}\ } & X^{\lfloor i \rfloor} \\
\uparrow{\scriptstyle \alpha} & & \uparrow{\scriptstyle \alpha^{i-1}} & & \uparrow{\scriptstyle \alpha^i} \\
Y' & \longleftarrow & Y' \times_Y X^{\lfloor i-1 \rfloor} & \xleftarrow{\ \varphi'^{i-1}_j\ } & Y' \times_Y X^{\lfloor i \rfloor}.
\end{array}$$

Both squares are pull-backs.

Thus we have:

$$\sum_{j=0}^{i-1} (-1)^j \; \mathfrak{L}_*(\varphi_j^{i-1})(\mathfrak{L}^*(\alpha^{i-1})(\langle z,u\rangle)) =$$

$$\sum_{j=0}^{i-1} (-1)^j \; \mathfrak{L}^*(\alpha^i)(\mathfrak{L}_*(\varphi'^{\,i-1}_j)(\langle z,u\rangle)) =$$

$$\mathfrak{L}^*(\alpha^i)(\langle z, \sum_{j=0}^{i-1} (-1)^j \; \mathfrak{N}_*(\varphi'^{\,i-1}_j)(u)\rangle) =$$

$$\mathfrak{L}^*(\alpha^i)(\langle z, \mathfrak{N}_*(\alpha^i)(x)\rangle) = \langle \mathfrak{M}^*(\alpha^i)(z), x\rangle =$$

$$\langle \mathfrak{M}^*(\alpha)(z), x\rangle = \langle y, x\rangle, \text{ q.e.d.}$$

Now consider the special case, where $\mathfrak{M} = \mathfrak{G}$ is a "Green-functor on $\mathfrak{C}$" (i.e. a Mackey-functor with an inner composition such that any $\mathfrak{G}(S)$ becomes a (commutative) ring with a unit and any $\mathfrak{G}_*(\varphi) : \mathfrak{G}(S) \to \mathfrak{G}(S')$ ($\varphi : S' \to S$ a morphism in $\mathfrak{C}$) a unit-preserving homomorphism) and $\mathfrak{N} = \mathfrak{L}$ a Green-module over $\mathfrak{G}$. Then for any $\varphi : X \to Y$ the groups $H^i(\mathfrak{N},\varphi)$ can be considered as $\mathfrak{G}(Y)$-modules and for any $\alpha : Y' \to Y$ the kernels of the maps $H^i(\alpha,\mathfrak{N}) : H^i(\mathfrak{N},\varphi) \to H^i(\mathfrak{N},\varphi')$ are annihilated by the ideal $I_{\mathfrak{G}}(\alpha) \subseteq \mathfrak{G}(Y)$, especially $I_{\mathfrak{G}}(\varphi)$ annihilates $H^i(\mathfrak{N},\varphi)$, in particular $H^i(\mathfrak{G},\varphi)$, and we have $H^i(\mathfrak{N},\varphi) = 0$, in particular $H^i(\mathfrak{G},\varphi) = 0$, if $\mathfrak{G}^*(\varphi) : \mathfrak{G}(X) \to \mathfrak{G}(Y)$ is surjective, which generalizes Prop. 8.2 and underlines anew the importance of the surjectivity of induction-maps.

But even if nothing is known about the surjectivity of the induction-map, one can use the results of § 7, to get some restrictions on the structure of the groups $H^i(\mathfrak{N},\varphi)$ in case $\mathfrak{C} = G^*$ is the category of

G-sets. More precisely we have as a consequence of Prop. 8 A,1:

<u>Proposition 8 A,2:</u>

If $\mathfrak{G} : \mathfrak{C} \to \underline{\underline{\mathfrak{Ab}}}$ is a Green-functor on $\mathfrak{C}$, $\mathfrak{M} : \mathfrak{C} \to \underline{\underline{\mathfrak{Ab}}}$ a Green-module over $\mathfrak{G}$ and if $\varphi : X \to Y$ and $\alpha : Y' \to Y$ are morphisms in $\mathfrak{C}$ with $n \cdot 1_{\mathfrak{G}(Y)} \in K_{\mathfrak{G}}(\varphi) + I_{\mathfrak{G}}(\alpha)$ for some natural number $n \in \mathbb{N}$, then the kernel of $H^i(\alpha,\mathfrak{M}) : H^i(\mathfrak{M},\varphi) \to H^i(\mathfrak{M},\varphi')$ is annihilated by n for all $i \geqslant 1$.

Proof:

If $n \cdot 1_{\mathfrak{G}(Y)} = u + v$ with $u \in K_{\mathfrak{G}}(\varphi)$ and $v \in I_{\mathfrak{G}}(\alpha)$ and if $x \in H^i(\mathfrak{M},\varphi)$ $(i \geqslant 1)$, then $n \cdot x = \langle u,x\rangle + \langle v,x\rangle = 0$, ($\langle u,x\rangle = 0$, because $\mathfrak{G}_*(\varphi)(u) = 0$, and $\langle v,x\rangle = 0$ by Prop. 8 A,1).

Of course, one can state many corollaries for Mackey-functors on $G^\wedge$, using $|G|_{\pi'} \cdot 1_{\Omega(G)} \in K(S) + I(S_\pi)$.
I do not want to go into this in detail and just want to remark, that we get among other results for instance:
If $\mathfrak{M} : G^\wedge \to \underline{\underline{\mathfrak{Ab}}}$ is a Mackey-functor and $\varphi : X \to Y$ a G-map, then $H^i(\mathfrak{M},\varphi)$ is annihilated by $|G|$ for $i \geqslant 1$.

Finally I want to remark, that, using the covariant part of a Mackey-functor $\mathfrak{M} : \mathfrak{C} \to \underline{\underline{\mathfrak{Ab}}}$, one can as well define homology-groups $H_i(\mathfrak{M},\varphi)$ and even can put together the two sequences:

$$\cdots \to \mathfrak{M}(X^{[i]}) \to \mathfrak{M}(X^{[i-1]}) \to \cdots \to \mathfrak{M}(X) \longrightarrow \mathfrak{M}(X) \to \mathfrak{M}(X^{[2]}) \to \cdots,$$

$$\mathfrak{M}(X) \xrightarrow{\mathfrak{M}^*(\varphi)} \mathfrak{M}(Y) \xrightarrow{\mathfrak{M}_*(\varphi)} \mathfrak{M}(X)$$

to define Tate-cohomology-groups $\hat{H}^i(\mathfrak{N},\varphi)$. I have not studied these definitions, but I guess, that one can dualize the proof of Prop. 8 A,1 in some way, to get similar results for the homology-groups $H_i(\mathfrak{N},\varphi)$ and that Prop. 8 A,1 might be the first step for a general construction of cup-products of Tate-cohomology-groups, associated to any pairing $\mathfrak{M} \times \mathfrak{N} \to \mathfrak{L}$ of Mackey-functors.

Anyway - as will be shown in the next section - there exists to any $\mathbb{Z}G$-module M a Mackey-functor: $\mathfrak{M}_M \colon G^{\bullet} \to \underline{\underline{\mathfrak{Ab}}}$ and for this Mackey-functor one can show:

$$\hat{H}^i(G,M) = \hat{H}^i(\mathfrak{M}_M,\ \varphi : G/E \to *),$$

$$H^i(G,M) = H^{i+1}(\mathfrak{M}_M,\ \varphi : G/E \to *), (i \geq 1).$$

Thus in any case our theory generalizes the usual cohomology-theory of $\mathbb{Z}G$-modules. An application to Witt rings will be given in Appendix A.

§ 9 Examples

The theory of Mackey functors grew out of an attempt, to understand certain phenomena, which had been observed in the study of induced modules in classical, modular and integral representation theory.

Now, to apply our theory to this subject, we have to show, that Mackey functors indeed occur naturally in representation theory, i.e. that there are well-known representation-theoretic constructions, which give rise to Mackey and Green functors. Unfortunately it seems to be necessary for this purpose, to consider these constructions from a point of view, slightly different from the one generally taken by group theorists: See for instance the introduction of the characterring in § 6, using K_G-theory (which of course is suggested by the work of Atiyah).

In this section now this example is generalized, not by generalizing the base space - e.g. from finite G-sets to compact G-spaces - but instead by generalizing the fibers from $\mathbb{C}$-vectorspaces to arbitrary R-modules (R a commutative ring without any specified topological structure, the most interesting case being $R = \mathbb{Z}$, $R = \mathbb{Z}_p$, $R = \hat{\mathbb{Z}}_p$, $R = \mathbb{F}_p$). This can be done, just because the base-sets have no topological significance.

Furthermore because of certain further applications (permutation and monomial representations and transfer) it seems reasonable, to proceed even more general at first and to admit the fibers to be objects in nearly arbitrary categories.
I hope that this procedure will not horrify the reader too much and will justify itself in the end.

Anyway before giving this general construction I want to describe another class of Mackey functors, which is associated to $\mathbb{Z}G$-modules and their cohomology. So let G be a finite group. A (left) $\mathbb{Z}G$-module M is an abelian group (i.e. a $\mathbb{Z}$-module) together with a (left) linear action of G on M, i.e. a map $G \times M \to M : (g,m) \mapsto gm$ such that $g(m+n) = gm + gn$, $g(hm) = (gh)m$, $em = m$ for $m,n \in M$, $g,h \in G$, e the neutral element in G. For a $\mathbb{Z}G$-module M and a G-set S define $\mathfrak{H}(S,M) = \mathfrak{H}_G(S,M)$ to be set of all G-maps: $f : S \to M$, i.e. all maps $f : S \to M$ with $f(gs) = gf(s)$ $(g \in G,\ s \in S)$. Obviously $\mathfrak{H}(S,M)$ can be considered as an abelian group (with composition defined argumentwise: $(f_1 + f_2)(s) = f_1(s) + f_2(s)$). For a G-map $\varphi : T \to S$ between two G-sets T and S one can define two homomorphisms:

$\varphi_* = \mathfrak{H}_*(\varphi,M) : \mathfrak{H}(S,M) \to \mathfrak{H}(T,M) : f \mapsto f\varphi$ and

$\varphi^* = \mathfrak{H}^*(\varphi,M) : \mathfrak{H}(T,M) \to \mathfrak{H}(S,M) : f \mapsto \varphi^* f$ with

$$\varphi^* f(s) = \sum_{t \in \varphi^{-1}(s)} f(t) \quad (s \in S).$$

One verifies easily, that $\mathfrak{H}(\cdot,M) : G^\wedge \to \underline{\underline{\mathfrak{Ab}}}$:
$S \mapsto \mathfrak{H}(S,M)$, $\varphi \mapsto (\mathfrak{H}^*(\varphi,M), \mathfrak{H}_*(\varphi,M))$ is a bifunctor.
We have moreover

<u>Proposition 9.1:</u>

The bifunctor $\mathfrak{H}(\cdot,M) : G^\wedge \to \underline{\underline{\mathfrak{Ab}}}$ is a Mackey functor.

Proof:

Let
$$\begin{array}{ccc} T_1 \underset{S}{\times} T_2 & \xrightarrow{\Phi} & T_2 \\ \downarrow{\scriptstyle\Psi} & & \downarrow{\scriptstyle\psi} \\ T_1 & \xrightarrow[\varphi]{} & S \end{array}$$
be a pullback diagram in $G^\wedge$

and $f \in \mathfrak{H}(T_1,M)$. We have to show:
$\psi_*\varphi^* f(t) = \Phi^*\Psi_* f(t)$ for any $t \in T_2$. But
$$(\psi_*\varphi^* f)(t) = (\varphi^* f)(\psi(t)) = \sum_{x\in\varphi^{-1}(\psi(t))} f(x),$$
$$(\Phi^*\Psi_* f)(t) = \sum_{(x,y)\in\Phi^{-1}(t)} (\Psi_* f)(x,y) = \sum_{(x,y)\in\Phi^{-1}(t)} f(\Psi(x,y)) =$$
$$= \sum_{(x,y)\in\Phi^{-1}(t)} f(x) \text{ and } (x,y) \in \Phi^{-1}(t) \subseteq T_1 \underset{S}{\times} T_2 \Leftrightarrow y =$$
$= \Phi(x,y) = t$ and $\varphi(x) = \psi(y) = \psi(t)$, thus the two sums coincide.- In other words: $\psi_*\varphi^* f(t)$ is the sum of the values of f on the fiber of $\psi(t) \in S$ w.r.t. $\varphi: T_1 \to S$ and $\Phi^*\Psi_* f(t)$ is the sum of the values of f, lifted to $T_1 \underset{S}{\times} T_2$, on the fiber of $t \in T_2$ w.r.t. $\Phi : T_1 \underset{S}{\times} T_2 \to T_2$. But $T_1 \underset{S}{\times} T_2$ being the fiberproduct of φ and ψ these two fibers are essentially and the values of f on these fibers are identically the same.
The additivity of $\mathfrak{H}(\cdot,M)$ is trivial.

It is easy to see, that for $U \leq G$ one has a natural isomorphism $\mathfrak{H}(G/U,M) \xrightarrow{\cong} M^U = \{m \in M \mid um = m \text{ for all } u \in U\}$. (cf. p.11, l.32). Thus $\mathfrak{H}(G/U,M) \cong H^0(U,M)$ and one may ask, whether one gets the higher cohomology-groups by considering $M \mapsto \mathfrak{H}(\cdot,M)$ as a covariant functor from the category of $\mathbb{Z}G$-modules into the category of Mackey functors and taking its derivatives. This is indeed the case: of course first one has to assure, that $M \mapsto \mathfrak{H}(\cdot,M)$ can be considered as a covariant functor. But any $\mathbb{Z}G$-homomorphism $\mu : M \to N$ defines a natural transformation:

$\mu : \mathfrak{H}(\cdot,M) \to \mathfrak{H}(\cdot,N) : \mathfrak{H}(S,M) \to \mathfrak{H}(S,N) : f \mapsto \mu f$,

thus a morphism in the category of Mackey functors. Moreover $M \mapsto \mathfrak{H}(\cdot,M)$ is easily seen to be left exact, thus the higher derivatives exist. Using either the explicit construction via injective resolutions of M or the wellknown axiomatic description of cohomology as a δ-functor (cf. [10], chap. VII, §2 for instance) one gets easily:

Proposition 9.2:

The functor $\mathfrak{H}$ from the category of $\mathbb{Z}G$-modules into the category of Mackey-functors is left-exact. If $\mathfrak{H}^i$ is its i^{th} derivative, then one has a natural isomorphism: $\mathfrak{H}^i(G/U,M) \xrightarrow{\cong} H^i(U,M)$, especially $\mathfrak{H}^i(*,M) \xrightarrow{\cong} H^i(G,M)$.

Moreover for $U \leq V \leq G$ and $\varphi: G/U \to G/V : gU \mapsto gV$ the maps

$\mathfrak{H}^i_*(\varphi,M) : \mathfrak{H}^i(G/V,M) = H^i(V,M) \to \mathfrak{H}^i(G/U,M) = H^i(U,M)$

and

$\mathfrak{H}^{i*}(\varphi,M) : \mathfrak{H}^{i}(G/U,M) = H^{i}(U,M) \to \mathfrak{H}^{i}(G/V,M) = H^{i}(V,M)$
are the usual restiction- and oorestriction-maps.

I think, that this way of introducing restriction and corestriction in cohomology might be the most natural one, especially because one gets the pullback (= Mackey) property gratuitously (which is rather useful for instance for establishing the action of the Hecke-ring on cohomology groups, cf. [5] or [12]). Thus it might be interesting to develop cohomology theory of finite groups, starting with this point of view. But for the purpose of these notes this would take too much time, especially because - as far as I have seen - no essentially new results would occur; - it is just a matter of presentation.

Instead[1)] I want to indicate, how to get the other functors, considered in homological algebra of finite groups, as Mackey functors too: For this purpose it is reasonable, to consider for any G-set S and any $\mathbb{Z}G$-module M the set M(S) of all (settheoretic) maps from S into M as a $\mathbb{Z}G$-module with composition defined argumentwise as above and with the G-action $G \times M(S) \to M(S) : (g,f) \mapsto gf$ defined by $(gf)(s) = g\cdot f(g^{-1}s)$ for any $s \in S$. ($\mathfrak{H}(S,M)$ of course is easily seen to be just the subgroup of G-invariant elements in M(S)).

1) Any reader not familiar with homological algebra better may skip this part.

As above one has for any G-map: $\varphi: T \to S$ two homomorphisms:

$\varphi_* = M_*(\varphi) : M(S) \to M(T) : f \mapsto f\varphi$ and

$\varphi^* = M^*(\varphi) : M(T) \to M(S) : f \mapsto g^* f$ with

$$\varphi^* f(s) = \sum_{t \in \varphi^{-1}(s)} f(t),$$

which even are $\mathbb{Z}G$-homomorphisms. Thus to any $\mathbb{Z}G$-module M there is associated a Mackey functor $M(\cdot)$ from $G^\wedge$ into the abelian category of $\mathbb{Z}G$-modules, given by: $S \mapsto M(S)$, $\varphi \mapsto (M^*(\varphi), M_*(\varphi))$.

Moreover we get this way a covariant functor from the category $\underline{\underline{\mathbb{Z}G\text{-mod}}}$ of $\mathbb{Z}G$-modules into the category of Mackey functors from $G^\wedge$ into $\underline{\underline{\mathbb{Z}G\text{-mod}}}$: $M \mapsto M(\cdot)$; $(\mu : M \to N) \mapsto (\Theta_\mu : M(\cdot) \to N(\cdot) : \Theta_\mu(S) : M(S) \to N(S) : f \mapsto \mu f)$.

This functor is exact.

Now let $F : \underline{\underline{\mathbb{Z}G\text{-mod}}} \to \underline{\underline{\mathfrak{Ab}}}$ be any right - or left - exact functor and define for any $\mathbb{Z}G$-module M the bifunctor $F(\cdot, M) : G^\wedge \to \underline{\underline{\mathfrak{Ab}}}$ by composing $M(\cdot)$ with $F : F(S,M) = F(M(S))$, $F(\varphi, M) = (F(M^*(\varphi)), F(M_*(\varphi)))$.

Obviously $F(\cdot, M)$ is a Mackey functor and again $M \mapsto F(\cdot, M)$ can be naturally extended to right-, resp. left - exact functor F from $\underline{\underline{\mathbb{Z}G\text{-mod}}}$ into the category of Mackey functors from $G^\wedge$ into $\underline{\underline{\mathfrak{Ab}}}$.

Its derivatives F_i (for F right-exact) or F^i (for F left-exact) again are Mackey functors. The important examples are:

$F = \mathfrak{H} : M \mapsto M^G = \{m \in M | gm = m \text{ for all } g \in M\}$,

$F : M \mapsto M_G = M/IG\cdot M$, where $IG = \{ \sum_{g\in G} n_g g \in \mathbb{Z}G | \sum n_g = 0\}$

is the augmentation ideal in the groupring $\mathbb{Z}G$,

more generally one may consider for any $\mathbb{Z}G$-module N the functors:

$$M \mapsto \mathrm{Hom}_G(M,N),$$

$$M \mapsto \mathrm{Hom}_G(N,M),$$

$$M \mapsto (M \otimes_{\mathbb{Z}} N)^G,$$

$M \mapsto M \otimes_{\mathbb{Z}G} N =_{\mathfrak{D}f} M \otimes_{\mathbb{Z}} N/\langle gm\otimes n - m\otimes g^{-1}n | g\in G,\ m\in M,\ n\in N\rangle$

and so on.

Now we come to the other examples of Mackey functors, generalizing K_G-theory: At first let $\mathfrak{C}$ be an arbitrary categor(y) (finite dimensional $\mathbb{C}$-vector spaces in K_G-theory) and S a G-set. A $\mathfrak{C}$-bundle - or more precisely a G-equivariant $\mathfrak{C}$-bundle - ξ over S associates to any $s \in S$ an object ξ_s in $\mathfrak{C}$ (the fiber over S) and to any $g \in G$ and $s \in S$ a morphism $\xi(g,s) : \xi_s \to \xi_{gs}$ in $\mathfrak{C}$, such that $\xi(e,s)$ is the identity on ξ_s (e the neutral element in G) and $\xi(h,gs)\circ\xi(g,s) : \xi_s \to \xi_{gs} \to \xi_{hgs}$

equals $\xi(hg,s)$. This can be described in more convenient form by first associating to any G-set S a category $\underline{\underline{S}}$, whose objects are just the elements in S with morphisms $[s,t]_{\underline{\underline{S}}} = \{(s,g,t) | g \in G,\ gs = t\}$ for any two $s,t \in S$ and obvious composition $((s,g,t)\circ(x,h,s) = (x,gh,t))$. Then a $\mathfrak{C}$-bundle ξ over S is nothing else than a covariant functor $\xi : \underline{\underline{S}} \to \mathfrak{C} : s \mapsto \xi_s$, $(s,g,gs) \mapsto \xi(g,s) : \xi_s \to \xi_{gs}$.

Now its trivial, to define the category of $\mathfrak{C}$-bundles over S as the category $[\underline{\underline{S}},\mathfrak{C}]$ of covariant functors from $\underline{\underline{S}}$ into $\mathfrak{C}$: A morphism μ from a $\mathfrak{C}$-bundle ξ over S into another one ξ' is a natural transformation from the functor $\xi : \underline{\underline{S}} \to \mathfrak{C}$ into the functor $\xi' : \underline{\underline{S}} \to \mathfrak{C}$, i.e. μ is a familiy $\mu(s):\xi_s \to \xi'_s$ $(s \in S)$ of morphisms in $\mathfrak{C}$, such that for any $s \in S$ and $g \in G$ the diagram

$$\begin{array}{ccc} \xi_s & \xrightarrow{\mu(s)} & \xi'_s \\ \downarrow{\scriptstyle \xi(g,s)} & & \downarrow{\scriptstyle \xi'(g,s)} \\ \xi_{gs} & \xrightarrow{\mu(gs)} & \xi'_{gs} \end{array}$$

commutes.

Note, that the category $\underline{\underline{S}}$ can also be realized as the full subcategory of $G^{\wedge}/S$, whose objects are all maps $T \to S$ with $T = G/E$, the isomorphism being given by $S \ni s \mapsto (f_s : G/E \to S : g \mapsto gs)$,

$(s,g,t) \mapsto (\rho(g) : G/E \to G/E)$ with $\rho(g)(x) = xg^{-1}$
$f_s \searrow S \swarrow f_t$

for all $x \in G/E$.

As we have seen already: For $\mathfrak{C} = \underline{\underline{\mathbb{C}\text{-mod}}}^f$ the category of finite dimensional $\mathbb{C}$-vectorspaces our category $[\underline{\underline{S}},\mathfrak{C}]$ can be identified with the category of G-equivariant $\mathbb{C}$-vector-bundles over S.

Let us also consider the case $\mathfrak{C} = \underline{\underline{Ens}}^f$, the category of finite sets. In this case a $\mathfrak{C}$-bundle ξ over a G-set S is nothing else than a G-set over S: obviously G acts in a natural way on the disjoint union $\dot{\bigcup}_{s \in S} \xi_s$ of the fibers by : $gx = \xi(g,s)x$ for

$x \in \xi_s \subset \bigcup_{s\in S} \xi_s$, so $\bigcup_{s\in S} \xi_s$ can be considered as a G-set and the map $\bigcup_{s\in S} \xi_s \to S : x \mapsto s$, if $x \in \xi_s \subseteq \bigcup_{s\in S} \xi_s$ obviously then is a G-map. On the other hand, if $\varphi : S' \to S$ is a G-set over S, then $\varphi : S' \to S$ can be considered as $\mathfrak{C}$-bundle ξ over S with fibers $\xi_s = \varphi^{-1}(s)$ and $\xi(g,s) : \xi_s \to \xi_{gs}$ defined by $x \mapsto gx (x \in \xi_s = \varphi^{-1}(s))$.
Also a $\mathfrak{C}$-bundle-morphism $\mu : \xi \to \xi'$ between two $\mathfrak{C}$-bundles over S is easily verified to be just a G-map over S between the corresponding G-sets over S - and vice versa. Thus the category $\lfloor \underline{\underline{S}}, \underline{\underline{Ens}}^f \rfloor$ can be identified with the category $G^\wedge/S$ of G-sets over S.

Especially we have $\lfloor \underline{\underline{*_G}}, \underline{\underline{Ens}}^f \rfloor \cong G^\wedge/* \cong G^\wedge$, the category of (finite) G-sets.
This of course holds more generally: Because $\underline{\underline{*_G}}$ is just the category considered in § 2 , which has exactly one object, whose endomorphism-semigroup is the group G, we have for any category $\mathfrak{C}$ an isomorphism between $\lfloor \underline{\underline{*_G}}, \mathfrak{C} \rfloor$ and the category of G-objects in $\mathfrak{C}$, i.e. the category, whose objects are pairs (X,ε) with X an object in $\mathfrak{C}$ and ε a grouphomomorphism from G into $Aut_{\mathfrak{C}}(X) \subseteq End_{\mathfrak{C}}(X)$, and suitable morphisms.

Now let $\varphi : T \to S$ be a G-map. φ induces a functor $\underline{\underline{\varphi}} : \underline{\underline{T}} \to \underline{\underline{S}} : t \mapsto \varphi(t), (t,g,gt) \mapsto (\varphi(t),g,\varphi(gt))$, thus we get a functor $\underline{\underline{\varphi}}_* : \lfloor \underline{\underline{S}}, \mathfrak{C} \rfloor \to \lfloor \underline{\underline{T}}, \mathfrak{C} \rfloor : \xi \mapsto \xi\underline{\underline{\varphi}}$ by composing any $\xi : \underline{\underline{S}} \to \mathfrak{C}$ with $\underline{\underline{\varphi}}$. Interpreting ξ as a $\mathfrak{C}$-bundle over S as in the beginning, $\xi_* = \underline{\underline{\varphi}}_*(\xi) = \xi\underline{\underline{\varphi}}$

is just the bundle ξ restricted to T via φ, i.e. the fiber of ξ_* over some $t \in T$ is the fiber of ξ over $\varphi(t) \in S$ and the map $\xi_*(g,t) : \xi_{*t} = \xi_{\varphi(t)} \to \xi_{*gt} = \xi_{\varphi(gt)} = \xi_{g\varphi(t)}$ is the map $\xi(g,\varphi(t))$.

Especially for $\mathfrak{C} = \underline{\underline{\mathbb{C}\text{-mod}}}^f$ the restriction ξ_* of a $\mathfrak{C}$-bundle, i.e. a G-equivariant $\mathbb{C}$-vectorbundle ξ over S is just the restricted bundle as defined in equivariant K-theory, for $\mathfrak{C} = \underline{\underline{\text{Ens}}}^f$ the restriction ξ_* of a $\mathfrak{C}$-bundle ξ, i.e. a G-set $\dot{\bigcup}_{s\in S} \xi_s$ over S is just the resticted G-set over T, i.e. the pull back $T \underset{S}{\times} \dot{\bigcup}_{s\in S} \xi_s$, considered as a G-set over T.

Now to get a functor $\underline{\underline{\varphi}}^* : [\underline{\underline{T}},\mathfrak{C}] \to [\underline{\underline{S}},\mathfrak{C}]$ in the other direction, the most natural idea of course is to take a left- or a right-adjoint to $\underline{\underline{\varphi}}_*$. It is well known, that such an adjoint exists only under certain conditions on the category $\mathfrak{C}$ (existence of limits) and can then be gotten by a construction due to Kan (cf[13]). In our special case it is easy to see, that the existence of finite sums $\sum_{i=1}^{n} X_i$ in $\mathfrak{C}$ is (necessary and) sufficient for the existence of a left adjoint:

For ξ a $\mathfrak{C}$-bundle over T define $\underline{\underline{\varphi}}^*(\xi) = \xi^*$ to have the fibers $\xi^*_s = \sum_{t\in\varphi^{-1}(s)} \xi_t$ with

$$\xi^*(g,s) = \sum_{t\in\varphi^{-1}(s)} \xi(g,t) : \xi^*_s = \sum_{t\in\varphi^{-1}(s)} \xi_t \to \xi^*_{gs} = \sum_{t\in\varphi^{-1}(s)} \xi_{gt},$$

for a morphism $\mu : \xi \to \eta$ of $\mathfrak{C}$-bundles over T define $\underline{\underline{\varphi}}^*(\mu) = \mu^*$ by $\mu^*(s) = \sum_{t\in\varphi^{-1}(s)} \mu(t) : \xi^*_s \to \eta^*_s$.

Now let η be $\mathfrak{C}$-bundle over S and ξ a $\mathfrak{C}$-bundle over T. Then to any morphism $\mu : \xi \to \underline{\underline{\varphi}}_*(\eta) = \eta_*$, i.e. to any family of maps $\mu(t) : \xi_t \to \eta_{*t} = \eta_{\varphi(t)}$ compatible with the G-action, there corresponds exactly one morphism $\mu' : \underline{\underline{\varphi}}^*(\xi) = \xi^* \to \eta$, given by $\mu'(s) = \sum_{t \in \varphi^{-1}(s)} \mu(t) : \xi^*_s = \sum_{t \in \varphi^{-1}(s)} \xi_t \to \eta_s$, and vice versa. Thus $\underline{\underline{\varphi}}^*$ is the left adjoint of $\underline{\underline{\varphi}}_*$.

Again for $\mathfrak{C} = \underline{\underline{\mathbb{C}\text{-mod}}}^f$, the bundle ξ^* over S is just the direct image of the bundle ξ over T, defined in equivariant K-theory for finite maps (and any map between finite sets of course is finite), for $\mathfrak{C} = \underline{\underline{\text{Ens}}}^f$ the bundle ξ^* over S corresponds to the G-set $\dot{\bigcup}_{s \in S} \xi^*_s = \dot{\bigcup}_{s \in S} \dot{\bigcup}_{t \in \varphi^{-1}(s)} \xi_t = \dot{\bigcup}_{t \in T} \xi_t$ over S, i.e. the same G-set over S, which one gets by composing the map $\dot{\bigcup}_{t \in T} \xi_t \to T$ with $\varphi : T \to S$.

Because the definition of $\underline{\underline{\varphi}}^*$ is rather similar to the definition of $\mathfrak{S}^*(\varphi, M)$, one can expect:

<u>Proposition 9.3:</u>

Let $\mathfrak{C}$ be a category with finite sums, G a finite group and

$$\begin{array}{ccc} T_1 \underset{S}{\times} T_2 & \xrightarrow{\Phi} & T_2 \\ \downarrow \Psi & & \downarrow \psi \\ T_1 & \xrightarrow[\varphi]{} & S \end{array}$$

a pull back diagram of G-sets. Then the two functors $\underline{\underline{\psi}}_* \underline{\underline{\varphi}}^*$ and $\underline{\underline{\Phi}}^* \underline{\underline{\Psi}}_* : [\underline{\underline{T}}_1, \mathfrak{C}] \to [\underline{\underline{T}}_2, \mathfrak{C}]$ are naturally equivalent (or even identically equal, once for any finite family of objects in $\mathfrak{C}$ a unique object, representing the sum, has been choosen).

The proof is analogous to the proof of Prop. 9.1, i.e. it is based on the fact, that for any $t \in T_2$ the two subsets $\varphi^{-1}(\psi(t)) \subseteq T_1$ and $\psi(\Phi^{-1}(t)) \subseteq T_1$ coincide. The details are left to the reader.

Let us observe, that - by dualizing $\mathfrak{C}$ - we could as well consider the right adjoint of $\underline{\underline{\varphi}}_*$, which of course exists if and only if $\mathfrak{C}$ contains finite products and is then defined analogously to $\underline{\underline{\varphi}}^*$ - just substituting products for sums. For $\mathfrak{C} = \underline{\underline{\mathbb{C}\text{-mod}}}^f$ this does not lead to anything new, since sum and product coincide in the abelian category $\underline{\underline{\mathbb{C}\text{-mod}}}^f$, for $\mathfrak{C} = \underline{\underline{\text{Ens}}}^f$ we get indeed another functor, which does not respect sums (disjoint union), but products, and thus is called multiplicative induction. Its properties will be studied extensively in chapter 3.

More generally, any construction which associates to any finite family $\{X_i \mid i \in I\}$ of objects in $\mathfrak{C}$ an object $F(X_i \mid i \in I)$ in $\mathfrak{C}$ and to any finite family of morphisms $\{\mu_i : X_i \to Y_i \mid i \in I\}$ a morphism $F(\mu_i \mid i \in I) : F(X_i \mid i \in I) \to F(Y_i \mid i \in I)$ and has similar formal properties as finite sums and products (essentially $F(\mu_i \cdot \nu_i \mid i \in I) = F(\mu_i \mid i \in I) \cdot F(\nu_i \mid i \in I)$, independence of the index set I up to canonical isomorphisms and "associativity", i.e. canonical isomorphisms $F(F(X_i^j \mid i \in I_j) \mid j \in J) \cong$ $\cong F(X_i^j \mid j \in J,\ i \in I_j)$, to get functoriality) can be used, to define functors $\underline{\underline{\varphi}}_F^* : [\underline{\underline{T}}, \mathfrak{C}] \to [\underline{\underline{S}}, \mathfrak{C}]$, such that Prop. 9.3 holds. Instead of explicitely stating all these formal properties, let us remark (and this will be enough for our purposes), that for $\mathfrak{C}$ the category of modules over some commutative ring R the tensorproduct $F(X_i \mid i \in I) = \bigotimes_{i \in I} X_i$ is such a construction (with $\bigotimes_{i \in I} X_i = R$ for $I = \emptyset$!).

Since moreover for a disjoint union $S \dot{\cup} T$ (with) imbeddings $i : S \to S \dot{\cup} T$ and $j : T \to S \dot{\cup} T$ the two functors $\underline{\underline{i}}_* : [\underline{\underline{S \dot{\cup} T}}, \mathfrak{C}] \to [\underline{\underline{S}}, \mathfrak{C}]$ and $\underline{\underline{j}}_* : [\underline{\underline{S \dot{\cup} T}}, \mathfrak{C}] \to [\underline{\underline{T}}, \mathfrak{C}]$ together define an isomorphism:

$$\underline{\underline{i}}_* \times \underline{\underline{j}}_* : [\underline{\underline{S \dot{\cup} T}}, \mathfrak{C}] \to [\underline{\underline{S}}, \mathfrak{C}] \times [\underline{\underline{T}}, \mathfrak{C}],$$

one could say, that for any category $\mathfrak{C}$ and any "construction" F in $\mathfrak{C}$, associating to any finite family of objects, resp. morphisms in $\mathfrak{C}$ an object, resp. a morphism in $\mathfrak{C}$ with similar formal properties as Σ, π or $\bigotimes_R$, there exists a Mackey-functor from G^* into the "category of categories", mapping any G-set S onto the category $[\underline{\underline{S}}, \mathfrak{C}]$ of $\mathfrak{C}$-bundles over S and any G-map $\varphi : T \to S$ onto the pair of functors $(\underline{\underline{\varphi}}_*, \underline{\underline{\varphi}}_F^*)$.

Since we do not want to get lost in categories, it seems more appropriate, to consider only small categories, i.e. such categories, for which the isomorphism-classes of objects in $[\underline{\underline{S}}, \mathfrak{C}]$ form a set for any G-set S (e.g. the category of finite sets $\underline{\underline{Ens}}^f$ or finitely generated R-modules for some ring R $\underline{\underline{R\text{-}mod}}^f$). Then we can associate to any

G-set S the set $k(S,\mathfrak{C})$ of isomorphism classes of $\mathfrak{C}$-bundles over S and - after choosing a "construction" F - to any G-map $\varphi : T \to S$ the pair of maps

$$\varphi_* = k_*(\varphi,\mathfrak{C}) : k(S,\mathfrak{C}) \to k(T,\mathfrak{C}),$$
$$\varphi_F^* = k_F^*(\varphi,\mathfrak{C}) : k(T,\mathfrak{C}) \to k(S,\mathfrak{C}),$$

induced by $\underline{\underline{\varphi}}_*$, resp. $\underline{\underline{\varphi}}_F^*$. Thus we get:

<u>Proposition 9.4:</u>

Let G be a finite group, $\mathfrak{C}$ a small category and F a "construction" in $\mathfrak{C}$, associating to any finite family of objects, resp. morphisms in $\mathfrak{C}$ an object, resp. a morphism in $\mathfrak{C}$ with similar formal properties as sum or product or tensorproduct. Then there exists a Mackey functor $k(\cdot,\mathfrak{C})$ from $G^\wedge$ into the category of sets, which associates to any G-set S the set of isomorphism classes of $\mathfrak{C}$-bundles over S and to any G-map $\varphi : T \to S$ a pair of maps (φ_*,φ_F^*), the first being given by restriction, the second depending on F.

Of course, to apply our foregoing results, we are mostly interested in Mackey functors from $G^\wedge$ into $\underline{\underline{\mathfrak{Ab}}}$, not into the category of sets. But following the example of K_G-theory one easily defines at least an abelian semigroup structure on $k(S,\mathfrak{C})$: Let ξ^i $(i = 1,2)$ be two $\mathfrak{C}$-bundles over S. Define $\xi = \xi^1 \underset{F}{+} \xi^2$ to be the $\mathfrak{C}$-bundle over S with fibers $\xi_s = F(\xi_s^i | i \in \{1,2\})$ and maps $\xi(g,s) = F(\xi^i(g,s)|i\in\{1,2\}: \xi_s \to \xi_{gs}$.

Of course the isomorphism class of ξ depends only on the isomorphism class of ξ^1 and ξ^2.

Moreover the independence of $F(X_i | i \in I)$ from the indexset I shows $\xi^1 \underset{F}{+} \xi^2 \cong \xi^2 \underset{F}{+} \xi^1$, the associativity of F implies $(\xi^1 \underset{F}{+} \xi^2) \underset{F}{+} \xi^3 \cong \xi^1 \underset{F}{+} (\xi^2 \underset{F}{+} \xi^3)$, thus $\underset{F}{+}$. defines a composition on the set $k(S,\mathfrak{C})$ of isomorphism classes of $\mathfrak{C}$-bundles over S, which is commutative and associative, i.e. defines an abelian semigroup structure on $k(S,\mathfrak{C})$. We write $k_F(S,\mathfrak{C})$ for the set $k(S,\mathfrak{C})$, considered as a semigroup w.r.t. $\underset{F}{+}$.

For $\mathfrak{C} = \underline{\underline{\mathbb{C}\text{-mod}}}^f$ we can choose F to be either the direct sum $\oplus$ or the tensorproduct $\otimes$ (over $\mathbb{C}$).

Of course for two $\mathfrak{C}$-bundles ξ^1 and ξ^2 over S the bundle $\xi^1 \underset{\oplus}{+} \xi^2 =_{df} \xi^1 \oplus \xi^2$ is just the direct sum, the bundle $\xi^1 \underset{\otimes}{+} \xi^2 =_{df} \xi^1 \otimes \xi^2$ is just the tensorproduct of these two bundles as defined in K_G-theory.

For $\mathfrak{C} = \underline{\underline{\text{Ens}}}^f$ we can choose F to be either the sum (disjoint union) or the product $\times$ and get for two $\mathfrak{C}$-bundles (G-sets) ξ^1 and ξ^2 over S either their disjoint union $\xi^1 \underset{\dot\cup}{+} \xi^2 = \xi^1 \dot\cup \xi^2 = \xi^1 + \xi^2$ or their (fiber-)product $\xi^1 \underset{\times}{+} \xi^2 = \xi^1 \underset{S}{\times} \xi^2$ over S.

Of course, finally we will have to consider both structures at the same time to get Green functors, but at first let us consider the case, where we have just one such construction F on a category $\mathfrak{C}$.

Let $\varphi : T \to S$ be a G-map. It is trivial, that $\underline{\underline{\varphi}}_* : \lfloor \underline{\underline{S}}, \mathfrak{C} \rfloor \to \lfloor \underline{\underline{T}}, \mathfrak{C} \rfloor$ commutes with $\underset{F}{+}$ for any such F,

i.e. that for two $\mathfrak{C}$-bundles ξ^1 and ξ^2 over S we have $\underline{\varphi}_*(\xi^1 \underset{F}{+} \xi^2) \cong \underline{\varphi}_*(\xi^1) \underset{F}{+} \underline{\varphi}_*(\xi^2)$.
Moreover using the associativity of F one can verify (by checking at the fibers)
$\underline{\varphi}_F^*(\xi^1 \underset{F}{+} \xi^2) \cong \underline{\varphi}_F^*(\xi^1) \underset{F}{+} \underline{\varphi}_F^*(\xi^2)$ for any two $\mathfrak{C}$-bundles ξ^1, ξ^2 over T, thus $\underline{\varphi}_F^*$ commutes with $\underset{F}{+}$, too, and we get:

<u>Proposition 9.4':</u>

Let G be a finite group, $\mathfrak{C}$ a small category and F a construction in $\mathfrak{C}$ (as in Prop. 9.4). Then there exists a Mackey functor $k_F(\cdot,\mathfrak{C})$ from $G^\wedge$ into the category of abelian semigroups, which associates with any G-set S the set $k_F(S,\mathfrak{C})$ of isomorphism classes of $\mathfrak{C}$-bundles over S, considered as an abelian semigroup with composition $\underset{F}{+}$ defined by F, and with any G-map $\varphi : T \to S$ the two maps:

$$k_*(\varphi,\mathfrak{C}) : k_F(S,\mathfrak{C}) \to k_F(T,\mathfrak{C}),$$
$$k_F^*(\varphi,\mathfrak{C}) : k_F(T,\mathfrak{C}) \to k_F(S,\mathfrak{C}),$$

defined by restriction ($\underline{\varphi}_*$) of $\mathfrak{C}$-bundles, resp. by induction w.r.t. F ($\underline{\varphi}_F^*$).
Composing this functor with the canonical functor from the category of abelian semigroups into the category of abelian groups, described in § 5, which associates to any abelian semigroup its universal (Grothendieck)group (the left adjoint to the im-

bedding (forget functor) of abelian groups into the category of semigroups), we finally get a Mackey functor: $K_F(\cdot,\mathfrak{C}) : G^\wedge \to \underline{\underline{\mathfrak{Ab}}}$, which associates to any G-set S the Grothendieck group $K_F(S,\mathfrak{C})$ of $\mathfrak{C}$-bundles over S, taken w.r.t. $\overset{+}{F}$, and to any G-map $\varphi: T \to S$ the maps:

$$K_*(\varphi,\mathfrak{C}) : K_F(S,\mathfrak{C}) \to K_F(T,\mathfrak{C}),$$

$$K_F^*(\varphi,\mathfrak{C}) : K_F(T,\mathfrak{C}) \to K_F(S,\mathfrak{C}),$$

defined by restriction, resp. induction w.r.t. F.

Note that contrary to $k_*(\varphi,\mathfrak{C})$ the map $K_*(\varphi,\mathfrak{C})$ depends on F, since its domain $K_F(S,\mathfrak{C})$ and its range $K_F(T,\mathfrak{C})$ depends on F. Usually we will use the last part of this proposition only if $\mathfrak{C}$ contains finite sums and F is the sum construction. In this case we write $K(S,\mathfrak{C})$ and $K^*(\varphi,\mathfrak{C})$ instead of $K_F(S,\mathfrak{C})$ and $K_F^*(\varphi,\mathfrak{C})$ and call $K(S,\mathfrak{C})$ just the Grothendieck group of $\mathfrak{C}$-bundles over S.
For $\mathfrak{C} = \underline{\underline{\mathbb{C}\text{-mod}}}^f$ of course the functor $K(\cdot,\mathfrak{C})$ $(=K_\oplus(\cdot,\mathfrak{C}))$ is just the functor $K_G : G^\wedge \to \underline{\underline{\mathfrak{Ab}}}$, for $\mathfrak{C} = \underline{\underline{\text{Ens}}}^f$ the functor $K(\cdot,\mathfrak{C}) = (K_\cup(\cdot,\mathfrak{C}))$ is the functor $\Omega : G^\wedge \to \underline{\underline{\mathfrak{Ab}}}$, considered as functors into the category of abelian groups (i.e. forgetting the ringstructure on K_G and Ω), whereas the existence of the 0-vectorspace, resp. the empty G-set implies, that $K_\otimes(\cdot,\underline{\underline{\mathbb{C}\text{-mod}}}^f)$ and $K_\times(\cdot,\underline{\underline{\text{Ens}}}^f)$ are the zero functors. But there are other ways, to exploit the additional structure on $k(S,\mathfrak{C})$, derived from $\otimes$ (in case $\mathfrak{C} = \underline{\underline{\mathbb{C}\text{-mod}}}^f$) or from $\times$ (in case $\mathfrak{C} = \underline{\underline{\text{Ens}}}^f$). The first is - following usual

K-theory - to forget the functors $\underline{\varphi}_F$ (for F not being the sum), but to use the fact, that in many cases F is distributive w.r.t. sums, i.e. that $\underset{F}{+} : k(S,\mathfrak{C}) \times k(S,\mathfrak{C}) \to k(S,\mathfrak{C})$ is bilinear, to put a ringstructure an $K(S,\mathfrak{C})$, which makes $K(\cdot,\mathfrak{C})$ to a Green functor. This will be done in just a moment. Another way is to prove, that - again for a distributive construction F - the map $k_F^*(\varphi,\mathfrak{C}) : k(T,\mathfrak{C}) \to k(S,\mathfrak{C}) \to K(S,\mathfrak{C})$ is an "algebraic map" and thus extends uniquely to an algebraic map $K(T,\mathfrak{C}) \to K(S,\mathfrak{C})$, commuting with the multiplicative semigroup-structure, defined on the rings $K(T,\mathfrak{C})$, $K(S,\mathfrak{C})$, and thus enabling one, to define further "multiplicative" Mackey functors. This will be done in the next chapter.

Still I want to give one nontrivial example of a "multiplicative" Mackey functor right now: Let $\mathfrak{C}$ be the category of 1-dimensional $\mathbb{C}$-vector-spaces and F the tensorproduct $\otimes$. Then $k_F(S,\mathfrak{C})$ is already a group - we write Pic(S) for this group of 1-dimensional $\mathbb{C}$-vectorbundles over S - and for any $\varphi : T \to S$ we get homomorphisms:
φ_*: Pic(S) $\to$ Pic(T), $\varphi_\otimes^* :$ Pic(T) $\to$ Pic(S).

We leave it to the reader, to verify, that for S = G/U one has a canonical isomorphism between Pic(S) and the abelian dual (of U) $\hat{U} = \mathrm{Hom}(U,\mathbb{C}^*) = \mathrm{Hom}(U/\lfloor U,U\rfloor,\mathbb{C}^*)$ and that for $\varphi : G/U \to *$ the associated induction map $\varphi_\otimes^* : \hat{U} \to \hat{G}$ is just the dual of the transfer-map

$G/[G,G] \to U/[U,U]$ (cf. [6]).
The application of the general theory of Mackey functors to Pic then gives results, closely related to the theorems of Grün on transfer in finite groups.
Now we go on in our construction and want to get Green functors. So let $\mathfrak{C}$ be a small category with finite sums and a construction F, such that F is distributive w.r.t. sums, i.e. if $\{X_i \mid i \in I\}$ is finite family of objects in $\mathfrak{C}$, $j \in I$ and $X_j = X_j' + X_j''$, then there exists a canonical isomorphism between $F(X_i \mid i \in I)$ and $F(Y_i \mid i \in I) + F(Z_i \mid i \in I)$ with $Y_i = Z_i = X_i$ for $i \neq j$, $Y_j = X_j'$, $Z_j = X_j''$.
For $\mathfrak{C} = \underline{\underline{\mathbb{C}\text{-mod}}}^f$ or more generally $\mathfrak{C} = \underline{\underline{R\text{-mod}}}^f$ for any commutative ring R we may take $F = \otimes = \otimes_R$, for $\mathfrak{C} = \underline{\underline{Ens}}^f$ we may take $F = X$, the cartesian product.
Then for any G-set S the set $k(S,\mathfrak{C})$ of $\mathfrak{C}$-bundles over S carries two different commutative and associative compositions, one which we now call "addition" and write "+" instead of "$+_\Sigma$" defined by taking sums, and one which we now call "multiplication" and write "$\times$" or "$\otimes$" or just "$\cdot$" instead of "$+_F$", defined by F, and the multiplication is distributive w.r.t. the addition, in other words: $k(S,\mathfrak{C})$ can be considered as a half ring and its Grothendieckgroup $K(S,\mathfrak{C}) = K_\Sigma(S,\mathfrak{C})$ taken w.r.t. addition as the associated universal ring (cf. § 5).

We want to show, that the multiplication defined on $K(S,\mathfrak{C})$ defines an inner composition $K(\cdot,\mathfrak{C}) \times K(\cdot,\mathfrak{C}) \to K(\cdot,\mathfrak{C})$ of the Mackey functor $K(\cdot,\mathfrak{C})$: So let $\varphi : T \to S$ be a G-map. As we have seen already (first part of Prop. 9.4') $k_*(\varphi,\mathfrak{C})$ and thus $K_*(\varphi,\mathfrak{C})$ commutes with our multiplication. It remains to show, that for $x \in K(S,\mathfrak{C})$ and $y \in K(T,\mathfrak{C})$ we have $\varphi^*(\varphi_*(x)\cdot y) = x \cdot \varphi^*(y)$. W.l.o.g. we may assume x to be represented by a $\mathfrak{C}$-bundle ξ over S and y to be represented by a $\mathfrak{C}$-bundle η over T. Then $\varphi^*(\varphi_*(x)\cdot y)$ is represented by the bundle $(\xi_* \cdot \eta)^*$, whose fibers at some $s \in S$ are given by $\sum_{t\in\varphi^{-1}(s)} (\xi_*\cdot\eta)_t = \sum_{t\in\varphi^{-1}(s)} F(\xi_{*t},\eta_t) =$

$$= \sum_{t\in\varphi^{-1}(s)} F(\xi_s,\eta_t) \cong F(\xi_s, \sum_{t\in\varphi^{-1}(s)} \eta_t) = F(\xi_s,\eta_s^*) =$$

$= (\xi\cdot\eta^*)_s$ (with $F(X,Y) =_{Df} F(Z^i \mid i \in \{1,2\})$, $Z^1=X$, $Z^2=Y$), i.e. are isomorphic to the fibers of $\xi\circ\eta^*$. Moreover one checks easily, that these isomorphisms of the fibers are compatible with the G-structure, thus $(\xi_*\cdot\eta)^* \cong \xi\cdot\eta^*$ and $(x_*\cdot y)^* = x\cdot y^*$, q.e.d. .
The commutativity of the multiplication now gives as well $(y\cdot x_*)^* = y^*\cdot x$, i.e. our multiplication indeed is an inner composition of Mackey functors.
Finally consider the value $X_1 = F(\emptyset)$ of F on the empty family of objects in $\mathfrak{C}$. Associativity of F implies $F(X_1,X) \cong X$ for any object X in $\mathfrak{C}$, thus the trivial bundle η_1 over S, which associates to any

$s \in S$ the object X_1 with $\eta_1(g,s) = Id_{X_1}$ for any $g \in G$, $s \in S$, represents a neutral element $1_{K(S,\mathfrak{C})}$ w.r.t. the multiplication in $K(S,\mathfrak{C})$ and $\varphi_*: K(S,\mathfrak{C}) \to K(T,\mathfrak{C})$ of course maps $1_{K(S,\mathfrak{C})}$ onto $1_{K(S,\mathfrak{C})}$. Thus we have altogether:

<u>Proposition 9.5:</u>

Let G be a finite group, $\mathfrak{C}$ a small category with finite sums and with a construction F, which associates to any finite family of objects, resp. morphisms in $\mathfrak{C}$ an object, resp. a morphism in $\mathfrak{C}$ and is "associative" and"distributive".
Then there exists a Green functor $K(\cdot,\mathfrak{C}) : G^\wedge \to \underline{\underline{\mathfrak{Ab}}}$, which associates to any G-set S the Grothendieck group $K(S,\mathfrak{C})$ of $\mathfrak{C}$-bundles over S with multiplication derived from F and to any G-map: $\varphi : T \to S$ the restriction map $\varphi_* = K_*(\varphi,\mathfrak{C}) : K(S,\mathfrak{C}) \to K(T,\mathfrak{C})$, derived from the functor $\underline{\varphi}_*: [\underline{S},\mathfrak{C}] \to [\underline{T},\mathfrak{C}]$, and the induction map: $\varphi^* = K^*(\varphi,\mathfrak{C}) : K(T,\mathfrak{C}) \to K(S,\mathfrak{C})$, derived from the left adjoint $\underline{\varphi}^*: [\underline{T},\mathfrak{C}] \to [\underline{S},\mathfrak{C}]$ to $\underline{\varphi}_*$.
Of course for $\mathfrak{C} = \underline{\underline{Ens}}^f$ the Green functor $K(\cdot,\mathfrak{C})$ is just Ω, for $\mathfrak{C} = \underline{\underline{\mathbb{C}\text{-mod}}}^f$ it is K_G. It will be one of the main objectives of these lectures, to compute the defect base $\mathfrak{D}(G,\mathfrak{C})$ of these Green-functors for various categories $\mathfrak{C}$. Note that $\mathfrak{D}(G,\mathfrak{C})$ does not depend on the construction F, which is needed only to ensure the existence of a defect base. As is suggested by our notation $\mathfrak{D}(G,\mathfrak{C})$ already, one way to compute $\mathfrak{D}(G,\mathfrak{C})$ will be to vary the group G,

especially to compare the defect base $\mathfrak{D}(G,\mathfrak{C})$ of G with the defect base $\mathfrak{D}(U,\mathfrak{C})$ for subgroups and quotient groups U of G and to use induction w.r.t. $|G|$.

To state the main result we get this way still for arbitrary $\mathfrak{C}$, write $\mathfrak{D}_R(G,\mathfrak{C})$, resp. $\mathfrak{D}_\pi(G,\mathfrak{C})$ in case $R = \mathbb{Z}[\frac{1}{p} | p \notin \pi] \subseteq \mathbb{Q}$, for the defect base of $R \underset{\mathbb{Z}}{\otimes} K(\cdot,\mathfrak{C}): G^* \to \underline{\underline{\mathfrak{Ab}}}$ (R a commutative ring) and $\mathfrak{D}_R(\mathfrak{C})$ for the class of all finite groups U with $U \in \mathfrak{D}_R(U,\mathfrak{C})$. Then we have:

<u>Theorem 9.1:</u>

Let $\mathfrak{C}$ and F be as in Prop. 9.5. Then:

(a) If $\varepsilon : G \to H$ is a grouphomomorphism, then $\mathfrak{D}_R(G,\mathfrak{C}) \subseteq \{V \leqslant G | \varepsilon(V) \in \mathfrak{D}_R(H,\mathfrak{C})\}$, especially $\mathfrak{D}_R(\mathfrak{C})$ is closed w.r.t. quotients, i.e. if $G \in \mathfrak{D}_R(\mathfrak{C})$ and $N \leqslant G$, then $G/N \in \mathfrak{D}_R(\mathfrak{C})$.

(b) $\mathfrak{D}_\pi(\mathfrak{C})$ is closed w.r.t. subgroups, i.e. if $G \in \mathfrak{D}_\pi(\mathfrak{C})$ and $U \leqslant G$, then $U \in \mathfrak{D}_\pi(\mathfrak{C})$ (π any set of primes).

(c) $\mathfrak{D}_R(G,\mathfrak{C}) = \{U \leqslant G | U \in \mathfrak{D}_R(\mathfrak{C})\}$.

Remark:

The last part of Thm 9.1 shows, that to compute the defect base $\mathfrak{D}_R(G,\mathfrak{C})$ for arbitrary groups it is enough, to determine the class of finite groups $\mathfrak{D}_R(\mathfrak{C})$, associated with any category $\mathfrak{C}$ with finite sums and an appropriate construction F (and any commutative ring R). The first two parts state, that this class is closed w.r.t. quotients and subgroups, which sometimes can be used to determine this class by induction. I don't know, whether on the other hand any class of finite groups, which is closed w.r.t. subgroups and quotients, can occur as such a class $\mathfrak{D}_R(\mathfrak{C})$ for an appropriate category $\mathfrak{C}$ and ring R. All classes, I know to appear, are either the class of all finite groups ($\mathfrak{C}$ = $\underline{\underline{Ens}}^f$) or of rather special type, always contained in (and much smaller than) the class of all finite solvable groups ($\mathfrak{C}$ more or less abelian).

For the proof of Thm 9.1 we have to study relations (particularly restriction and induction) between G-sets (and bundles over G-sets) and H-sets (and bundles over H-sets), associated with a given grouphomomorphism $\varepsilon : G \to H$. These relations of course are rather fundamental and could have been studied much earlier. I have postponed this until now, since it is closely related and rather similar to the study of restriction and induction of G-sets (and bundles) over G-sets, associated with a given G-map $\varphi : T \to S$, and I didn't want the reader to get mixed up between these two concepts (restriction and induction associated with either a G-map $\varphi: T\to S$ or a grouphomomorphism $\varepsilon : G \to H$), particularly because the second one being closer to usual representation-theory might have prevented the reader from getting familiar with the first one and its many advantages. But now we have to consider both concepts and to relate them to each other. Since we have to consider various groups at a time, let us write $K_G(\cdot,\mathfrak{C})$ for the functor $K(\cdot,\mathfrak{C}):G^*\to\underline{\underline{\mathfrak{Ab}}}$, defined above.

Let $\varepsilon : G \to H$ be a grouphomomorphism. Restricting the action of H on an H-set S to G via ε we get a G-set $S|_G = S|_\varepsilon = \underline{\underline{\varepsilon}}_*(S)$. Also any H-map $\varphi : T \to S$ between two H-sets T,S is as well a G-map $\varphi|_G = \varphi|_\varepsilon = \underline{\underline{\varepsilon}}_*(\varphi)$ between $T|_G$ and $S|_G$. Thus $\varepsilon : G \to H$ defines a functor $\underline{\underline{\varepsilon}}_*: H^* \to G^*$. It is easy to see, that $\underline{\underline{\varepsilon}}_*$ commutes with finite sums, products and pull backs (mòre generally with finite injective and projective limits). Especially ε defines a ring -homomorphism $\Omega_*(\varepsilon) : \Omega(H) \to \Omega(G)$.

Now let $\mathfrak{M} : G^* \to \underline{\underline{\mathfrak{Ab}}}$ be a Mackey functor. Composing $\mathfrak{M}$ with $\underline{\underline{\varepsilon}}_*$ we get a Mackey functor $\mathfrak{M}\varepsilon = \mathfrak{M}\underline{\underline{\varepsilon}}_* : H^* \to \underline{\underline{\mathfrak{Ab}}}$. Obviously for a given H-map $\varphi : T \to S$ the induction map $\mathfrak{M}\varepsilon^*(\varphi) : \mathfrak{M}\varepsilon(T) \to \mathfrak{M}\varepsilon(S)$ is surjective if and only if $\mathfrak{M}(\varphi|_G) : \mathfrak{M}(T|_G) \to \mathfrak{M}(S|_G)$ is surjective (since both maps are identical). Thus, if S is a defect set of $\mathfrak{M}\varepsilon$, we have a surjective map $\mathfrak{M}(\eta_{S|_G}) : \mathfrak{M}(S|_G) \to \mathfrak{M}(*_G)$, i.e.
$\mathfrak{D}_{\mathfrak{M}} \subseteq \{V \leqslant G \mid (S|_G)^V = S^{\varepsilon(V)} \neq \emptyset\} = \{V \leqslant G \mid \varepsilon(V) \in \mathfrak{D}_{\mathfrak{M}\varepsilon}\}$.

To prove Thm 9.1, (a), it remains to show that $\mathfrak{D}_{(R\otimes K_G(\cdot,\mathfrak{C}))\varepsilon} \subseteq \mathfrak{D}_{R\otimes K_H(\cdot,\mathfrak{C})}$. But by Cor(P.8.2)1 this follows from the fact, that $R \otimes K_G(\cdot,\mathfrak{C})\varepsilon$ can be considered as an $R \otimes K_H(\cdot,\mathfrak{C})$- algebra: associating to any $\mathfrak{C}$-bundle or - more precisely - H-equivariant $\mathfrak{C}$-bundle ξ over S the G-equivariant $\mathfrak{C}$-bundle $\xi|_G$ over $S|_G$, which has the same fibers as ξ with the G-action defined by restricting the H-action to G via ϵ, we get a functor: $[\underline{S},\mathfrak{C}] \to [\underline{\underline{S|_G}},\mathfrak{C}]$, which commutes nicely with everything necessary. (Of course this functor can as well be defined using the canonical functor $\underline{\underline{S|_G}} \to \underline{S} : s \to s$, $(s,g,\epsilon(g)\cdot s) \to (s,\epsilon(g),\epsilon(g)\cdot s)$). This defines the wanted natural transformation $R\otimes K_H(\cdot,\mathfrak{C}) \to R\otimes K_G(\cdot,\mathfrak{C})\epsilon$, which makes $R \otimes K_G(\cdot,\mathfrak{C})\epsilon$ to an $R \otimes K_H(\cdot,\mathfrak{C})$-algebra, and thus proves Thm 9.1 (a), which will be applied mostly to surjective homomorphisms:

$\epsilon : G \to H \cong G/N$, $N = \text{Ke } \epsilon$.

The proof of the second part (b) of Thm 9.1 will be one of the main applications of the theory of (algebraic and) multiplicative induction maps, to be developed in the next chapter.

The proof of the last part is based on Prop 8.1 (b) and on

<u>Lemma 9.1:</u>

Let $\mathfrak{C}$ be an arbitrary category and $U \leqslant G$ a sub - group of G. Then the two categories $[\underline{\underline{G/U}},\mathfrak{C}]$ and $[\underline{\underline{*_U}},\mathfrak{C}]$ are equivalent, an canonical explicite equivalence $[\underline{\underline{G/U}},\mathfrak{C}] \to [\underline{\underline{*_U}},\mathfrak{C}]$ being given by restricting any $G\mathfrak{C}$-bundle ξ over G/U and the action of G on its fibers to its single fiber over $U = *_U \in G/U$ and the action of U on this fiber. (In other words $[\underline{\underline{G/U}},\mathfrak{C}] \to [\underline{\underline{*_U}},\mathfrak{C}]$ is the composition of first restricting G to U, to get a functor $[\underline{\underline{G/U}},\mathfrak{C}] \to [\underline{\underline{G/U|_U}},\mathfrak{C}]$, and then using the U-map: $*_U \to G/U|_U : *_U \to U \in G/U$, to get a functor $[\underline{\underline{G/U|_U}},\mathfrak{C}] \to [\underline{\underline{*_U}},\mathfrak{C}]$). Moreover if F is a construction on $\mathfrak{C}$, the derived constructions on $[\underline{\underline{G/U}},\mathfrak{C}]$ and on $[\underline{\underline{*_U}},\mathfrak{C}]$ correspond to each other w.r.t. this canonical equivalence, especially if $\mathfrak{C}$ and F are as in Prop 9.5, then the restriction of $\mathfrak{C}$-bundles over G/U onto their fibers over $U = *_U \in G/U$ defines an isomorphism: $K_G(G/U,\mathfrak{C}) \tilde{\to} K_U(*_U,\mathfrak{C})$.

Proof:

Trivial: the functor $[\underline{\underline{G/U}},\mathfrak{C}] \to [\underline{\underline{*_U}},\mathfrak{C}]$ is defined by the functor $\underline{\underline{*_U}} \to \underline{\underline{G/U}} : *_U \to U \in G/U$, $(*_U,u,*_U) \to (U,u,U)$, which is an equivalence of categories, since any object in $\underline{\underline{G/U}}$ is isomorphic to $*_U = U \in G/U$ and the endo-(=auto-)morphism group of $*_U$ in $\underline{\underline{*_U}}$ and in $\underline{\underline{G/U}}$ is the same. Thus this functor defines an equivalence of functor categories: $[\underline{\underline{G/U}},\mathfrak{C}] \to [\underline{\underline{*_U}},\mathfrak{C}]$.

Now let $\mathfrak{C} = \underline{\underline{Ens}}^f$. Then we get an equivalence between the categories $G^\wedge/G/U$ and $\hat{U} = \hat{U}/*_U$, which is defined by associating to any G-set S over G/U the preimage of $*_U = U \in G/U$, considered as an U-set.

Now let $\mathfrak{M} : G^\wedge \to \underline{\underline{\mathfrak{Ab}}}$ be a Green functor. Since the canonical forget functor $V : G^\wedge/G/U \to G^\wedge : (S \to G/U) \to S$ commutes with sums (disjoint unions) and pull backs (not products!), $\mathfrak{M}$ induces a Green functor $\mathfrak{M}|_U : \hat{U} \cong G^\wedge/G/U \overset{V}{\to} G^\wedge \overset{\mathfrak{M}}{\to} \underline{\underline{\mathfrak{Ab}}}$. Obviously is G/U without defect w.r.t. $\mathfrak{M}$ if and only if $U \in \mathfrak{D}(\mathfrak{M}|_U)$, i.e. $*_U$ is without defect w.r.t. $\mathfrak{M}|_U$.

Thus by Prop. 8.1 (b) for the proof of Thm 9.1 (c) it remains to show, that for $\mathfrak{M} = K_G(\cdot,\mathfrak{C})$ ($\mathfrak{C}$ a category with a construction F as in Prop. 9.5) we have an isomorphism $\mathfrak{M}|_U \cong K_U(\cdot,\mathfrak{C})$.
So let $\varphi : S \to G/U$ be a G-set over G/U and $S' = \varphi^{-1}(*_U)$ the fiber of φ over $*_U = U \in G/U$, considered as a G-set. We have to find a canonical isomorphism: $K_G(S,\mathfrak{C}) \stackrel{\sim}{\to} K_U(S',\mathfrak{C})$, such that for any diagram

$$\begin{array}{ccc} S & \stackrel{\alpha}{\to} & T \\ & \varphi\searrow \quad \swarrow\psi & \\ & G/U & \end{array}$$

of G-maps, i.e. any morphism in $G^\wedge/G/U$,

we have commutative diagrams

$$\begin{array}{ccc} K_G(S,\mathfrak{C}) & \stackrel{\sim}{\to} & K_U(S',\mathfrak{C}) \\ \downarrow\alpha^* & & \downarrow\alpha'^* \\ K_G(T,\mathfrak{C}) & \stackrel{\sim}{\to} & K_U(T',\mathfrak{C}) \end{array} \qquad \text{and}$$

$$\begin{array}{ccc} K_G(T,\mathfrak{C}) & \stackrel{\sim}{\to} & K_U(T',\mathfrak{C}) \\ \downarrow\alpha_* & & \downarrow\alpha'_* \\ K_G(S,\mathfrak{C}) & \stackrel{\sim}{\to} & K_U(S',\mathfrak{C}) \end{array} \qquad \text{with } T' = \psi^{-1}(*_U)$$

and $\alpha' = \alpha|_{S'} : S' \to T'$, a U-map.

But for $\varphi = \mathrm{Id}_{G/U} : G/U \to G/U$ we have already an isomorphism $K_G(G/U,\mathfrak{C}) \stackrel{\sim}{\to} K_U(*_U,\mathfrak{C})$, defined in Lemma 9.1 by restricting $\mathfrak{C}$-bundles over G/U onto their fibers over $*_U = U \in G/U$. This can of course be generalized: If ξ is a G-equivariant $\mathfrak{C}$-bundle over S, then the restriction of ξ onto $S' = \varphi^{-1}(*_U)$ can be considered as an U-equivariant $\mathfrak{C}$-bundle over S'. Or in other words: We have a functor $\underline{\underline{S'}} \to \underline{\underline{S}}$ defined by imbedding S' into S (i.e. $s' \mapsto s'$, $(s',u,us') \mapsto (s',u,us')$) which defines contravariantly a functor: $[\underline{\underline{S}},\mathfrak{C}] \to [\underline{\underline{S'}},\mathfrak{C}]$ by composition. Obviously this restriction functor commutes with sums and our construction F, resp. the derived constructions on $[\underline{\underline{S}},\mathfrak{C}]$ and on $[\underline{\underline{S'}},\mathfrak{C}]$, and thus defines a (ring-)homomorphism: res = res$(G \to U)$: $K_G(S,\mathfrak{C}) \to K_U(S',\mathfrak{C})$ for any G-set over G/U: $\varphi : S \to G/U$.

One checks easily, that for the homomorphism res and for any commutative triangel 

the above diagrams commute.

Thus it remains to show, that res : $K_G(S,\mathfrak{C}) \to K_U(S',\mathfrak{C})$ is indeed an isomorphism. This can be done either by generalizing our proof in case $\varphi = Id_{G/U}: S = G/U \to G/U$, i.e. by realizing, that the functor $\underline{\underline{S'}} \to \underline{\underline{S}}$ is an equivalence of categories (any object in $\underline{\underline{S}}$ is isomorphic to one in the image of $\underline{\underline{S'}} \to \underline{\underline{S}}$ and for $s_1, s_2 \in S' \subseteq S$ we have for the set of morphisms in $\underline{\underline{S'}}$ and $\underline{\underline{S}}$ respectively: $[s_1,s_2]_{\underline{\underline{S'}}} = \{(s_1,u,s_2) | u \in U, us_1 = s_2\}$ $= \{(s_1,g,s_2) | g \in G,\ gs_1 = s_2\} = [s_1,s_2]_{\underline{\underline{S}}}$, because $gs_1 = s_2$ implies $\varphi(gs_1) = g\varphi(s_1) = g*_U = \varphi(s_2) = *_U$, i.e. $g \in U$), which implies

<u>Lemma 9.2:</u>

Let $\varphi : S \to G/U$ be a G-map and $S' = \varphi^{-1}(*_U)$ the associated U-set. Then the functor $\underline{\underline{S'}} \to \underline{\underline{S}}$, defined by the imbedding $S' \subseteq S$, is an equivalence and thus the functor res : $[\underline{\underline{S}},\mathfrak{C}] \to [\underline{\underline{S'}},\mathfrak{C}]$ is an equivalence of categories for any $\mathfrak{C}$, especially it defines isomorphisms $K_G(S,\mathfrak{C}) \xrightarrow{\sim} K_U(S',\mathfrak{C})$.

Or we can reduce our statement to the case, where S is simple, thus w.l.o.g. $S = G/V$ with $V \leqslant U$ and $\varphi : G/V \to G/U : gV \to gU$, especially $S' = U/V$, in which case we have a commutative diagram

$$\begin{array}{ccc} K_G(G/V,\mathfrak{C}) & \xrightarrow{\text{res}(G \to U)} & K_U(U/V,\mathfrak{C}) \\ & \searrow^{\text{res}(G \to V)} \quad \swarrow_{\text{res}(U \to V)} & \\ & K_V(*_V,\mathfrak{C}) & \end{array}$$

with isomorphisms res $(G \to V)$ and res $(U \to V)$, thus res $(G \to U)$ is an isomorphism.

Altogether we have proved $K_G(\cdot,\mathfrak{C})|_U \cong K_U(\cdot,\mathfrak{C})$, which together with the fact, that G/U has no defect w.r.t. $K_G(\cdot,\mathfrak{C})$ if and only if $*_U$ has no defect w.r.t. $K_G(\cdot,\mathfrak{C})|_U$, and with Prop. 9.1 (b) implies Thm 9.1 (c).

Let us finally consider relative Grothendieck rings. We have to study exact and split exact sequences of $\mathfrak{C}$-bundles over S with $\mathfrak{C}$ an appropriate category. Of course one could work with rather general categories again, but for our purposes it is sufficient to consider the category $\mathfrak{C} = \underline{\underline{R\text{-mod}}}$ of R-modules for a commutative ring R with $1 \in R$, in which case we call $\mathfrak{C}$-bundles also R-bundles, finitely generated, finitely presented, R-torsion-free (for integral domains R) or R-projective if and only if all fibers are finitely generated,

resp. finitely presented, R-torsionfree or R-projective.
We write 0 for the zero-bundle, all of whose fibers are 0, and as well for the zero-morphism $\xi \to \eta$, which is the zero-map on every fiber. Furthermore for two R-bundles ξ, η we write $\xi \oplus \eta$, resp. $\xi \otimes \eta = \xi \underset{R}{\otimes} \eta$ for the direct sum, resp. the tensor product, taken fiberwise.

We want to consider finite complexes $C: \xi^0 \xrightarrow{\mu_1} \xi^1 \xrightarrow{\mu_2} \xi^2 \longrightarrow \cdots \xrightarrow{\mu_n} \xi^n$ of R-bundles over a G-set S, i.e. sequences of R-bundles $\xi^0, \xi^1, \cdots, \xi^n$ and R-bundle-maps: $\mu_1 : \xi^0 \to \xi^1, \cdots, \mu_n : \xi^{n-1} \to \xi^n$ with $\mu_{i+1}\mu_i = 0$ $(i=1,\cdots,n-1)$. Two such complexes $C: \xi^0 \xrightarrow{\mu_1} \xi^1 \longrightarrow \cdots \xrightarrow{\mu_n} \xi^n$ and $C': \eta^0 \xrightarrow{\nu_1} \eta^1 \to \cdots \xrightarrow{\nu_n} \eta^n$ we have the notion of a homomorphism (of degree 0) $\alpha : C \to C'$, i.e. a sequence of bundle-maps: $\alpha_i : \xi^i \to \eta^i$ $(i=0,\ldots,n)$ with

$$\begin{array}{ccc} \xi^i & \xrightarrow{\mu_{i+1}} & \xi^{i+1} \\ \downarrow{\scriptstyle \alpha_i} & & \downarrow{\scriptstyle \alpha_{i+1}} \\ \eta^i & \xrightarrow{\nu_{i+1}} & \eta^{i+1} \end{array}$$

commutative $(i=0,\ldots,n-1)$, which is called an isomorphism, if all α_i are isomorphisms, and the notion of the direct sum

$$C \oplus C': \quad \xi^0 \oplus \eta^0 \xrightarrow{\mu_1 \oplus \nu_1} \xi^1 \oplus \eta^1 \to \cdots \to \xi^n \oplus \eta^n.$$

For a complex $C: \xi^0 \xrightarrow{\mu_1} \xi^1 \to \cdots \to \xi^n$ and an arbitrary R-bundle η we can define the tensorproduct

$$C \otimes \eta = C \underset{R}{\otimes} \eta : \xi^0 \otimes \eta \xrightarrow{\mu_1 \otimes Id_\eta} \xi^1 \otimes \eta \to \cdots \to \xi^n \otimes \eta,$$

which can be generalized to the notion of the tensor-product of complexes $C \otimes C'$ in the usual way:

For $C : \xi^0 \xrightarrow{\mu_1} \xi^1 \xrightarrow{\mu_2} \ldots \to \xi^n$ and

$C' : \eta^0 \xrightarrow{\nu_1} \eta^1 \xrightarrow{\nu_2} \eta^2 \ldots \xrightarrow{\nu_k} \eta^k$ we define

$C \otimes C' = C \underset{R}{\otimes} C' : \zeta^0 \xrightarrow{\sigma_1} \zeta^1 \xrightarrow{\sigma_2} \ldots \xrightarrow{\sigma_{n+k}} \zeta^{n+k}$

by: $\zeta^r = \bigoplus_{i+j=r} \xi^i \otimes \eta^j$ $(r=0,\ldots,n+k)$ and

$\sigma_r : \zeta^{r-1} = \bigoplus_{i+j=r-1} \xi^i \otimes \eta^j \to \zeta^r = \bigoplus_{i+j=r} \xi^i \otimes \eta^j$ by

$\sigma_r|_{\xi^i \otimes \eta^j} = \mu_{i+1} \otimes \mathrm{Id}_{\eta^j} \oplus (-1)^j \, \mathrm{Id}_{\xi^i} \otimes \nu_{j+1}$.

(Observe: $C \otimes C'$ is generally not isomorphic to $C' \otimes C$).

Finally a complex $C : \xi^0 \xrightarrow{\mu_1} \xi^1 \to \ldots \to \xi^n$ is called elementary, if $\xi^i = 0$ for all i except two consecutive ones, say $i = r-1, r$, with $0 \neq r-1$, $r \neq n$ and if $\xi^{r-1} = \xi^r = \xi$ with some R-bundle ξ for these two indices and $\mu_r : \xi^{r-1} = \xi \to \xi^r = \xi$ is the identity.
A complex $C : 0 = \xi^0 \to \xi^1 \to \ldots \to \xi^{n-1} \to \xi^n = 0$ is called split, if it is isomorphic to a direct sum of elementary complexes.
A complex: $C : \xi^0 \xrightarrow{\mu_1} \xi^1 \to \ldots \xrightarrow{\mu_n} \xi^n$ is called exact at some i, if the kernel of μ_{i+1} equals the image of μ_i at every fiber of ξ^i $(i \in \{1,\ldots,n-1\})$, and it is called exact, if it is exact at every $i = 1,\ldots,n-1$.
We state

<u>Lemma 9.3:</u>

(1) The direct sum of two split, resp. exact complexes is split, resp. exact.

(2) Any split complex is exact.

(3) For a short complex $C : 0 \to \xi^1 \xrightarrow{\mu_2} \xi^2 \xrightarrow{\mu_3} \xi^3 \to 0$ the following statements are equivalent:

(i) C is split.

(ii) There exists an isomorphism $\xi^2 \cong \xi^1 \oplus \xi^3$ in such a way, that $\mu_2 : \xi^1 \to \xi^2$ becomes the imbedding $\xi^1 \to \xi^1 \oplus \xi^3$ and μ_3 becomes the projection $\xi^1 \oplus \xi^3 \to \xi^3$.

(iii) C is exact and there exists a map $\nu_1 : \xi^2 \to \xi^1$ with $\nu_1\mu_2 = Id_{\xi^1}$.

(iv) C is exact and there exists a map $\nu_2 : \xi^3 \to \xi^2$ with $\mu_3\nu_2 = Id_{\xi^3}$.

(4) For any split complex C and any R-bundle η the complex $C \otimes \eta$ is split.

(5) If $S = S_1 \cup S_2$ then a complex of R-bundles over S is split, if and only if its restrictions to S_1 and S_2 are split.

Proof:

(1) and (5) are trivial, (2) follows from (1) and the exactness of elementary complexes, (3) is standard, (4) follows from $(C \oplus C') \otimes \eta \cong C \otimes \eta \oplus C' \otimes \eta$ and from the fact, that for an elementary complex C the complex $C \otimes \eta$ is elementary.

Because of later applications we generalize Lemma 9.3,(4) in the usual way by replacing η by an arbitrary complex:

Lemma 9.4:

If C or C' is split, then $C \otimes C'$ is split.

The proof is a purely technical exercise and left to the reader.

(One reduces to $C : 0 \to \xi^1 = \xi \xrightarrow{\mu = Id_\xi} \xi^2 = \xi \to 0$ elementary and has with $C' : \eta^o \xrightarrow{\nu_1} \eta^1 \to \ldots \xrightarrow{\nu_n} \eta^n$ an explicite isomorphism between

$C \otimes C' : 0 \to \zeta^1 \to \zeta^2 \to \ldots \to \zeta^{n+2} \to 0 \quad (\zeta^i = \xi^1 \otimes \eta^{i-1} \oplus \xi^2 \otimes \eta^{i-2})$

and $\bigoplus_{i=0}^{n} C_i$ with

$C_i : 0 \to 0 \to \ldots \to 0 \to \xi^1 \otimes \eta^i \xrightarrow{Id} \xi^2 \otimes \eta^i \to 0 \to \ldots \to 0$

elementary given by

$\alpha_i : \xi^1 \otimes \eta^{i-1} \oplus \xi^2 \otimes \eta^{i-2} \to \xi^1 \otimes \eta^{i-1} \oplus \xi^2 \otimes \eta^{i-2}$

with $\alpha_i = \begin{pmatrix} Id & \mu^{-1} \otimes \nu_{i-1} \\ 0 & Id \end{pmatrix}$, using an obvious convention concerning matrix notation).

Now let S,T be G-sets and $\varphi : T \to S$ a G-map. The functors $\underline{\varphi}_* : \lfloor \underline{S}, \underline{\underline{R\text{-mod}}} \rfloor \to \lfloor \underline{T}, \underline{\underline{R\text{-mod}}} \rfloor$ and $\underline{\varphi}^* : \lfloor \underline{T}, \underline{\underline{R\text{-mod}}} \rfloor \to \lfloor \underline{S}, \underline{\underline{R\text{-mod}}} \rfloor$ obviously carry complexes into complexes and one verifies easily:

Lemma 9.5:

$\underline{\varphi}_*$ and $\underline{\varphi}^*$ carry direct sums of complexes into direct sums and elementary, resp. split, resp. exact complexes into elementary, resp. split, resp. exact complexes. Furthermore $\underline{\varphi}_*$ commutes with tensorproducts of complexes.

Now let Y be a further G-set. A complex $C : \xi^o \to \xi^1 \to \ldots \to \xi^n$ of R-bundles over a G-set S is called Y-split, if the complex $\underline{\psi}_*(C)$ with

$\psi : Y\times S \to S$ the projection onto S is split.

We claim

<u>Lemma 9.6:</u>

(1) The direct sum of two Y-split complexes is Y-split.

(2) If C is Y-split and η an arbitrary R-bundle, then $C \otimes \eta$ is Y-split. More generally the tensor-product $C \otimes C'$ of two complexes is Y-split, if C or C' is Y-split.

(3) For a given complex C the class of G-sets $\{Y | C \text{ is } Y\text{-split}\}$ is r-closed, i.e.

(i) $Y' < Y$ and C Y-split $\Rightarrow C$ Y'-split

(ii) C Y-split and Y'-split $\Leftrightarrow C$ $Y \dot\cup Y'$-split.

(4) If C is Y-split and C' Y'-split, then $C \otimes C'$ is $Y \dot\cup Y'$-split.

(5) If $\varphi : T \to S$ is a G-map, then $\underline{\underline{\varphi}}_*$ and $\underline{\underline{\varphi}}^*$ carry Y-split complexes into Y-split complexes.

Proof:

(1) is trivial, (2) follows from the last part of Lemma 9.5 and Lemma 9.4, (3),(i) is trivial, (3),(ii) follows from Lemma 9.3, (5), (4) follows from (3),(ii) and (2), (5) follows from the fact, that

$$\begin{array}{ccc} Y\times T & \xrightarrow{\Phi} & Y\times S \\ \Psi\downarrow & & \downarrow\psi \\ T & \xrightarrow[\varphi]{} & S \end{array}$$

is a pull back diagram, thus $\underline{\underline{\psi}}_*\underline{\underline{\varphi}}^* \cong \underline{\underline{\Phi}}^*\underline{\underline{\Psi}}_*$ and (anyway)

$\underline{\Psi}_*\underline{\Phi}_* \cong \underline{\Phi}_*\underline{\Psi}_*$, and from Lemma 9.5.

Now consider finitely generated, resp. presented R-bundles and write $K_G(S,R)$, resp. $K'_G(S,R)$ instead of $K_G(S,\mathfrak{C})$ with $\mathfrak{C}$ the category of finitely generated, resp. presented R-modules, considered as commutative rings w.r.t. $\oplus$ and $\underset{R}{\otimes}$. If $C : \xi^o \to \xi^1 \to \ldots \to \xi^n$ is a complex of finitely generated, resp. presented R-modules over S, define

$\chi_C = \xi^o - \xi^1 + \xi^2 + - \ldots + (-1)^n\xi^n \in K_G(S,R)$,

resp. $\in K'_G(S,R)$ to be the Euler-characteristic of C. (We identify an R-bundle η with the element, represented by it in $K_G^{(')}(S,R)$, even if $k_G^{(')}(S,R) \to K_G^{(')}(S,R)$ is not necessarily injective.) We state

<u>Lemma 9.7:</u>

(1) C split implies $\chi_C = 0$.

(2) $\chi_{C\oplus C'} = \chi_C + \chi_{C'}$.

(3) $\chi_{C\otimes C'} = \chi_C \cdot \chi_{C'}$, $\chi_{C\otimes\eta} = \chi_C \cdot \eta$.

(4) If $\varphi : T \to S$ is a G-map, then $\varphi_*(\chi_C) = \chi_{\underline{\Phi}_*(C)}$ and $\varphi^*(\chi_{C'}) = \chi_{\underline{\Phi}^*(C')}$ for complexes C, C' over S, resp. T.

Now for a G-set Y define $J_G^{(')}(S,R;Y)$ to be the ideal in $K_G^{(')}(S,R)$, generated by the Euler-characteristics of all Y-split complexes of finitely generated (presented) R-bundles over S and

$K_G^{(')}(S,R;Y) = K_G^{(')}(S,R)/J_G^{(')}(S,R;Y)$ to be the quotient.

We have:

Proposition 9.6:

(1) $J_G^{(')}(S,R;Y)$ is the additive subgroup of $K_G^{(')}(S,R)$, generated by the Euler-characteristics of all short Y-split complexes of finitely generated (presented) R-bundles:

$$C : 0 \to \xi^1 \to \xi^2 \to \xi^3 \to 0.$$

(2) For a G-map $\varphi : T \to S$ we have

$$\varphi_*(J_G^{(')}(S,R;Y)) \subseteq J_G^{(')}(T,R;Y),$$

$$\varphi^*(J_G^{(')}(T,R;Y)) \subseteq J_G^{(')}(S,R;Y),$$

i.e. $J_G^{(')}(\cdot,R;Y) : G^\wedge \to \underline{\underline{\mathfrak{Ab}}}$ can be considered as a sub-Mackey-functor of $K_G^{(')}(\cdot,R)$ and thus $K_G^{(')}(\cdot,R;Y) : G^\wedge \to \underline{\underline{\mathfrak{Ab}}}$ as a Green functor.

Proof:

(1) Because of Lemma 9.7,(3) and Lemma 9.6,(2) the ideal $J_G^{(')}(S,R;Y)$ equals the additive subgroup, generated by the Euler-characteristics of Y-split complexes of arbitrary (finite) length. Now let $C : 0 \overset{\mu_1}{\to} \xi^1 \overset{\mu_2}{\to} \xi^2 \to \dots \to \xi^n \to 0$ be such a complex and consider the short complexes

$$C_i : 0 \to \mathrm{Im}(\mu_i) \to \xi^i \to \mathrm{Im}(\mu_{i+1}) \to 0 \quad (i = 1,\dots,n).$$

Because $\chi_C = \sum_{i=1}^{n} (-1)^i \chi_{C_i}$ and because all C_i are Y-split, if C is, we are reduced to short Y-split complexes, q.e.d.

(2) follows from Lemma 9.6,(5) and Lemma 9.7,(4).

Now let $\mathfrak{D}_A(G,R;Y)$ be the defect basis of $A \underset{\mathbb{Z}}{\otimes} K_G(\cdot,R;Y) : G^\wedge \to \underline{\underline{\mathfrak{Ab}}}$, A a commutative ring with $1 \in A$, and write $\mathfrak{D}_\pi(G,R;Y)$ instead of $\mathfrak{D}_A(G,R;Y)$, if $A = \mathbb{Z}[\frac{1}{p} | p \in \pi'] \subseteq \mathbb{Q}$. Similarly to Thm 9.1 we can state:

<u>Theorem 9.2:</u>

(i) $\mathfrak{D}_A(G,R) = \mathfrak{D}_A(G,R;*_G)$;

(ii) if there exists a ringhomomorphism $R \to R'$, then $\mathfrak{D}_A(G,R';Y) \subseteq \mathfrak{D}_A(G,R;Y)$;

(iii) if there exists a G-map $Y' \to Y$, then $\mathfrak{D}_A(G,R;Y') \subseteq \mathfrak{D}_A(G,R;Y)$;

(iv) if Y_1, Y_2 are G-sets, then $\mathfrak{D}_\pi(G,R;Y_1 \dot{\cup} Y_2) = \mathfrak{D}_\pi(G,R;Y_1) \cup \mathfrak{D}_\pi(G,R;Y_2)$;

(v) if $\alpha : G \to H$ is a grouphomomorphism and Y an H-set, then $\mathfrak{D}_A(G,R;Y|_G) \subseteq \{U \leqslant G | \alpha(U) \in \mathfrak{D}_A(H,R;Y)\}$

(vi) if $G \in \mathfrak{D}_\pi(G,R;Y)$, $U \leqslant G$, then $U \in \mathfrak{D}_\pi(U,R;Y|_U)$;

(vii) $\mathfrak{D}_\pi(G,R;Y) = \{U \leqslant G | U \in \mathfrak{D}_\pi(U,R;Y|_U)\}$.

Proof:

(i) is obvious; (ii) and (iii) follow from Cor.(P.8.2)1, because our assumptions imply the existence of Greenfunctor-homomorphisms $K_G(\cdot,R;Y) \to K_G(\cdot,R';Y)$ (tensoring all fibers of R-bundles with R' over R), resp. $K_G(\cdot,R;Y) \to K_G(\cdot,R;Y')$ (Lemma 9.6,(3),(i));

(iv): by (iii) we know already

$$\mathfrak{D}_\pi(G,R;Y_1) \cup \mathfrak{D}_\pi(G,R;Y_2) \subseteq \mathfrak{D}_\pi(G,R;Y_1 \dot{\cup} Y_2).$$

To prove the opposite direction we use the fact

$$J_G(*_G,R;Y_1) \cdot J_G(*_G,R;Y_2) \subseteq J_G(*_G,R;Y_1 \dot{\cup} Y_2)$$

(Lemma 9.6,(4) and Lemma 9.7,(3)).

Let S_i be a defect set of $\mathbb{Z}[\frac{1}{p} | p \in \pi'] \otimes K_G(\cdot,R;Y_i)$ $(i = 1,2)$, i.e. $\mathfrak{D}_\pi(G,R;Y_i) = \mathfrak{U}(S_i)$. By the definition of a defect set and Lemma 8.6,(a) we have elements

$x_i \in J(*_G,R;Y_i)$ and $y_i \in I_{K_G(\cdot,R)}(S_i)$ with $x_i + y_i = n_i$ and n_i a π'-number $(i=1,2)$. Thus we have $n = n_1 n_2 = (x_1 + y_1)(x_2 + y_2) =$
$= x_1 \cdot x_2 + (y_1 x_2 + y_2 x_1 + y_1 y_2)$ for a π'-number n and $x_1 \cdot x_2 \in J_G(*_G,R;Y_1 \dot{\cup} Y_2)$,
$y_1 x_2 + y_2 x_1 + y_1 y_2 \in I_{K_G(\cdot,R)}(S_1) + I_{K_G(\cdot,R)}(S_2) =$
$= I_{K_G(\cdot,R)}(S_1 \dot{\cup} S_2)$, thus

$$\mathfrak{D}_\pi(G,R;Y_1 \dot{\cup} Y_2) \subseteq \mathfrak{U}(S_1 \dot{\cup} S_2) = \mathfrak{U}(S_1) \cup \mathfrak{U}(S_2) =$$
$$= \mathfrak{D}_\pi(G,R;Y_1) \cup \mathfrak{D}_\pi(G,R;Y_2).$$

(v) is proved completely analogously to Thm 9.1,(a): one can restrict everywhere from H to G via α and gets $1 \in I_{A \otimes K_G(\cdot,R;Y|_G)}(S|_G)$, if $1 \in I_{A \otimes K_H(\cdot,R;Y)}(S)$ for an H-set S.

(vi) again will be one of the main results of the second part of these notes (Chapter 3 and 4), concerning multiplicative induction maps.

(vii): Using (vi) and (v), one can prove, that

$$\{U \leq G \mid U \in \mathfrak{D}_\pi(U,R;Y|_U)\}$$

is subconjugately closed. Thus it is enough to prove the weaker and much more elementary statement (than (vi)), that

$$\mathfrak{D}_\pi(G,R;Y) = \overline{\{U \leq G \mid U \in \mathfrak{D}_\pi(U,R;Y|_U)\}},$$

which follows from Prop. 8.1,(b), once we know, that "$U \in \mathfrak{D}_\pi(U,R;Y|_U)$" is equivalent with "G/U without defect w.r.t. $K_G(\cdot,R;Y)$".

But we know already, that for a G-set over G/U, say $\varphi : S \to G/U$, the restriction of a G-equivariant

R-bundle over S to the U-equivariant sub-R-bundle over the U-set $S' = S|_{*_U} = \varphi^{-1}(*_U)$ defines an isomorphism of categories:

$$[\underline{\underline{S}}, \underline{\underline{R\text{-mod}}}] \to [\underline{\underline{S}}', \underline{\underline{R\text{-mod}}}],$$

especially a G-equivariant R-bundle over G/U is uniquely determined by its fiber over $*_U \in G/U$, considered as an RU-module (and any RU-module can be gotten this way) and a complex of such bundles over G/U is Y-split, if and only if the corresponding complex over $*_U$ is $Y|_U$-split. Thus $K_G(G/U,R;Y) \cong K_U(*_U,R;Y|_U)$.

More generally we have - using the same argument - a canonical isomorphism

$$K_G(S,R;Y) \cong K_U(S',R;Y|_U)$$

for any G-set over G/U : $\varphi : S \to G/U$ with $S' = S|_{*_U} = \varphi^{-1}(*_U)$ (considered as a U-set) or - even more precisely -: the two Green functors defined on $G^\wedge/G/U \cong U^\wedge$ by $K_G(\cdot,R;Y)|_U : \varphi : S \to G/U \to K_G(S,R;Y)$ and by $\varphi : S \to G/U \to K_U(\varphi^{-1}(*_U),R;Y|_U)$ are canonically isomorphic.

Thus we have: $U \in \mathfrak{D}_\pi(U,R;Y|_U) \Leftrightarrow *_U$ is a defect set of $K_U(\cdot,R;Y|_U) : U^\wedge \cong G^\wedge/G/U \to \underline{\underline{\mathfrak{Ab}}} \Leftrightarrow$ G/U is without defect w.r.t. $K_G(\cdot,R;Y)$, q.e.d.

Finally let us relate the functors $K_G(\cdot,R;Y)$ to the relative Grothendieck ring as defined by I. Reiner and T.Y. Lam: Using just the last argument one can see, that a complex C of G-equivariant R-bundles over G/U is split, if and only if the complex

$C|_{*_U}$ of fibers at $*_U = U \in G/U$ is split, considered as a complex of RU-modules, especially a complex of RG-modules (over $*_G$) is G/U-split, if and only if the complex restricted to U is split. Thus $K_G(*_G,R;G/U)$ can as well be considered as the Grothendieck ring $a(G,R;U)$ of (finitely generated) RG-modules, relative to U-split exact sequences, more generally (cf. Lemma 9.6,(3)) $K_G(*_G,R;Y)$ as the Grothendieck ring $a(G,R;\mathfrak{U})$ of RG-modules, relative to $\mathfrak{U}$-split exact sequences with $\mathfrak{U} = \{U \leqslant G | Y^U \neq \emptyset\}$, i.e. complexes, which are split restricted to any $U \in \mathfrak{U}$.
Finally we have identified already $K_G(G/H,R;Y)$ with $K_H(*_H,R;Y|_H)$ and thus we can identify $K_G(G/H,R;Y)$ with the Grothendieck ring $a(H,R;H\cap\mathfrak{U})$ of all RH-modules, relative to $H\cap\mathfrak{U}$-split exact sequences, with $H\cap\mathfrak{U} = \{H\cap U | U \in \mathfrak{U}\} = \{V\leqslant H | Y^V \neq \emptyset\} = \{V\leqslant H | V \in \mathfrak{U}\}$.

Our way of introducing relative K_G-functors has the advantage of getting the Mackey- and Green-functor-structure and some of the results concerning their defect basis as a trivial byproduct, it also explains very well, why it is not sufficient to consider relative Grothendieck rings only w.r.t. one single subgroup, resp. one transitive G-set (namely: a transitive G-set restricted to a subgroup generally does not stay transitive!), but it may have the disadvantage of looking rather unconventional and complicated. Yet I hope, that the reader slowly might get used to our a bit complicated, but for many purposes convenient definitions and notations and finally even may profit by them.

§ 10 Some relations with classical representation theory

In this section I want to relate our theory to classical representation theory, especially I want to indicate a proof of Brauer's result, which states, that the Cartan-matrix of a finite group w.r.t. a field Λ of characteristic p has a determinant a power of p, $\neq 0$. This proof will later be generalized to arbitrary relative Grothendieck rings.

We know already, that the theory of complex G-equivariant vectorbundles over G-sets and the theory of complex linear representations of G and its subgroups are closely related, and we have used this fact, to motivate our procedure.

As pointed out already in § 6 and in the beginning of § 7 the general induction lemma for Mackey functors (Thm 7.1) together with the fact, that the restriction

$$K_G(*_G) = X(G) \to K_G(\dot{\bigcup_{C\in\mathcal{Z}}} G/C) = \prod_{C\in\mathcal{Z}} X(C)$$

($\mathcal{Z} = \{C \leqslant G \mid C \text{ cyclic}\}$) is injective, implies already

$|G|\cdot K_G(*_G) = |G|\cdot X(G) \subseteq \mathrm{Image}(K_G(\dot{\bigcup}_{C\in\mathcal{Z}} G/C)\to K_G(*_G)) = \sum_{C\in\mathcal{Z}} X(C)^{C\to G}$

($X(C)^{C\to G}$ the image of $X(C)$ in $X(G)$ w.r.t. the induction map $X(C) \to X(G)$), i.e. Artin's Induction-Theorem for complex representations.

More generally we get for any field L with char $L = 0$ using the same argument the following result, which occurs already in the work of Brauer, Bermann, Witt, Swan and others (cf. for instance Banaschewski, [1] 🌐; his proof is principally rather close to our proof):
Let $X(G,L)$ be the ring of (generalized) characters of L-rational representations of G (i.e. $X(G,L) \cong K_G(*_G,L)$), π a set of primes and $\mathfrak{H}_\pi = \mathfrak{H}_\pi\mathfrak{Z} = \{V \leq G \mid \exists\, C \leq V,\ C \text{ cyclic},\ V/C \text{ a } p\text{-group with } p \in \pi\}$ the set of all p-hyperelementary subgroups of G. Then

$$|G|_{\pi'} \cdot X(G,L) \subseteq \sum_{V \in \mathfrak{H}_\pi} X(V,L)^{V\to G},$$

especially

$$|G| \cdot X(G,L) \subseteq \sum_{C \in \mathfrak{Z}} X(C,L)^{C\to G},$$

$$X(G,L) = \sum_{V \in \mathfrak{H}} X(V,L)^{V\to G}$$

($\mathfrak{H} = \mathfrak{H}_\pi\mathfrak{Z}$ with π the set of all primes, i.e. $\pi' = \emptyset$).

Thus the defect basis $\mathfrak{D}_\pi(K_G(\cdot,L)) = \mathfrak{D}_\pi(G,L)$ of the Green functor $Z_{\pi'} \otimes K_G(\cdot,L)$ is contrained in $\mathfrak{H}_\pi\mathfrak{Z}$.
Equality holds, as one knows, for $\pi = \emptyset$ ($\mathfrak{D}_{\mathbb{Q}}(G,L) = \mathfrak{Z}$) and for arbitrary π, but $L = \mathbb{Q}$. The explicite computation of $\mathfrak{D}_\pi(G,L)$ as given by Berman and Witt needs the consideration not only of ordinary permutation-representations, but also of monomial permutation representations.

Now let Λ be a field with char $\Lambda = p \neq 0$. The ring $B(G,\Lambda)$ of modular or Brauer characters of G w.r.t. Λ can be defined as the Grothendieck ring $a(G,\Lambda) \cong K_G(*_G,\Lambda)$ of all (finite-dimensional!) ΛG-modules modulo the ideal, generated by the Euler characteristics $M_o - M_1 + M_2 \in a(G,\Lambda)$ of all exact sequences $0 \to M_o \to M_1 \to M_2 \to 0$ of ΛG-modules, - thus it is $\mathbb{Z}$-free with a $\mathbb{Z}$-basis the simple (i.e. irreducible) ΛG-modules (Jordan-Hölder - Theorem!).

But such a sequence $0 \to M_o \to M_1 \to M_2 \to 0$ is exact, if and only if it is split as a sequence of Λ-modules, resp. ΛE-modules ($E \leqslant G$ the trivial subgroup), i.e. if and only if it is G/E-split (cf. the end of § 9). Thus we have a canonical identification

$$B(G,\Lambda) \cong K_G(*_G,\Lambda;G/E)$$

more generally:

$B(U,\Lambda) \cong K_U(*_U,\Lambda;U/E) = K_U(*_U,\Lambda;G/E|_U) \cong K_G(G/U,\Lambda;G/E)$

for any subgroup $U \leqslant G$.

Moreover the results of § 9 show, that w.r.t. this identification the usual restriction $B(G,\Lambda) \to B(U,\Lambda)$ and induction $B(U,\Lambda) \to B(G,\Lambda)$ coincide with the maps $K_G(*_G,\Lambda;G/E) \to K_G(G/U,\Lambda;G/E)$, $K_G(G/U,\Lambda;G/E) \to K_G(*_G,\Lambda;G/E)$, associated to $\eta_{G/U} : G/U \to *_G$.

Now let R be a local Dedekind ring with quotient field L, char L = 0, maximal ideal $\mathfrak{m}$ and residue class field $R/\mathfrak{m} \cong \Lambda$. (It is known, that for any given Λ one can always find such a ring R!). The ringhomomorphism $R \to L$, $R \to \Lambda$ of course induces homomorphisms of Green functors $K_G(\cdot,R) \to K_G(\cdot,L)$, $K_G(\cdot,R) \to K_G(\cdot,\Lambda) \to K_G(\cdot,\Lambda;G/E)$ in the usual way (tensoring the fibers of R-bundles with L, resp. Λ over R). It is classical (cf. e.g. Serre, [11], p. III-9), that (at least at $*_G$, but the argument generalizes immediately!) in the triangel of Green functors

$$\begin{array}{ccc} & & K_G(\cdot,L) \\ & \nearrow & \\ K_G(\cdot,R) & & \Big\downarrow d \\ & \searrow & \\ & & K_G(\cdot,\Lambda;G/E) \end{array}$$

there exists axactly one homomorphism (of Green functors) $d : K_G(\cdot,L) \to K_G(\cdot,\Lambda;G/E)$, which makes the above diagram commutative.

By Cor(P.8.2)1 this implies

$\mathfrak{D}_Q(K_G(\cdot,\Lambda;G/E)) = \mathfrak{D}_Q(G,\Lambda;G/E) \subseteq \mathfrak{D}_Q(G,L) = \mathfrak{g}$.

Furthermore for $C \in \mathfrak{g}$ cyclic with $C = C_p \times C_{p'} = C_p \times D$ (C_p the p-Sylow-subgroup of C, $C_{p'} = D$ the direct product of all q-Sylow-subgroups of C with $q \neq p$) one knows that the restriction

$B(C,\Lambda) = K_G(G/C,\Lambda;G/E) \to B(D,\Lambda) = K_G(G/D,\Lambda;G/E)$

is injective (note for instance, that $C_p \leqslant C$ acts trivial on any simple ΛC-module). Thus we have for any $C \in \mathfrak{g}$:

$$C \in \mathfrak{D}_Q(C,\Lambda;G/E|_C) \Rightarrow C_p = E$$

and therefore, using the weaker form of Thm 9.2,(vii) as stated in its proof:

$$\mathfrak{D}_Q(G,\Lambda;G/E) \subseteq \overline{\mathfrak{Z} \cap \{U \leq G | U \in \mathfrak{D}_Q(U,\Lambda;G/E|_U)\}} \subseteq$$
$$\subseteq \{C \leq G | C \text{ cyclic, } (|C|,p) = 1\} = \mathfrak{Z}_{p'} .$$

One knows, that indeed

$$\mathfrak{D}_Q(G,\Lambda;G/E) = \mathfrak{Z}_{p'} .$$

Since $K_G(G,\Lambda;G/E) \cong B(G,\Lambda)$ is torsionfree, the inclusion above already implies:

$$\mathfrak{D}_{p'}(G,\Lambda;G/E) \subseteq \mathfrak{H}_{p'}\mathfrak{Z}_{p'} = \{V | V \in \mathfrak{H} \text{ and } (|V|,p) = 1\},$$

more precisely (using Thm 7.1):

$$|G|_p \cdot B(G,\Lambda) \subseteq \sum_{V \in \mathfrak{H}, (|V|,p) = 1} B(V,\Lambda)^{V \to G} .$$

Now observe, that for $V \leq G$, $(|V|,p) = 1$ all ΛG-modules, which are induced from ΛV-modules, are projective (Maschke - Gaschütz - D.G.Higman). Thus the above relation implies, that $|G|_p \cdot B(G,\Lambda)$ is contained in the image of the Cartan map $p(G,\Lambda) \to B(G,\Lambda)$, which is defined as the restriction of the canonical epimorphism

$K_G(*_G,\Lambda) = a(G,\Lambda) \to K_G(*_G,\Lambda;G/E) = B(G,\Lambda)$

to the ideal (additive span) $p(G,\Lambda)$ of all projective ΛG-modules (considered as elements in $a(G,\Lambda)$). It is easy to see, that this can be considered as the main step in proving, that the Cartan matrix (cf. CR,p.593) has a determinant a power of p, $\neq 0$.

I want to point out, that in our proof of this fact we have used essentially only our knowledge of the primeideal structure of the Burnside ring $\Omega(G)$ and some very elementary facts on linear representations over fields of characteristic 0 and p. All other results concerning

Mackey- and Green functors, which have been used, can also immediately be verified directly in these special cases.
It is one of the main goals of the following two chapters, to develop techniques, which allow to generalize the result concerning the Cartan map to arbitrary relative Grothendieck rings $K_G(*_G,\Lambda;S)$. This will depend heavily on the proof of Thm 9.2,(vi).

It would be tempting, also to write down the results on transfer and cohomology, which follow by applying the general induction lemma for Mackey functors to Pic and the cohomology-functors $\mathfrak{H}^i$, to underline the wideranging applicability of our techniques. But I fear, it may have the opposite effect of boring the reader, especially because - as far as I have tried - nothing essentially new will occur. So I leave this as an exercise to anyone, who might be interested in this special matter (the main point is to observe, that in these special cases the composition of restriction and induction w.r.t. $\eta_S : S \to *_G$ is just multiplication with $|S|$, i.e. that these functors are cohomology functors in the sense of Green, [4]).
Further applications to modular representation theory, especially to the theory of blocks and defect groups may also be found in the work of Green, [4] (as observed above, his G-functors can be identified with Mackea functors $G^* \to$ k-mod,

k a commutative ring with $1 \in k$, together with an arbitrary inner composition).

Finally an application of our theory to the study of Witt rings, which has been written down already half a year ago in two separate papers, will be given in an appendix to the first part of these notes.

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- Appendix A -

The Wittring as Mackey-Functor

Ernst Witt zum 60. Geburtstag

1. In a series of papers W. Scharlau used induction-techniques to study Wittrings (cf [10], [11], [6]), based on the axiomatic theory of such techniques, developed in T.Y. Lam's thesis [8]. The main tool of this theory is an axiomatic formulation of the Frobenius-reciprocity-law, which states an important relation between induction-(transfer-) and restriction-maps.

Meanwhile it has been noticed, that in the case of finite groups a more sophisticated axiomatic theory can be developed by including an axiomatic formulation of the "Mackey-subgroup-theorem" (cf [1], §44, p.324) into the system of axioms (see J.A. Green's theory of G-functors [4] and the closely related theory of Mackey-functors developed in [2] and [7]). An outline of a unified theory will be given in §3, a more thorough treatment in [3].

The main point of this paper is the fact, that even this more sophisticated theory applies to Wittrings, casting new light on Scharlau's induction-techniques for Wittrings, on the relation of Burnside- and Wittrings, studied in [13] and on the theorem of Rosenberg and Ware (cf. [9] and [6]). Especially this last theorem appears to be a special case of an axiomatic generalization of Brauer's characterization of generalized characters among classfunctions (cf. [1], §40, p.291).

Since this last aspect might be the most interesting part of this paper, it will be developed in §2, even before the general theory of Mackey-functors is applied to Wittrings in §4.

2. Let us start with a rather general Lemma on bilinear forms over commutative rings: If R is a commutative ring with $1 \in R$, then a bilinear form over R (or R-bilinear form) is just a pair (M,f) with M an R-module and $f : M \times M \to R$ an R-bilinear map.

If $\alpha : R \to A$ is a ringhomomorphism into a commutative ring A (with $1 \in A$ and $\alpha(1) = 1$, of course), then $\alpha(M,f) = (A \otimes M, \mathrm{Id}_A \otimes f)$ defines a bilinear form over A. Moreover if $\rho : A \to R$ is a map back into R, R-linear with respect to the R-module-structure on A, induced by α, and if (N,h) is an A-bilinear form, then $\rho(N,h) = (N|_R, \rho h)$ is an R-bilinear form.

Now let $\alpha : R \to A$, $\beta : R \to B$ be two ringhomomorphisms as above and $\rho : A \to R$ an R-linear map back into R. ρ induces a B-linear map

$\sigma : B \otimes_R A \to B : b \otimes a \mapsto b \cdot \beta(\rho(a))$ and we have a ringhomomorphism $\gamma : A \to B \otimes_R A : a \mapsto 1 \otimes a$ such that the diagramm

$$\begin{array}{ccc} R & \xrightarrow{\beta} & B \\ \rho\uparrow & & \uparrow\sigma \\ A & \xrightarrow[\gamma]{} & B \otimes_R A \end{array}$$

commutes.

For any A-bilinear form (M,f) we get thus two B-bilinear forms $\beta(\rho(M,f))$ and $\sigma(\gamma(M,f))$.

The Lemma states:

<u>Lemma 2.1.:</u> There exists a natural isomorphism between $\beta(\rho(M,f))$ and $\sigma(\gamma(M,f))$.

Proof: Trivial verification:

$\beta(\rho(M,f)) = \beta(M|_R, \rho f) = (B \otimes_R M, Id_B \otimes_R \rho f)$, $\sigma(\gamma(M,f)) =$
$\sigma((B \otimes_R A) \otimes_A M, Id_{B\otimes A} \otimes_A f) = \sigma(B \otimes_R M, Id_B \otimes_R f) = (B \otimes_R M, Id_B \otimes_R \rho f)$.

In spite of its triviality the above Lemma can be very usefull. At first I want to give an interpretation of Lemma 2.1. in terms of Wittrings, following the definition of Wittrings of bilinear forms over commutative rings, given in [5] : An R-bilinear form (M,f) may be called regular, if M is a finitely generated projective (f.g.p.) R-module and if f is symmetric and induces an isomorphism

$$M \to M' = Hom_R(M,R) : x \mapsto (y \mapsto f(x,y)).$$

One has the following usefull

<u>Criterion:</u> A symmetric R-bilinear form (M,f) is regular if and only if there exists $x_i, y_i \in M (i = 1,\dots,n)$

with $x = \sum_1^n f(x,y_i)x_i$ for all $x \in M$.

The Wittring $W(R)$ is the Grothendieckring $W_o(R)$ of regular bilinear forms over R modulo the subgroup (= ideal), generated by metabolic forms, where (M,f) is called metabolic if it is regular and $M = N \oplus N'$, such that N equals its own orthogonal complement in M.

Any ringhomomorphism $\alpha : R \to A$ induces a ringhomomorphism $\alpha_*: W(R) \to W(A)$. Moreover if $R = R_1 \times R_2$ with projections $\pi_i : R \to R_i$ $(i = 1,2)$, then $\pi_{1*} \times \pi_{2*}: W(R) \to W(R_1) \times W(R_2)$ is an isomorphism.

Now let $R \underset{\rho}{\overset{\alpha}{\rightleftarrows}} A$ be a pair of maps as above. I define ρ to be regular, if the R-bilinear form (A,ρ) (with: $\rho : A \times A \to R : (a,b) \to \rho\,(ab))$ is regular.

Remarks: (1) If A is f.g.p. as R-module, then there exists a regular map $\rho\ A \to R$ if and only if A and $\mathrm{Hom}_R\,(A,R)$ are isomorphic as A-modules. Moreover the well-defined tracemap $t_{A/R} : A \to R$ is regular if and only if A is a separabel R-algebra (cf. [12]).

(2) If $\rho : A \to R$ is regular and $\tau : A \to R$ linear, then there exists exactly one $a \in A$ with $\tau(x) = \rho(ax)$ for all $x \in A$. Especially any two regular maps differ only by a unit in A.

The above criterion implies easily:

<u>Lemma 2.2.:</u> If ρ is regular, then $\rho\,(M,f)$ is regular (metabolic) for any regular (metabolic) A-bilinear form (M,f).

Thus ρ induces an additive map back $\rho^*: W(A) \to W(R)$, which is easily seen to be $W(R)$-linear with respect to α_* (Frobenius-reciprocity-law, cf. [10]).

We now go back to the situation of Lemma 2.1. and assume ρ to be regular. Then σ is regular as well and we get

<u>Lemma 2.1':</u> With the above notations we have a commutative diagramm

$$\begin{array}{ccc} W(R) & \xrightarrow{\beta^*} & W(B) \\ \rho^* \uparrow & & \uparrow \sigma^* \\ W(A) & \xrightarrow[\gamma_*]{} & W(A \underset{R}{\otimes} B). \end{array}$$

To state the main result, let $\alpha : R \to A$, $\beta : R \to B$ again be two ringhomomorphisms. From β we get the Amitsurkomplex:

$$\mathfrak{C}(B/R) \;:\; R \xrightarrow{\beta=\beta^1} B \underset{\beta_2^2}{\overset{\beta_1^2}{\rightrightarrows}} B \otimes B \overset{\beta_1^3}{\Rrightarrow} \cdots$$

with $\beta_i^n\,(b_1 \otimes \ldots \otimes b_{n-1}) = b_1 \otimes \ldots \otimes b_{i-1} \otimes 1 \otimes b_i \otimes \ldots \otimes b_{n-1}$.

Applying W and taking cohomology-groups, we get the groups $H^i(B/R,\, W)\,(i \geqslant 0)$. It is easy to see, that $W(R)$ acts naturally on $H^i(B/R, W)$, this action being induced from

$\beta^n : R \to B \otimes \ldots \otimes B : r \mapsto \beta(r) \otimes 1 \otimes \ldots \otimes 1 = \ldots = 1 \otimes \ldots \otimes 1 \otimes \beta(r).$

Moreover tensoring with A over R we get

$$A \underset{R}{\otimes} \mathfrak{C}(B/R) = \mathfrak{C}(A \underset{R}{\otimes} B/A) : A \xrightarrow{\gamma=\gamma_1^1} A \underset{R}{\otimes} B \underset{\gamma_2^2}{\overset{\gamma_1^2}{\rightrightarrows}} A \underset{R}{\otimes} B \underset{R}{\otimes} B \Rrightarrow \cdots$$

and we have a natural transformation:

$$\begin{array}{ccccccc}
\mathfrak{C}(B/R) & : & R & \longrightarrow & B & \rightrightarrows & B \otimes B \Rrightarrow \cdots \\
\downarrow \alpha & & \downarrow \alpha & & \downarrow \alpha^1 = \alpha\otimes Id_B & & \downarrow \alpha^2 = \alpha\otimes Id_{B\otimes B} \\
\mathfrak{C}(A\otimes B/A) & : & A & \longrightarrow & A\otimes B & \rightrightarrows & A \otimes B \otimes B \Rrightarrow \cdots
\end{array}$$

thus we have natural maps

$H^i(\alpha) : H^i(B/R,W) \to H^i(A \otimes B/A,W)$.

For $A = B$, $\alpha = \beta$ one verifies easily $H^i(\beta) = 0$. Thus a general statement, concerning the kernel of $H^i(\alpha)$, allways implies results on $H^i(B/R,W) = Ke(H^i(\beta))$.

Now let $\rho : A \to R$ be a regular R-linear map. The main result is:

<u>Theorem 2.1.:</u> $H^i(\alpha)$ is $W(R)$-linear for the natural action of $W(R)$ on $H^i(B/R,W)$ and $H^i(A \otimes B/A,W)$ and its kernel is annihilated by $\rho^*(W(A)) \subseteq W(R)$.

Remarks: (1) For $i = 0$ and $A = B$, $\alpha = \beta$ this is just one of the induction-principles of T.Y.Lam, [8].

(2) Because any two regular maps ρ, ρ': $A \to R$ differ by a unit, the image $\rho^*(W(A))$ is independent from the map ρ, actually taken.

Before proving Theorem 2.1. I want to state some corollaries:

Corollary:2.1.: If $n \cdot 1_{W(R)} \in \rho^*(W(A))$, then $Ke(H^i(\alpha))$ is an n-torsiongroup, $(i \geq o)$. If $n \cdot 1_{W(R)} \in \rho^*(W(A)) + Ke(\beta : W(R) \to W(B))$, then $Ke(H^i(\alpha))$ is an n-torsiongroup for $i \geq 1$.

Corollary 2.2.: If $\rho^* : W(A) \to W(R)$ is surjective (i.e. $1 \in Im\ \rho^*$), then all $H^i(\alpha)$ are injective $(i \geq 0)$. If $W(R) = \rho^*(W(A)) + Ke(\beta_* : W(R) \to W(B))$, then all $H^i(\alpha)$ are injective for $i \geq 1$.

Corollary 2.3.: Especially $n \cdot 1_{W(R)} \in \rho^*(W(A))$ implies $n \cdot H^i(A/R,W) = 0$ $(i \geq o)$, $n \cdot 1_{W(R)} \in \rho^*(W(A)) + Ke(\alpha_* : W(R) \to W(A))$ implies $n \cdot H^i(A/R,W) = 0$ for $i \geq 1$, surjectivity of ρ^* implies the triviality of $H^i(A/R,W)$ for all $i \geq o$, $\rho^*(W(A)) + Ke(\alpha_* : W(R) \to W(A)) = W(R)$ implies $H^i(A/R,W) = 0$ for $i \geq 1$.

Examples of regular maps, which induce epimorphisms, have been given by W.Scharlau (of [10]). Using his argument, one can prove:

Lemma 2.3.: Let R be a commutative ring with $1 \in R$, $f(x) = x^{2n+1} + a_{2n} x^{2n} + \ldots + a_o \in R[x]$ with a_o a unit in R and $A = R[x]/f(x)$. Then

$\rho : A \to R : \sum_{0}^{2n} r_\nu x^\nu \mapsto r_o$ is a regular map, which maps $1_A \in W(A)$ onto $1_R \in W(R)$.

This Lemma is especially usefull in combination with

<u>Lemma 2.4.:</u> Let $\alpha : R \to A$ and $\beta : A \to B$ be ring-homomorphisms and $\rho: A \to R$, $\sigma : B \to A$ be regular R-, resp. A-linear maps back. Then $\rho\sigma : B \to R$ is a regular R-linear map and $(\rho\sigma)^* = \rho^*\sigma^*$. Especially $(\rho\ \sigma)^*$ is surjective, if ρ^* and σ^* are surjective.

Now let R be a field K and A a finite field extension L. Then any K-linear nonzero map $\rho : L \to K$ is regular. Let m be the "Pfisterideal" in $W(K)$, i.e. the ideal generated by all regular bilinear forms of even degree. Then one has:

<u>Lemma 2.5.:</u> Let $\rho : L \to K$ be a nonzero K-linear map. Then $\rho : W(L) \to W(K)$ is surjective, resp. has image in m, if and only if $(L : K)$ is odd, resp. even.

<u>Proof:</u> Combine Lemma 2.3. and Lemma 2.4.
Thus for $(L : K)$ odd all the Amitsur-cohomology-groups $H^i(L/K,W)$ $(i \geq o)$ are trivial. For $i = o$ this is a theorem of T.A. Springer, for $i = 1$ and L a Galois-extension it is the theorem of Rosenberg and Ware [9] and the above proof is in this special case just a variation of the proof, given by Knebusch and Scharlau in [6].

It is possible to generalize the above result to arbitrary finite extensions L/K, resp. separable finite extensions L/K if char $K \neq 2$:

<u>Theorem 2.2.:</u> Let L/K be a finite extension and L_1/K a separabel subextension such that $(L : L_1)$ is odd. Let G be the Galoisgroup of the normal closure of L_1/K and $|G| = 2^n m$ with m odd. Then 2^n annihilates all $H^i(L/K,W)$ $(i \geq 1)$.

Remark: If char $K \neq 2$, L/K finite and L_1 the separabel closure of K in L, then $(L : L_1)$ is odd and Theorem 2.2. can be applied. If $(L : K)$ is odd, choose $L_1 = K$, to get the above mentioned results.

<u>Proof:</u> Let $\alpha_1 : K \to L_1$, $\alpha_2 : L_1 \to L$ be the embeddings, $\rho_2 : L \to L_1$, $\rho_1 : L_1 \to K$ nonzero L_1-, resp. K-linear maps. By Corollary 2.3. it is enough to show
$2^n \in (\rho_1\rho_2)^*(W(L)) + Ke(\alpha_2\alpha_1)$. But $(\rho_1\rho_2)^* = \rho_1^*\rho_2^*$ and ρ_2^* is surjective and $(\alpha_2\alpha_1)_* = \alpha_{1*}\alpha_{2*}$ and α_{2*} is injective. Thus it is enough to show
$2^n \in \rho_1(W(L_1)) + Ke(\alpha_{1*})$ for L_1/K separabel and this will be one of the main applications of § 3 and § 4.

I come now to the proof of Theorem 2.1. We use Lemma 2.1.' for all the diagramms of the following form (with $B^{(n)}$ for $B \otimes \ldots \otimes B$, n times):

$$\begin{array}{ccccc}
B^{(n-1)} & \xrightarrow{\beta^n_j} & B^{(n)} & \xrightarrow{\beta^{n+1}_i} & B^{(n+1)} \\
{\scriptstyle \alpha^{n-1}}\downarrow \ \uparrow{\scriptstyle \rho^{n-1}} & & {\scriptstyle \alpha^{n}}\downarrow \ \uparrow{\scriptstyle \rho^{n}=\rho \otimes Id_{B(n)}} & & {\scriptstyle \alpha^{n+1}}\downarrow \ \uparrow{\scriptstyle \rho^{n+1}} \\
A \otimes B^{(n-1)} & \xrightarrow[\gamma^n_j]{} & A \otimes B^{(n)} & \xrightarrow[\gamma^{n+1}_i]{} & A \otimes B^{(n+1)}
\end{array}$$

For any $x \in W(B^{(n)})$, such that $\alpha^n_*(x) = \sum_{i=1}^{n} (-1)^i \gamma^n_{i\,*}(y)$ for some $y \in W(A \otimes B^{(n-1)})$, and for any $z \in W(A)$ we have

$$\sum_{j=1}^{n} (-1)^j \beta^n_{j*} (\rho^{n-1*}(z \cdot y)) =$$

$$\sum_{j=1}^{n} (-1)^j \rho^{n*} (\gamma^n_{j*}(z \cdot y)) =$$

$$\sum_{j=1}^{n} (-1)^j \rho^{n*} (z \cdot \gamma^n_{j*}(y)) =$$

$$\rho^{n*}(z \cdot \sum_{j=1}^{n} (-1)^j \gamma^n_{j*}(y)) =$$

$$\rho^{n*}(z \cdot \alpha^n*(x)) = \rho^n*(\gamma^n(z)) \cdot x$$

(with γ^n the canonical map $W(A) \to W(A \otimes B^{(n)})$ and using Frobenius reciprocity)

$= \rho^*(z) \cdot x$ (using Lemma 2.1.' for

$$\begin{array}{ccc}
R & \longrightarrow & B^{(n)} \\
\uparrow{\scriptstyle \rho} & & \uparrow{\scriptstyle \rho^n} \\
A & \xrightarrow[\gamma^n]{} & A\otimes B^{(n)}
\end{array}$$

)

Thus $\rho*(z) \cdot x = 0$ in $H^n(B/R,W)$, q.e.d.

For $n = o$ just the last part of this proof is enough.

Remark: We have not used, that x is a cycle in $H^n(B/R,W)$, - only that $\alpha^n*(x)$ is a boundary in $H^n(A \otimes B/B,W)$.

3. Now I want to give an outline of the theory of Mackey-functors. This will be applied in § 4 to Witt-rings over fields K and finite commutative separable K-Algebras. The group G, considered in this section, will then be the Galoisgroup of a finite or infinite algebraic Galoisextension of K.

So let G be a finite or profinite group. A G-set is a finite set, on which G acts (continuously) by permutations from the left. G-sets from a category $G^\wedge$ with an initial and a final object (ϕ and ★, the one-point-G-set, respectively), with sums $S_1 + S_2$ (disjoint union), products $S_1 \times S_2$ (cartesian product with diagonal G-action) and - more generally - pull-backs

$$\begin{array}{ccc} S_1 \underset{T}{\times} S_2 & \longrightarrow & S_2 \\ \downarrow & & \downarrow \\ S_1 & \longrightarrow & T \end{array}$$

Let $\underline{\underline{\mathfrak{Ab}}}$ be the category of abelian groups. We consider bifunctors $\mathfrak{M} : G^\wedge \to \underline{\underline{\mathfrak{Ab}}}$, i.e. a pair of functors $\mathfrak{M}_* : G^\wedge \to \underline{\underline{\mathfrak{Ab}}}$ and $\mathfrak{M}^* : G^\wedge \to \underline{\underline{\mathfrak{Ab}}}$, such that $\mathfrak{M}_*$ is contra-variant, $\mathfrak{M}^*$ is covariant and $\mathfrak{M}_*$ and $\mathfrak{M}^*$ coincide on the objects in $G^\wedge$. Thus for any G-map $\rho : S \to T$ in $G^\wedge$ there are two maps:

$\mathfrak{M}_*(\rho) = \rho_* : \mathfrak{M}(T) \cdot (= \mathfrak{M}_*(T) = \mathfrak{M}^*(T)) \to \mathfrak{M}(S)$

and

$\mathfrak{M}^*(\rho) = \rho^* : \mathfrak{M}(S) \to \mathfrak{M}(T)$.

ρ_* will be called the restriction map, ρ^* the induction map associated with ρ.

We define $\mathfrak{M}$ to be a Mackey-functor, if $\mathfrak{M}$ satisfies the following two conditions:

(M1) (The "Mackey-subgroup-theorem", of [1], § 44, p. 324)

For any pull-back-diagramm

$$\begin{array}{ccc} S_1 \times_T S_2 & \xrightarrow{\Phi} & S_2 \\ \Psi \downarrow & & \downarrow \psi \\ S_1 & \xrightarrow[\varphi]{} & T \end{array}$$

the following diagramm commutes:

$$\begin{array}{ccc} \mathfrak{M}(S_1 \times_T S_2) & \xrightarrow{\Phi^*} & \mathfrak{M}(S_2) \\ \Psi_* \uparrow & & \uparrow \psi_* \\ \mathfrak{M}(S_1) & \xrightarrow[\varphi^*]{} & \mathfrak{M}(T) \end{array}$$

(M2) (Additivity) For any disjoint union $S + T$ with imbeddings $i : S \to S + T$, $j : T \to S + T$ any circle in the triangel

$$\begin{array}{ccc} \mathfrak{M}(S+T) & \xrightarrow{i_* \times j_*} & \mathfrak{M}(S) \times \mathfrak{M}(T) \\ & {\scriptstyle i^* \oplus j^*} \nwarrow \qquad \swarrow {\scriptstyle \cong} & \\ & \mathfrak{M}(S) \oplus \mathfrak{M}(T) & \end{array}$$

equals the identity.

Remark: Assuming (M1) the condition (M2) is equivalent with (M0): $\mathfrak{M}(\emptyset) = 0$ and either (M2') $i_* \times j_*$ is injective or (M2") $i^* \oplus j^*$ is surjective.

Moreover (M2) states, that a Mackey-functor is uniquely determined by its restriction to the subcategory of transitive G-sets G/U (U an open subgroup in G).

Examples: 1) All the various absolute and relative Grothendieck-groups of certain representations of a group G and its subgroups can be considered as Mackey-functors. For instance consider a G-set S as a discrete, compact topological G-space and let $K_G(S)$ be the Grothendieck-group (-ring) of equivariant $\mathbb{C}G$-vector bundles over S. For any $\varphi: S \to T$ define $\varphi_*: K_G(T) \to K_G(S)$ by the pull-back of vectorbundles and $\varphi^*: K_G(S) \to K_G(T)$ by the direct image ($\varphi: S \to T$ is a finite covering!).

One has $K_G(\star) = X(G)$, the characterring of G, more generally $K_G(G/U) \cong X(U)$ and the maps $X(G) \cong K_G(\star) \underset{\varphi_*}{\overset{\varphi^*}{\rightleftarrows}} K_G(G/U) \cong X(U)$, defined by $\varphi: G/U \to \star$ are just usual restriction and induction of characters.

2) For a G-module X consider $\mathfrak{M}_X(S)$, the set of G-equivariant maps f from S into X. $\mathfrak{M}_X(S)$ is an abelian group and for any $\varphi: S \to T$ one has a restriction $\varphi_*: \mathfrak{M}_X(T) \to \mathfrak{M}_X(S)$, defined by composition: $f \to f\varphi$, and an induction $\varphi^*: \mathfrak{M}_X(S) \to \mathfrak{M}_X(T) : f \mapsto (t \mapsto \sum_{\varphi(s)=t} f(s))$.

$X \mapsto \mathfrak{M}_X$ is a left-exact functor from the category of G-modules into the category of Mackey-functors (which is a nice abelian category, if one defines a natural transformation $\Theta : M \to N$ to be a family of maps $\Theta(S) : M(S) \to N(S)$ in just one direction such that Θ is compatibel with φ_* as well as with φ^*). Its derivations $\mathfrak{H}^i$ can be used to define the groupcohomology $(\mathfrak{H}^i_X(G/U) = H^i(U, X_{|U}))$ simultaneously with the restriction - and induction-(corestriction-) maps.

3) For any G-set S consider the set of isomorphism classes of the category $G^\wedge/S$ of G-sets over S. Disjoint union defines an additive structure on this set, giving it the structure of an abelian semigroup. Let $\Omega(S)$ be the associated universal abelian group. The map $S \mapsto \Omega(S)$ can be made to a Mackey-functor: for any $\varphi : S \to T$ define $\varphi^*: \Omega(S) \to \Omega(T)$ by composition: $(\varphi' : S' \to S) \mapsto (\varphi\varphi' : S' \to T)$ and $\varphi_*: \Omega(T) \to \Omega(S)$ by the pull back:
$(\psi: T' \to T) \to (\psi_{|\varphi}: T' \underset{T}{\times} S \to S)$.

I call Ω the Burnside- or the universal Mackey-functor.
4) see §4.

For any G-set S one has exactly one map $\varphi: S \to \star$ and therefore a canonical pair of maps : $\mathfrak{M}(S) \underset{\varphi_*}{\overset{\varphi^*}{\rightleftarrows}} \mathfrak{M}(\star)$.

Define $I_{\mathfrak{M}}(S) = I(S) = \mathrm{Im}(\varphi^*)$, $K_{\mathfrak{M}}(S) = K(S) = \mathrm{Ke}(\varphi_*)$. Then one has the following generalization of Artin's Induction-Theorem.

<u>Theorem 3.1.</u> : $|G : G_S| \cdot \mathfrak{M}(\star) \subseteq I(S) + K(S)$ with $|G : G_S|$ the index of $G_S = \{g \in G \mid gs = s \text{ for any } s \in S\}$ in G.

Sometimes the factor $|G : G_S|$ has to be eliminated. This can be done analogously to the induction-theorems of Brauer and Swan: For any finite family $\mathfrak{U}$ of open subgroups of G define $S_{\mathfrak{U}} = \dot{\bigcup}_{U \in \mathfrak{U}} G/U$ (any G-set can be realized this way!) $K_{\mathfrak{M}}(\mathfrak{U}) = K(\mathfrak{U}) = K(S_{\mathfrak{U}})$, $I_{\mathfrak{M}}(\mathfrak{U}) = I(\mathfrak{U}) = I(S_{\mathfrak{U}})$. For any set π of primes define

$\mathfrak{H}_\pi \mathfrak{U} = \{V \leqslant G \mid \exists\, p \in \pi, N \trianglelefteq V, g \in G, U \in \mathfrak{U}$ with N open in G, V/N a p-group and $gNg^{-1} \leqslant U\}$. If G is finite, $\mathfrak{Z}$ the set of cyclic subgroups and $\pi = \{p\}$, then $\mathfrak{H}_\pi \mathfrak{Z}$ is the set of p-hyperelementary subgroups of G (see [1] [redacted]). Then one has the following generalization of Theorem 3.1.:

<u>Theorem 3.1'</u>: Let $\mathfrak{U}$ be a finite set of open subgroups of G, $N = \bigcap gUg^{-1} (g \in G, U \in \mathfrak{U})$, let π be a set of primes and $|G : N|_{\pi'}$ the maximal divisor of $|G : N|$,

which contains no primedivisor in π. Then for any Mackey-functor $\mathfrak{M}$ on $G^{\wedge}$ we have

$$|G : N|_{\pi}, \mathfrak{M}(\star) \subseteq I(\mathfrak{S}_{\pi}\mathfrak{U}) + K(\mathfrak{U}).$$

(cf. [3], chapter II, §7, Thm 7.1 and replace G by G/G_S, which is definitely finite).

For any map $\varphi : S \to T$ one can define a semisimplizial complex $\mathfrak{C}(\varphi)$,

$$\mathfrak{C}(\varphi): T \overset{\varphi}{\longleftarrow} S \underset{\varphi_2^2}{\overset{\varphi_1^2}{\Longleftarrow}} S \underset{T}{\times} S \overset{\varphi_1^3}{\Lleftarrow} S \underset{T}{\times} S \underset{T}{\times} S \ldots$$

with $\varphi_i^n : S^{(n)} = S \underset{T}{\times} S \underset{T}{\times} \ldots \underset{T}{\times} S \to S^{(n-1)}$

defined by $(s_1,\ldots,s_n) \mapsto (s_1,\ldots s_{i-1}, s_{i+1},\ldots s_n)$.

Thus for any Mackey-functor $\mathfrak{M}$ (even for any contravariant functor $G^{\wedge} \to \underline{\underline{\mathfrak{Ab}}}$) one gets a complex:

$$\mathfrak{C}(\varphi,\mathfrak{M}) : \mathfrak{M}(T) \xrightarrow{\varphi} \mathfrak{M}(S) \underset{\varphi_2^2}{\overset{\varphi_1^2}{\rightrightarrows}} \mathfrak{M}(S \underset{T}{\times} S) \ldots .$$ with cohomology-groups $H^i(\varphi,\mathfrak{M})$ $(i \geq 0)$. If $T = \star$ we write also $H^i(S,\mathfrak{M})$ instead of $H^i(\varphi,\mathfrak{M})$.

Moreover for any $\psi : Y \to T$ one can take the pull-back of $\mathfrak{C}(\varphi)$ with respect to ψ:

$$\mathfrak{C}(\varphi|_{\psi}) = \mathfrak{C}(\varphi)|_{\psi}: Y \overset{\Phi}{\longleftarrow} Y \underset{T}{\times} S \underset{\Phi_2^2}{\overset{\Phi_1^2}{\Longleftarrow}} Y \underset{T}{\times} S \underset{T}{\times} S \overset{\Phi_1^3}{\Lleftarrow} \ldots .$$

and one has a natural transformation

$$\begin{array}{ccccccc}
\mathfrak{C}(\varphi) : & T & \xleftarrow{\varphi} & S & \overset{\varphi_1^2}{\underset{\varphi_2^2}{\Longleftarrow}} & S \underset{T}{\times} S & \Lleftarrow \cdots \\
\uparrow \bar{\Psi} & \uparrow \Psi & & \uparrow \psi = \psi^1 & & \uparrow \psi^2 & \\
\mathfrak{C}(\varphi|_{\psi}) : & Y & \xleftarrow{\Phi} & Y \underset{T}{\times} S & \overset{\Phi_1^2}{\underset{\Phi_2^2}{\Longleftarrow}} & Y \underset{T}{\times} S \underset{T}{\times} S & \Lleftarrow \cdots
\end{array}$$

Thus one gets maps : $H^i(\psi) : H^i(\varphi, \mathfrak{M}) \to H^i(\Phi, \mathfrak{M})$ $(i \geq o)$, which are trivial, if $S = Y$, $\varphi = \psi$.

An example: If S is a G-set, such that for any $g \in G$ there exists $s \in S$ with $gs = s$, then $H^1(S, K_G)$ can be interpreted as the group of classfunctions on G, which are generalized characters restricted to any subgroup $G_s = \{g \in G | gs = s\}$ $(s \in S)$, modulo the group of generalized characters, and for any $\psi : Y \to \star$ one can interpret $Ke\ (H^1(\psi) : H^1(S, K_G) \to H^1(S \times Y \to Y, K_G))$ as the subgroup of those classfunctions, which are generalized characters on any $G_s (s \in S)$ and $G_y (y \in Y)$, modulo the group generalized characters.

Thus the following theorem is closely related to Brauers characterisation of characters:

<u>Theorem 3.2:</u> With the notations of Theorem 3.1' let $S = S_{\mathfrak{U}}$, $Y = S_{\mathfrak{H}_\pi \mathfrak{U}}$, $\psi : Y \to \star$. Then $Ke\ (H^i(\psi))$ is annihilated by $|G : G_S|_\pi$, for $i \geq 1$. Especially $H^i(S_{\mathfrak{U}})$ is annihilated by $|G : G_S|$ for $i \geq 1$ $(\pi = \emptyset!)$.

Until now we have only used an axiomatization of Mackey's subgroup theorem. But the interesting Mackey-functors are those with an inner composition, because (by their definition) they are the basic objects for an axiomatic treatment of the consequences of the Mackey-subgroup-theorem and the Frobenius-reciprocity-law combined in a unified theory.

To define such functors we define first an outer composition of Mackey-functors: For three Mackey-functors $\mathfrak{M}, \mathfrak{N}, \mathfrak{L}$ we define an outer composition $\circledS : \mathfrak{M} \times \mathfrak{N} \to \mathfrak{L}$ to be a system of bilinear maps:

$\circledS(S) : \mathfrak{M}(S) \times \mathfrak{N}(S) \to \mathfrak{L}(S)$ such that for any $\varphi: S \to T$ in $G^{\wedge}$ the following diagramms commute:

(FO)
$$\begin{array}{ccc}
\mathfrak{M}(T) \times \mathfrak{N}(T) & \xrightarrow{\ \sim(T)\ } & \mathfrak{L}(T) \\
\Big\downarrow \mathfrak{M}_*(\varphi) \times \mathfrak{N}_*(\varphi) & & \Big\downarrow \mathfrak{L}_*(\varphi) \\
\mathfrak{M}(S) \times \mathfrak{N}(S) & \xrightarrow{\ \sim(S)\ } & \mathfrak{L}(S)
\end{array}$$

(F1)
$$\begin{array}{ccccc}
\mathfrak{M}(T) \times \mathfrak{N}(S) & \xrightarrow{\ Id \times \mathfrak{N}^*(\varphi)\ } & \mathfrak{M}(T) \times \mathfrak{N}(T) & & \\
& & & \searrow^{\sim(T)} & \\
\Big\downarrow \mathfrak{M}_*(\varphi) \times Id & & & & \mathfrak{L}(T) \\
& & & \nearrow_{\mathfrak{L}^*(\varphi)} & \\
\mathfrak{M}(S) \times \mathfrak{N}(S) & \xrightarrow{\ \sim(S)\ } & \mathfrak{L}(S) & &
\end{array}$$

(F2)
$$\begin{array}{ccccc}
\mathfrak{M}(S)\times\mathfrak{N}(T) & \xrightarrow{\mathfrak{M}^*(\varphi)\times \mathrm{Id}} & \mathfrak{M}(T)\times\mathfrak{N}(T) & & \\
\Big\downarrow{\scriptstyle \mathrm{Id}\times\mathfrak{N}_*(\varphi)} & & & \searrow{\scriptstyle \odot(T)} & \\
 & & & & \mathfrak{L}(T) \\
 & & & \nearrow{\scriptstyle \mathfrak{L}^*(\varphi)} & \\
\mathfrak{M}(S)\times\mathfrak{N}(S) & \xrightarrow{\odot(S)} & \mathfrak{L}(S) & &
\end{array}$$

If $\mathfrak{N} = \mathfrak{L}$, we say $\mathfrak{M}$ acts on $\mathfrak{N}$; if $\mathfrak{M} = \mathfrak{N} = \mathfrak{L}$, then $(\mathfrak{M}, \odot)$ is a Mackey-functor with an inner composition.

Examples: (1) The G-functors, defined and studied by J.A. Green [4], are equivalent to Mackey-functors with an inner composition.

(2) The tensor product of G-representations can often be used, to define an inner composition in the associated Grothendieckgroups.

(3) Ω acts on any Mackey-functor $\mathfrak{M}$ in a natural way: For any G-set S the map $(\varphi\colon S' \to S,\ x) \mapsto \varphi^*(\varphi_*(x))$ $(x \in \mathfrak{M}(S))$ induces a bilinear map $\Omega(S) \times \mathfrak{M}(S) \to \mathfrak{M}(S)$ which satisfies (F0), (F1), (F2).

Especially Ω acts on $\mathfrak{M} = \Omega$. This action defines an inner composition, which induces on any $\Omega(S)$ the structure of a commutive ring with $1 \in \Omega(S)$, such that the above action $\Omega(S) \times \mathfrak{M}(S) \to \mathfrak{M}(S)$ makes $\mathfrak{M}(S)$ into an (unitary, of course) $\Omega(S)$-module. (cf. [3], §7).

$\Omega(\star)$ is just the Burnsidering of G (cf. [13]), more generally $\Omega(G/U)$ the Burnsidering of U.

The action of Ω on any Mackey-functor M allows to reduce the proof of Theorem 3.1, 3.1', 3.2 to the case of Ω and plays thus a key-rôle in the proof of these theorems. (More generally in a rather formal way one can study Mackey-functors $\mathfrak{M}$, on which a Mackey-functors $\mathfrak{N}$ with an inner composition acts in a compatible way, and show, in which way special assumptions for $\mathfrak{N}$ imply special properties of $\mathfrak{M}$.) (cf. [3], §7).

Finally if $\mathfrak{M}$ is a Mackey-functor with an inner composition and a unit $1_{\mathfrak{M}(S)} = 1_S \in \mathfrak{M}(S)$ for any S and $\varphi_*(1_S) = 1_T$ for any $\varphi : T \to S$ in $G^\wedge$, then there exists a unique natural transformation $\textcircled{\sim} : \Omega \to \mathfrak{M}$, which is multiplicative and maps $1_{\Omega(S)}$ onto $1_{\mathfrak{M}(S)}$ - even if no assumptions oncommutativity and associativity in $\mathfrak{M}$ are made. We call such Mackey-functors also Mackey-algebras.

(4) See §4.

The basic fact for Mackey-functors $\mathfrak{M}$ with an inner composition is the following result of Green ([4], §3, Theorem 1, p.14 or [3], §8 in the case of finite groups).

<u>Theorem 3.3:</u> If $\mathfrak{M}(\star) \times \mathfrak{M}(\star) \to \mathfrak{M}(\star)$ is surjective (e.g. $\mathfrak{M}$ is a Mackey-algebra), then there exists a unique family $\mathfrak{D}_{\mathfrak{M}}$ of closed subgroups of G (the defect base of M) such that:

(i) $\mathfrak{D}_{\mathfrak{M}}$ is a closed subset of the topological (compact, totally disconnected) space $\mathfrak{S}(G)$ of all closed subgroups of G, described in [13]

(ii) If $U, V \leqslant G$, $g \in G$, $U \in \mathfrak{D}_{\mathfrak{M}}$ and $gVg^{-1} \subseteq U$, then $V \in \mathfrak{D}_{\mathfrak{M}}$.

(iii) For any $\varphi : S \to \star$ the map $\varphi^*: \mathfrak{M}(S) \to \mathfrak{M}(\star)$ is surjective if and only if for any $U \in \mathfrak{D}_{\mathfrak{M}}$ there exists $s \in S$ with $U \leqslant G_s$, i.e. if and only if $S^U \neq \emptyset$ for all $U \in \mathfrak{D}_{\mathfrak{M}}$.

Proof: The generalization from finite to profinite G is rather easy: For any open normal subgroup $N \trianglelefteq G$ one may restrict $\mathfrak{M}$ to the subcategory $(G/N)^\wedge$ of $G^\wedge$ of those G-sets, on which N acts trivial, and then has a defect base $\mathfrak{D}_{\mathfrak{M}}(G/N)$ at least for this restricted $\mathfrak{M}$. Define

$\mathfrak{D}_{\mathfrak{M}} = \{U \leqslant G \mid U \cdot N/N \in \mathfrak{D}_{\mathfrak{M}}(G/N)$ for all open normal subgroups $N \trianglelefteq G\}$.

The properties and the uniqueness of $\mathfrak{D}_{\mathfrak{M}}$ are easily verified.

One can show, that $\mathfrak{D}_{\Omega}$ contains all subgroups of G. Generally for a given $\mathfrak{M}$ the determination of its defect base $\mathfrak{D}_{\mathfrak{M}}$ is very often one of the most relevant problems connected with $\mathfrak{M}$ and allows many corollaries. Already upper and lower bounds for $\mathfrak{D}_{\mathfrak{M}}$ can be very usefull.
For instance one can prove the following usefull version of Theorem 3.1:

<u>Theorem 3.1":</u> Let $\mathfrak{M}$ be a Mackey-algebra, such that all groups in the defect base of $\mathfrak{M}$ are π-groups (π some set of primes) and let $\mathfrak{N}$ be a Mackey-functor, on which $\mathfrak{M}$ acts unitary (e.g. $\mathfrak{M}$ itsself). Then one has for any G-set S:
$|G : G_S|_\pi \cdot \mathfrak{N}(\star) \subseteq K_{\mathfrak{N}}(S) + I_{\mathfrak{N}}(S)$.

Proof: W.l.o.g. $G_S = E$. We know any way $|G| \cdot \mathfrak{M}(\star) \subseteq K_{\mathfrak{M}}(S) + I_{\mathfrak{M}}(S)$, thus it is enough to show, that also

$$n \cdot \mathfrak{M}(\star) \subseteq K_{\mathfrak{M}}(S) + I_{\mathfrak{M}}(S)$$

for some π-number n.

Consider the pull back diagramm

$$\begin{array}{ccc} T \times S & \xrightarrow{\psi} & S \\ \downarrow \Phi & & \downarrow \varphi \\ T & \xrightarrow[\Psi]{} & \star \end{array} \qquad \text{with } T = \dot{\bigcup_{U \in \mathfrak{D}_{\mathfrak{M}}}} G/U.$$

Since $\Psi^* \ (\mathrm{Im}(\Phi^* : \mathfrak{M}(T \times S) \to \mathfrak{M}(T)) \subseteq I_{\mathfrak{M}}(S)$

and $\Psi^*(\mathrm{Ke}(\Phi_*: \mathfrak{M}(T) \to \mathfrak{M}(T \times S)) \subseteq K_{\mathfrak{M}}(S)$ (by(M1))

and $\Psi^*(\mathfrak{M}(T)) = \mathfrak{M}(\star)$ (cf. [3], §8, Prop. 8.2) it is enough, to show

$$\prod_{U \in \mathfrak{D}_{\mathfrak{M}}} |U| \cdot \mathfrak{M}(T) \subseteq \mathrm{Ke}(\Phi_*) + \mathrm{Im}\ (\Phi^*).$$

But this follows easily from Theorem 3.1 and

<u>Lemma 3.1:</u> If U is an open subgroup of G, then one has a canonical equivalence between the category $G^{\wedge}/(G/U)$ of G-sets over G/U and $U^{\wedge}$, which associates to any $\varphi : S \to G/U$ the fiber $\varphi^{-1}(U)$ over $U \in G/U$, considered as a U-set. Especially any Mackey-functor $\mathfrak{M}$ on $G^{\wedge}$, restricted to G-sets over G/U, defines a Mackey-functor $\mathfrak{M}|_U$ on $U^{\wedge}$, thus $|U| \cdot \mathfrak{M}(G/U) \subseteq \mathrm{Ke}(\varphi_*) + \mathrm{Im}(\varphi^*)$ for any map $\varphi : S \to G/U$.

Proof: cf. [3], § 9, Lemma 9.1 .

One just has to observe, that the map

$\Phi : T \times S \to T = \biguplus_{U \in \mathfrak{D}_{\mathfrak{M}}} G/U$ is the sum of maps

$$\Phi_U : T \times S|_U = \Phi^{-1}(G/U) \to G/U$$

$(U \in \mathfrak{D}_{\mathfrak{M}})$ and that (using (M 2))

$$\mathrm{Ke}(\Phi_*) = \bigoplus_{U \in \mathfrak{D}_{\mathfrak{M}}} \mathrm{Ke}(\Phi_{U*}),$$

$$\mathrm{Im}(\Phi^*) = \bigoplus_{U \in \mathfrak{D}_{\mathfrak{M}}} \mathrm{Im}(\Phi_U{}^*).$$

The proof of Theorem 2.1 can be generalized to yield the following result:

<u>Theorem 3.4:</u> Let $\circledS : \mathfrak{M} \times \mathfrak{N} \to \mathfrak{L}$ be an outer composition. Let $\varphi : S \to T$, $\psi : Y \to T$ be two maps in $G^\wedge$. $\circledS$ induces a bilinear map:

$$\mathfrak{M}(T) \times H^i(\varphi, \mathfrak{N}) \to H^i(\varphi, \mathfrak{L})$$

which vanishes on

$$\psi^*(\mathfrak{M}(Y)) \times \mathrm{Ke}\ (H^i(\psi)),\ i \geq 0.$$

Of course there are a lot of corollaries, which I do not want to state here in detail. A more thorough treatment of the relations between cohomology, Mackey-functors and Hecke-rings might be given in another paper.

I just want to remark, that Theorem 3.2. follows from Theorem 3.4 with $\mathfrak{M} = \Omega$, $\mathfrak{N} = \Omega$, using the natural action of Ω on any Mackey-functor and Theorem 3.1' for $\mathfrak{M} = \Omega$. The same way Theorem 3.1" can be used, to prove:

<u>Theorem 3.2":</u> With the notations and assumptions of Theorem 3.1" one has

$$|G : G_S|_\pi \cdot H^i(S,N) = 0 \ (i \geq 1).$$

This will turn out to be some kind of generalization of Theorem 2.2.

Finally I want to state one rather special case of the general transfer-theorem for Mackey- or G-functors, proved by J.A. Green in [4] (§4.2, Thm 2, p.26 together with Remark 1 at the bottom of p.27):

<u>Theorem 3.5:</u> Let G be a finite group, $D = G_p$ a p-Sylow-subgroup, $H = N_G(D)$ and $\mathfrak{U}$ a family of subgroups of D, which contains the set $\{D \cap D^g \mid g \in G - H\}$.
Let $\mathfrak{M}$ be a Mackey-algebra with all defect-groups being p-groups. Consider $\mathfrak{M}(G/H,\mathfrak{U}) = \sum_{U \in \mathfrak{U}} K_U^*(\mathfrak{M}(G/U))$ with K_U the canonical map: $G/U \to G/H$: $gU \to gH$ $(U \leq D \leq H!)$.
Then the map $\lambda : G/H \to \star$ induces multiplicative isomorphisms, which are respectively inverse to each other:

$$\lambda^* : \mathfrak{M}(G/H) \,/\, \mathfrak{M}(G/H,\mathfrak{U}) \longrightarrow \mathfrak{M}(\star)/I_{\mathfrak{M}}(\mathfrak{U})$$

$$\lambda_* : \mathfrak{M}(\star) \,/\, I_{\mathfrak{M}}(\mathfrak{U}) \longrightarrow \mathfrak{M}(G/H)/\mathfrak{M}(G/H,\mathfrak{U})$$

Remark: (1) One has similar isomorphisms for any Mackey-functor $\mathfrak{N}$, on which $\mathfrak{M}$ acts unitary.

(2) Green's paper contains much more sophisticated results in case $D \leq H \leq G$ are arbitrary subgroups and $\mathfrak{M}$ an arbitrary Mackey-functor.

4. Now let K be a field and let E be a finite or infinite algebraic Galoisextension of K with Galois-group G. Define $\underset{\sim}{A}(K,E)$ to be the category of K-algebras R, such that $E \otimes_K R$ is a product of a finite number of copies of E, i.e. R is a product of finite number of finite subextensions L/K of E/K.

$\underset{\sim}{A}(K,E)$ is anti-equivalent to $G^{\wedge}$, this anti-equivalence being given by $S \mapsto R_S$ the set (= K-algebra) of G-equivariant maps $f: S \to E$, resp. $R \mapsto S_R$ the set of K-algebra-homomorphisms $R \to E$, (which becomes a G-set using the action of G on E), in the opposite direction and obvious definitions for morphisms (composition!). Moreover a pull back-diagramm

$$\begin{array}{ccc} S_1 \times_T S_2 & \to & S_2 \\ \downarrow & & \downarrow \\ S_1 & \longrightarrow & T \end{array}$$

corresponds to a tensorproduct-diagramm:

$$\begin{array}{ccc} B \otimes_R A & \longleftarrow & A = R_{S_2} \\ \uparrow & & \uparrow \\ B = R_{S_1} & \longleftarrow & R = R_T \end{array}$$

and the transitive G-sets correspond to field-extensions (G/U with the fixed field E^U).

Finally any $\alpha : R \to A$ in $\mathbb{A}(K,E)$ makes A to a separabel R-algebra, thus the trace: $t_{A/R}: A \to R$ is a regular map (cf §2 and [12]). Let α^* be the induced map $t^* : W(A) \to W(R)$. If α corresponds $\varphi: S_A \to S_R$, we write also φ_* for $\alpha_*: W(R) \to W(A)$ and φ^* for α^*. Also if R corresponds to $S(S = S_R)$, we write $W(S)$ for $W(R)$. With this definitions the first part of §2 implies:

<u>Theorem 4.1:</u> The "Witt-functor" $W: G^\wedge \to \underline{\underline{\mathfrak{Ab}}} : S \mapsto W(R_S) = W(S)$,
$(\varphi: S \to T) \mapsto (\varphi^*, \varphi_*) : W(S) \underset{\varphi_*}{\overset{\varphi^*}{\rightleftarrows}} W(T)$

is a Mackey-functor with an inner composition (the multiplication in the Wittring), such that $W(S)$ becomes a commutative ring with $1_S \in W(S)$ and $\varphi_*(1_T) = 1_S$ for any $\varphi : S \to T$ in $G^\wedge$, i.e. W is a commutative and associative Mackey-algebra.

The rest of this paper is just a collection of corollaries to Theorem 4.1:
At first, W being a Mackey-algebra, one gets a canonical ring-homomorphism: $\Omega(G) \to W(K)$, which turns out to be just the Scharlau-map, studied in [13].

Lemma 2.5 implies, that the defect-basis $\mathfrak{D}_W$ contains exactly all pro-2-subgroups of G, thus is "generated" by the pro-2-Sylow-subgroup of G. Therefore we can omit to state corollaries of Theorem 3.1, 3.1' and 3.2 and rather apply directly Theorem 3.1":

Let E/K be finite and let $L_1,..L_n$ be a family of subextensions of E.

Then one has:

$$(E : K)_2 \cdot W(K) \subseteq \bigcap_1^n Ke(W(K) \to W(L_i)) + \sum_1^n \rho_i^* (W(L_i))$$

with $(E : K)_2$ the maximal power of 2, dividing $(E : K)$, and $\rho_i : L_i \to K$ the trace $(i = 1,...,n)$.

For $n = 1$ this is just the result, promised in the proof of Theorem 2.2.

Of course the applications of Theorem 3.2." are already contained in Theorem 2.2.

Theorem 3.5 implies:

Let E/K be finite, let L/K be a maximal subextension of odd degree and F/K a minimal subextension in L/K such that L/F is normal (i.e. the fixfield of all K-automorphisms of L). Let $L_1,...,L_n$ be a family of subextensions of E/K with $L \subseteq L_i$, which contains the composition $L \cdot L^\sigma$ for any $\sigma \in G = Gal\ (E/K)$ with $L^\sigma \neq L$.

Let $\rho_i : L_i \to F$, $\sigma_i : L_i \to K$, $\sigma : F \to K$ be the trace respectively $(i = 1,...,n)$. Then the embedding: $W(K) \to W(F)$ defines an isomorphism:

$W(K)/ \sum_1^n \sigma_i^* (W(L_i)) \to W(F)/ \sum_1^n \rho_i^*(W(L_i))$, whose inverse is induced by $\sigma^*: W(F)/ \sum_1^n \rho_i^*(W(L_1)) \to W(K)/ \sum_1^n \sigma_i^*(W(L_1))$.

As pointed out by Green in [4], §5.3, p.37 D.L. Johnson's transfer theorem for cohomology of finite groups (see.[14]) follows the same way from Theorem 3.5.

Finally I want to remark, that the result of Leicht-Lorenz, concerning the prime ideals of W(K) (see [15]), can be used to determine the defect base of $\mathbb{Z}[1/2] \otimes_{\mathbb{Z}} W$ (in case this is nonzero, i.e. K formally real). If E is an algebraic closure of K and G = Gal (E/K) the full Galoisgroup of K, then $D_{\mathbb{Z}[1/2] \otimes W}$ contains exactly the subgroups of order 2 in G and of course the trivial subgroup.

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- Appendix B -

A relation between Burnside- and Wittrings.

Ernst Witt zum 60. Geburtstag

1. The arithmetic structure of the Wittring has been studied in a considerable number of recent papers (c. [2], [3], [4], [5], [6], [7]). It turns out to be rather simple:

A) The only torsion is 2-torsion

B) The torsiongroup is either the radical and the nilradical (Wittring of a formally real field) or the whole ring.

C) The factorring with respect to any prime ideal is either $\mathbb{Z}$ or a primefield of finite characteristic.

The proofs have been simplified more and more and the point of this note is not, to offer still another proof for these facts.

Instead I want to study certain analogies between the Wittring and the Burnsidering of a finite group G, as sonsidered in [9], [10], [11], which appeared to me quite surprising, when I first heard about the results of Leicht-Lorenz concerning primeideals in the Wittring and compared this with the results in [11]. In this note I want to give an explanation for these analogies and thereby try to offer a new systematic viewpoint, to look at some of the results and methods concerning Wittrings (see also [14], which contains a further development of this aspect)

More precisely let K be a commutative field with char $K \neq 2$. Following Scharlau (cf. [6]) one has for any finite separable extension L over K a nondegenerate quadratic form $\underline{L} = (L,t)$ over K, where the underlying K-space is just L as K-vectorspace and the quadratic form $t : L \to K$ is given by: $x \mapsto \text{Trace}_{L/K}(x^2)$.

This note elaborates on the following simple fact: let L_1, L_2 be two finite separable extensions over K; then the tensorproduct $\underline{L_1} \otimes \underline{L_2}$ of the two quadratic forms $\underline{L_1}$ and $\underline{L_2}$ can be computed by first splitting up the semisimple K-algebra $L_1 \otimes_K L_2$ into a product ("sum" in the traditional terminology!) of finite separable extensions (the components of $L_1 \otimes_K L_2$): $L_1 \otimes_K L_2 \cong \prod_{i=1}^{n} E_i$, and then taking the orthogonal sum $\perp_{i=1}^{n} \underline{E_i}$ of the corresponding quadratic forms $\underline{E_i}$ $(i = 1, \ldots, n)$.

This fact can be expressed easily in terms of Grothendieckring-constructions: Let $\Omega(K)$ be the Grothendieckring of finite, commutative, semisimple and separable K-algebras (with direct product ("sum") as sum and $\otimes_K$ as product) and W(K) the Wittring of K. Then the construction of Scharlau defines a ringhomomorphism $\Omega(K) \to W(K)$: $L \mapsto \underline{L}$

In § 2 $\Omega(K)$ is determined in terms of the Galoisgroup G of K and especially its primeideals are computed.
In § 3 the "Scharlau"-map: Sc: $\Omega(K) \to W(K) : L \to \underline{L}$ is considered.
It is easily seen to be surjective and thus to define an injective map Spec $(W(K)) \to$ Spec $(\Omega(K))$, which casts new light upon the results of Leicht-Lorenz on Spec $(W(K))$.

§ 4 finally contains an extension of the above method, which is adapted in an even better way to the special structure of Wittrings and especially allows a systematic treatment of some ideas of Scharlau.

2. Let G be a profinite group (definition cf. [8]). A G-set S is a finite set (with discrete topology!), on which G acts continuously by permutations from the left. G-sets form a category $G^{\wedge}$ with finite projective and injective limits, especially with sum (disjoint union) "$S_1 + S_2$" and product (cartesian product with diagonal G-action) "$S_1 \times S_2$". Because $S \times (S_1 + S_2) \cong (S \times S_1) + (S \times S_2)$ the isomorphism-classes of G-sets form a commutative "half-ring" $\Omega^{+}(G)$ with "+" as sum and "x" as product. Let $\Omega(G)$ be the corresponding Grothendieckring, the "Burnsidering" of G.

The additive structure of $\Omega(G)$ is as easily determined as in the case of a finite G. We state without proof:

Proposition 2.1: a) If S_1, S_2, S are G-sets with $S_1 + S \cong S_2 + S$, then $S_1 \cong S_2$, i.e. the canonical map $\Omega^{+}(G) \to \Omega(G)$ is injective.

b) $\Omega(G)$ is a free $\mathbb{Z}$-module with basis the isomorphism-classes of transitive G-sets.

For any closed subgroup $U \leqslant G$ we can define a homomorphism $\varphi_U : \Omega(G) \to \mathbb{Z}$, which maps a G-set S onto $|S^U|$, the number of U-invariant elements in S.

We have:

Theorem 2.1: Any homomorphism $\varphi : \Omega(G) \to R$ into some integral domain R factors through some φ_U.

Corrollary 2.1: For any primeideal $\mathfrak{p} \subseteq \Omega(G)$ there exists some closed subgroup $U \leqslant G$ and some characteristic p (a prime number or 0) with $\mathfrak{p} = \mathfrak{p}(U,p) = \{x \in \Omega(G) \mid \varphi_U(x) \equiv 0(p)\}$.

I postpone the proof to the end of this section and rather state some more results.

Corollary 2.1 leads to the question for necessary and sufficient conditions for the equality $p(U,p) = p(V,p)$ (U and V closed subgroups in G). Let us therefore define:

Definition 2.1: For two closed subgroups $U, V \leq G$ and a characteristic p we write $U \overset{p}{\sim} V$, if and only if $p(U,p) = p(V,p)$.

Then we have:

Theorem 2.2: a) $U \overset{o}{\sim} V$ if and only if U and V are conjugate in G.

b) For $p \neq o$ write $U^{(p)}$ for the intersection of all open normal subgroups N of U(short: $N \overset{\sigma}{\trianglelefteq} U$) with index a power of p. Then $U \overset{p}{\sim} V$ if and only if $U^{(p)}$ is conjugate to $V^{(p)}$ in G.

For the next statement let us consider the set $\mathcal{S}(G)$ of all closed subgroups of G. For any pair of open subgroups (N,U) with $N \overset{\sigma}{\trianglelefteq} G$, $N \leq U$ let us define $O_{N,U} = \{V \in \mathcal{S}(G) \mid V \cdot N = U\}$. Then $\mathcal{S}(G)$ can be considered as a topological space with the $O_{N,U}$'s as a subbasis of open sets. One has

Lemma 2.1: $\mathcal{S}(G)$ is a compact, totally disconnected Hausdorfspace. The set of open subgroups is dense in $\mathcal{S}(G)$. G acts continuously on $\mathcal{S}(G)$ by conjugation. Thus the orbitspace $G\backslash\mathcal{S}(G) = \mathcal{S}_c(G)$ of conjugacy-classes of closed subgroups is as well a compact, totally disconnected Hausdorf-space.

Theorem 2.3: a) There is a natural homeomorphism between Spec $(\mathbb{Q} \otimes \Omega(G))$ (with the Zariski-topology) and $\mathcal{S}_c(G)$. Moreover $\mathbb{Q} \otimes \Omega(G)$ is isomorphic to the ring of continuous(i.e. locally constant) functions from $\mathcal{S}_c(G)$ into $\mathbb{Q}$.

b) Spec $\Omega(G)$ is naturally homeomorphic to the quotient space of $\mathcal{S}(G) \times \mathrm{Spec}\ \mathbb{Z}$ with respect to the following equivalence-relation: $(U,p) \sim (V,q) \iff p = q$ and $U \overset{p}{\sim} V$ (p, q a primenumber or 0; $U, V \in \mathcal{S}(G)$).

All these results are simple generalizations of the corresponding facts for finite groups, as will be shown at the end of this section. Right now I want to give a field-theoretic interpretation in case G is the Galoisgroup of some (finite or infinite) algebraic, separable, Galois-fieldextension E/K. In this case the category $G^\wedge$ of G-sets is wellknown to be anti-equivalent to the category $\underset{\sim}{A}(K,E)$ of (finite, commutative, semisimple, separable) K-Algebras A with $A \otimes_K E$ isomorphic to a direct product of a finite number of copies of E. This antiequivalence is given by $S \mapsto \mathrm{Hom}_G(S,E)$, the set of G-equivariant maps from the G-set S into E with an obvious K-algebra-structure, -respectively: $A \mapsto \mathrm{Hom}_K(A,E)$, the set of K-algebra-homomorphisms from A into E with the G-action induced from the action of G on E.

Moreover transitive G-sets and simple algebras (i.e. fields) correspond to each other with respect to this anti-equivalence.
- All these facts are just simple Galoistheory.

Thus $\Omega(G)$ can as well be interpreted as the Grothendieckring $\Omega(K,E)$ of $\underset{\sim}{A}(K,E)$ with the sum given by the direct product and the product given by $\otimes_K$. For any extension L with $K \subseteq L \subseteq E$ there exists a ring-homomorphism $\varphi_L : \Omega(K,E) \to \mathbb{Z}$, which maps A onto the number of components, isomorphic to L, in $L \otimes_K A$. Theorem 2.1 reads now: "Any homomorphism $\Omega(K,E)$ into an integral domain factors through some φ_L" and Theorem 2.2:

" a) $\varphi_L = \varphi_M$ if and only if L and M are isomorphic K-extensions

b) $\varphi_L \equiv \varphi_M$ (p) if and only if the maximal normal p-extensions of L and M in E are isomorphic K-extensions". If E is a separable closure of K, we write simply $\Omega(K)$ instead of $\Omega(K,E)$. By Proposition 2.1 and the above remarks the isomorphismclasses of finite separable field extensions of K represent a free $\mathbb{Z}$-basis of $\Omega(K)$.

Now to the proofs: For finite G everything is wellknown (cf. [9] and [11]). For the extension to profinite G one just has to check, that everything behaves well with respect to projective (resp. injective) limits. Introducing notations does not take much less time then recalling definitions as well: So let us recall:

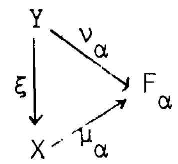
<u>Limits</u>: A filtered indexset I is a partially ordered set with the property: for any $\alpha, \beta \in I$ there exist $\gamma \in I$ with $\gamma \leqslant \alpha, \gamma \leqslant \beta$. For any category $\mathcal{K}$ a projectively (resp. injectively) filtered system of $\mathcal{K}$-objects (e.g. sets, groups, rings) is a covariant (resp. contravariant) functor $F : I \to \mathcal{K}$: from a filtered indexset I into $\mathcal{K}$, where I is made into a category in the obvious way: the objects are the elements in I and one has exactly one map from α to β, if $\alpha \leqslant \beta$, otherwise none (composition of maps is uniquely determined in this case and obvious).

A projective limit of a projectively filtered system F of $\mathcal{K}$-objects is an object $X \in \mathcal{K}$ together with maps $\mu_\alpha : X \to F_\alpha (\alpha \in I)$, such that the triangel

$$X \xrightarrow{\mu_\alpha} F_\alpha, \quad X \xrightarrow{\mu_\beta} F_\beta, \quad F_\alpha \xrightarrow{\varphi_{\alpha,\beta}} F_\beta$$

commutes for all $\alpha, \beta \in I$ with $\alpha < \beta$ ($\varphi_{\alpha,\beta}$ the map from F_α to F_β defined by the functor F) and such that for any other system $(Y, \nu_\alpha)_{\alpha \in I}$ with the same properties there exists exactly

one map $\xi : Y \to X$ with



commutative for any $\alpha \in I$.

The system $(X, \mu_\alpha) = \varprojlim F$ is then welldefined up to isomorphism by F. One has analogous definitions for the injective limit $\varinjlim F$ of an injectively filtered system F.

Of course all this could have been defined for arbitrary categories instead of **I**. But we need <u>filtered</u> indexsets anyway for most of the following results. E.g. we need the following fundamental lemma, which is an easy consequence of the fact, that arbitrary products of compact spaces are compact:

<u>Lemma 2.2:</u> Let F be a projectively filtered system of compact spaces. Then $\varprojlim F$ exists in any full category of topological spaces, containing the compact spaces, is compact itsself and is nonempty if and only if $F_\alpha \neq \emptyset$ for all $\alpha \in I$.

We have further:

<u>Lemma 2.3:</u> a) In the category of profinite groups (or any larger category of topological groups) there exists the projective limit for any filtered system of profinite groups and is itsself a profinite group.

b) Any profinite group can be represented as the projective limit of a projectively filtered system F of finite groups. This can be done even in a canonical way: take $I = \{N \mid N \overset{\sigma}{\triangleleft} G\}$, $F_N = G/N$, $\varphi_{N,M} : G/N \to G/M$ the canonical map for $N \subseteq M$.

Closed subgroups: Now let F be a projectively filtered system of profinite groups with indexset I, with $G = \varprojlim F$ and maps $\mu_\alpha : G \to F_\alpha$. For any closed subgroup U of G consider the system $F|_U : \alpha \mapsto \mu_\alpha(U) = U_\alpha$, $\varphi_{\alpha\beta}|_{U_\alpha} : U_\alpha \to U_\beta$ ($\alpha,\beta \in I$, $\alpha < \beta$). The maps $\mu_\alpha|_U : U \to U_\alpha$ define a map $U \to \varprojlim F|_U$, which is easily seen to be an isomorphism (using again the filtration of I). On the other hand any subsystem F' of F with $F'_\alpha \leqslant F_\alpha$ ($\alpha \in I$) and $\varphi_{\alpha,\beta}(F'_\alpha) = F'_\beta$ ($\alpha,\beta \in I$, $\alpha < \beta$) defines a subgroup $U = \varprojlim F'$ of G. Thus we have:

(1) $\mathcal{S}(G) = \varprojlim \mathcal{S}(F)$, where $\mathcal{S}(F)$ is considered as a projectively filtered system of sets $\mathcal{S}(F_\alpha)$ ($\alpha \in I$) with maps

$$\varphi_{\alpha,\beta} : \mathcal{S}(F_\alpha) \to \mathcal{S}(F_\beta) \ : \ U_\alpha \leqslant F_\alpha \mapsto \varphi_{\alpha,\beta}(U_\alpha) \leqslant F_\beta.$$

These maps are easily seen to be continuous in the topology of $\mathcal{S}(F_\alpha)$, resp. $\mathcal{S}(F_\beta)$ considered in Lemma 1, thus the above equality defines a continuous 1-1-map from $\mathcal{S}(G)$ onto $\varprojlim \mathcal{S}(F)$. This can be used at first to topologize $\mathcal{S}(G)$ by representing G in the canonical way as a projective limit of finite groups. This gives exactly the topology, considered in Lemma 1. But $\mathcal{S}(F_\alpha)$ is for finite F_α a finite (dicrete, compact) Hausdorfspace. Thus $\mathcal{S}(G)$ is - as a limit of discret compact Hausdorf-spaces - a compact, totally disconnected Hausdorfspace. But now (1) even gives a homeomorphism of topological spaces, because a continous 1-1 map from one compact space to another one necessarily is a homeomorphism. The rest of Lemma 2.1 ist trivial. But we need

(2) $\mathcal{S}_c(G) = \varprojlim \mathcal{S}_c(F)$

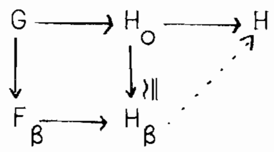
as topological spaces, whenever $G = \varprojlim F$. Of course there is a natural continuousmap from $\mathcal{S}_c(G)$ into $\varprojlim \mathcal{S}_c(F)$, induced from the maps $\mu_\alpha : G \to F_\alpha$. So we have to prove, that this map is 1-1 and onto. So let $U,V \leqslant G$ be closed subgroups, such that $U_\alpha = \mu_\alpha(U)$ is conjugate to $V_\alpha = \mu_\alpha(V)$ in F_α for all $\alpha \in I$. Consider the projectively filtered system $(U,V)_\alpha = \{x \in F_\alpha | x U_\alpha x^{-1} = V_\alpha\}$ of closed subsets of F_α. We have $(U,V)_\alpha \neq \emptyset$ for any $\alpha \in I$ by our assumption. Thus by Lemma 2.2 there exists $g \in \varprojlim (U,V)_\alpha \subseteq G$ and obviously $g U g^{-1} = V$. This proves, that the natural map $S_c(G) \to \varprojlim S_c(F)$ is 1-1.

Now consider an element $u \in \varprojlim S_c(F)$ with images u_α in $S_c(F_\alpha)$ $(\alpha \in I)$. Consider u_α as closed subset (orbit) in $S(F_\alpha)$. Thus $\alpha \mapsto u_\alpha \subseteq S(F_\alpha)$ defines a projectively filtered subsystem of nonmepty compact subsets of $S(F_\alpha)$. By Lemma 2.2 there exists $U \in \varprojlim u_\alpha \subseteq S(G)$ and obviously U represents an element in $S_c(G)$, which maps onto $u \in \varprojlim S_c(F_\alpha)$ (cf. [8], p. 1-4, proof of Proposition 3).

<u>Homomorphisms into finite groups:</u> Again let G be the projective limit of a projectively filtered system F of profinite groups with indexset I and let H be a finite group. The maps $\mu_\alpha: G \to F_\alpha (\alpha \in I)$ induce maps $\mathrm{Hom}(F_\alpha,H) \to \mathrm{Hom}(G,H)$ between the sets of continuous homomorphisms into H, which are compatibel with the maps: $\mathrm{Hom}(F_\beta,H) \to \mathrm{Hom}(F_\alpha,H)$ $(\alpha,\beta \in I, \alpha < \beta)$, induced by the $\varphi_{\alpha\beta}: F_\alpha \to F_\beta$, and thus define a map $\varinjlim \mathrm{Hom}(F,H) \to \mathrm{Hom}(G,H)$.

<u>Lemma 2.4</u>: The natural map $\varinjlim \mathrm{Hom}(F,H) \to \mathrm{Hom}(G,H)$ is an isomorphism.

Proof: a) Surjectivity: Let $\varphi : G \to H$ be a continuous homomorphism. Then $N = \mathrm{Ke}\, \varphi \overset{g}{\trianglelefteq} G$. Because I is filtered, the set $\{\mu_\alpha^{-1}(N_\alpha) \mid \alpha \in I, N_\alpha \overset{g}{\trianglelefteq} F_\alpha\}$ is a basis of neighbourhoods of $1 \in G$, thus there exists $\alpha \in I$ and $N_\alpha \overset{g}{\trianglelefteq} F_\alpha$ with $\mu_\alpha^{-1}(N_\alpha) \subseteq N$. Consider $H_\alpha = F_\alpha/N_\alpha$. For any $\beta \in I$ with $\beta \leqslant \alpha$ there exists a natural map $F_\beta \to H_\alpha$ (composition of $\varphi_{\beta\alpha}$ and the natural map $F_\alpha \to F_\alpha/N_\alpha$) with image H_β and there exists as well a natural map $G \to H_\alpha$ with image, say H_o. Using Lemma 2.2 one gets $H_o = \bigcap_{\beta \leqslant \alpha} H_\beta$. But because H_α is finite and I is filtered, this implies the existence of $\beta \in I$, $\beta \leqslant \alpha$ with $H_o = H_\beta$. Because $\mu_\alpha^{-1}(N_\alpha) = Ke(G \to H_\alpha) \subseteq N$ the map $\varphi : G \to H$ factors into the map $G \to H_o$ and some map $H_o \longrightarrow H$. Thus we have a commutative diagramm:



and get $\varphi \in Im(Hom(F_\beta, H) \to Hom(G,H))$.

b) Injectivity: We have to show, that for two indices $\alpha, \beta \in I$ and maps $\sigma_\alpha : F_\alpha \to H$, $\tau_\beta : F_\beta \to H$ with $\sigma_\alpha \mu_\alpha = \tau_\beta \mu_\beta : G \to H$ there exists $\gamma \in I$, $\gamma \leqslant \alpha, \beta$ with $\sigma_\alpha\, \varphi_{\gamma,\alpha} = \tau_\beta\, \varphi_{\gamma,\beta} : F_\gamma \to H$. Because I is filtered we may assume $\alpha = \beta$. If $\mu_\alpha : G \to F_\alpha$ is surjective (as in the case of the canonical representation of G as limit of finite groups), we get $\sigma_\alpha = \tau_\alpha$ immediately. In general consider the system $Y_\beta = \{x \in F_\beta \mid \varphi_{\beta,\alpha}\sigma_\alpha(x) \neq \varphi_{\beta,\alpha}\tau(x)\} (\beta \in I, \beta < \alpha)$. We have to show $Y_\beta = \emptyset$ for some β. But because H is finite, the complement of Y_β in F_β is an open subgroup of F_β, thus Y_β is compact. If $Y_\beta \neq \emptyset$ for all $\beta < \alpha$, Lemma 2.1 would imply : $\emptyset \neq \varprojlim Y \subseteq G = \varprojlim F$, the limit taken for the cofinal indexset $\{\beta \in I \mid \beta \leqslant \alpha\}$, and for any $g \in \varprojlim Y$ one would have $\sigma_\alpha\, \mu_\alpha(g) \neq \tau_\alpha\, \mu_\alpha(g)$, contradiction !

Now we come back to the rings $\Omega(G)$. If $\varphi : G \to H$ is a continuous homomorphism of profinite groups, restriction of the action of H on an H-set S to G via φ defines a functor $\varphi^{\wedge} : H^{\wedge} \to G^{\wedge}$ which commutes with sums and products and thus defines a ringhomomorphism $\Omega(H) \longrightarrow \Omega(G)$.

Now assume $G = \varprojlim F$ as above. The maps $\mu_\alpha : G \to F_\alpha$ define maps: $\Omega(F_\alpha) \to \Omega(G)$, which are compatible with the maps $\Omega(F_\beta) \to \Omega(F_\alpha)$ induced by the maps $\varphi_{\alpha,\beta} : F_\alpha \to F_\beta$, and thus define a map: $\varinjlim \Omega(F) \to \Omega(G)$.
We have a corollary to Lemma 2.4:

Lemma 2.5: The natural map $\varinjlim \Omega(F) \to \Omega(G)$ is an isomorphism.

Proof: a) Surjectivity: A G-set S can be described as a finite set S together with a homomorphism $\varphi : G \to \Pi_S$, the full group of permutations of S. By Lemma 2.4 there exists $\alpha \in I$ and $\varphi_\alpha : F_\alpha \to \Pi_S$ with $\varphi = \varphi_\alpha \mu_\alpha$, thus there is a F_α-set S_α with $S_\alpha|_G = S$, q.e.d.

b) Injectivity: Because of Proposition 2.1 it is enough to show, that for any two indices α and β, an F_α-set S_α and an F_β-set T_β with $S_\alpha|_G \cong T_\beta|_G$ there exists $\gamma \in I$ with $\gamma \leq \alpha, \beta$ and $S_\alpha|_{F_\gamma} \cong T_\beta \mid F_\gamma$. Using the isomorphism $S_\alpha \mid_G \cong T_\beta \mid_G$ we may identify the sets S_α and T_β in such a way, that $S_\alpha \mid_G = T_\beta \mid_G$. Thus we get maps $\sigma_\alpha : F_\alpha \to \Pi_{S_\alpha} = \Pi_{T_\beta}$ and $\tau_\beta : F_\beta \to \Pi_{S_\alpha} = \Pi_{S_\beta}$ with $\sigma_\alpha \mu_\alpha = \tau_\beta \mu_\beta$.

The existence of γ now follows from Lemma 2.4.

Remark: Using the fact, that any continuous homomorphism of a profinite group into a topological group without small subgroups factors through a finite group, and that $GL(n,\mathbb{R})$ or $GL(n,\mathbb{C})$ has no small subgroups, one gets the same way for the rings $X(G,\mathbb{R})$ or $X(G,\mathbb{C})$ of reell or complex characters: $G = \varprojlim F \Rightarrow X(G,\mathbb{R}) = \varinjlim X(F,\mathbb{R})$, $X(G,\mathbb{C}) = \varinjlim X(F,\mathbb{C})$.

After all these (essentially wellknown and mostly trivial) preparations I come now to the proof of Theorem 2.1:

Represent G as limit of finite groups: $G = \varprojlim F$. Then we have $\Omega(G) = \varinjlim \Omega(F)$ and thus $\mathrm{Hom}(\Omega(G),R) = \varprojlim \mathrm{Hom}(\Omega(F),R)$ for any ring R. Now assume R to be an integral domain and take some $\varphi : \Omega(G) \to R$. Because $\mathrm{Hom}(\Omega(G),R) = \varprojlim \mathrm{Hom}(\Omega(F),R)$ the map φ is determined by its components $\varphi_\alpha : \Omega(F_\alpha) \to R$. Because Theorem 1 is true for finite groups, the set $X_\alpha = \{U \leqslant F_\alpha \mid \varphi_\alpha$ factors through $\varphi_U\}$ is nonempty. Moreover the $X_\alpha (\alpha \in I)$ form in a natural way a projectively filtered system of subsets of $S(F_\alpha)$ and are finite, thus compact. Lemma 2.1 implies $\emptyset \neq \varprojlim X_\alpha \subseteq \varprojlim S(F) = S(G)$. Take a closed subgroup $U \in \varprojlim X_\alpha$. Because any φ_α factors through $\Omega(F_\alpha) \to \Omega(G) \xrightarrow{\varphi_U} \mathbb{Z}$, so does φ, q.e.d.

Proof of Theorem 2.2: Again we represent G as limit of finite groups: $G = \varprojlim F$.

a) Obviously conjugate subgroups define the same ringhomomorphism: $\Omega(G) \to \mathbb{Z}$. On the other hand assume $\varphi(U,o) = \varphi(V,o)$ for two closed subgroups $U,V \leqslant G$. Write U_α for $\mu_\alpha(U) \leqslant F_\alpha$, V_α for $\mu_\alpha(V) \leqslant F_\alpha$. The assumption implies $\varphi(U_\alpha,0) = \varphi(V_\alpha,0)$ and thus "U_α conjugate to V_α in F_α", because everything is true for F_α. But by formula (2) above this implies: U conjugate to V in G; q.e.d.

b) At first I want to show: For U a closed subgroup of G and $U_\alpha = \mu_\alpha(U) \leqslant F_\alpha$ one has $U_\alpha^{(p)} = \mu_\alpha(U^{(p)})$ and thus $U^{(p)} = \varprojlim U_\alpha^{(p)}$.

On the one hand $U_\alpha^{(p)}$ is a normal subgroup of U_α with p-power-index. Thus $\mu_\alpha^{-1}(U_\alpha^{(p)}) \cap U$ is a normal subgroup of U with p-power index and contains therefore $U^{(p)}$, which implies $\mu_\alpha(U^{(p)}) \subseteq U_\alpha^{(p)}$. On the other hand $\mu_\alpha(U^{(p)}) = \mu_\alpha(U^{(p)} \cdot \mathrm{Ke}(\mu_\alpha|_U))$ and $U^{(p)} \cdot \mathrm{Ke}\, \mu_\alpha|_U$ is an open subgroup of U containing $U^{(p)}$ and thus one of p-power-index. Thus $\mu_\alpha(U^{(p)})$ is normal and has p-power-index in U_α and therefore contains $U_\alpha^{(p)}$.

Now assume $p(U,p) = p(V,p)$ for two closed subgroups U and V in G and some p. With $U_\alpha = \mu_\alpha(U)$ and $V_\alpha = \mu_\alpha(V)$ this implies $p(U_\alpha,p) = p(V_\alpha,p)$ in $\Omega(F_\alpha)$ and thus $U_\alpha{}^{(p)}$ conjugate to $V_\alpha{}^{(p)}$ in F_α (see [10] § 5, Thm 5.1 (b) for finite F_α). Again this implies the conjugacy of $U^{(p)} = \varprojlim U_\alpha{}^{(p)}$ and $V^{(p)} = \varprojlim V_\alpha{}^{(p)}$ in G.

On the other hand it is easy to see, that the conjugacy of $U^{(p)}$ and $V^{(p)}$ in G implies $p(U,p) = p(V,p)$, because for any G-set S there exists an $N \trianglelefteq_o G$, which acts trivial on S, and thus we have

$$\varphi_U(S) = \varphi_{U\cdot N}(S) \equiv \varphi_{U^{(p)}\cdot N}(S) = \varphi_{V^{(p)}\cdot N}(S) \equiv \varphi_{V\cdot N}(S) = \varphi_V(S) \mod p.$$

Proof of Theorem 2.3: a) With $G = \varprojlim F$ and finite $F_\alpha (\alpha \in I)$ we have $\operatorname{Spec} \mathbb{Q} \otimes \Omega(G) = \operatorname{Spec} \varinjlim \mathbb{Q} \otimes \Omega(F) = \varprojlim (\operatorname{Spec} \mathbb{Q} \otimes \Omega(F)) = \varprojlim S_c(F) = S_c(G)$ with natural homeomorphisms and we have as well:

$$\mathbb{Q} \otimes \Omega(G) = \varinjlim \mathbb{Q} \otimes \Omega(F) = \varinjlim [\operatorname{Spec} \mathbb{Q} \otimes \Omega(F), \mathbb{Q}]$$
$$= \varinjlim [S_c(F), \mathbb{Q}] = [\varprojlim S_c(F), \mathbb{Q}] = [S_c(G), \mathbb{Q}],$$

where the second-last equality can be proved analogously to Lemma 2.4.

b) Of course $(U,p) \mapsto p(U,p)$ defines a continuous map $S(G) \times \operatorname{Spec} \mathbb{Z} \to \operatorname{Spec} \Omega(G)$, which is surjective by Corollary 2.1 and factors into a surjective and injective map through the quotientspace described in Theorem 2.3, b) by Theorem 2.2, Thus we get a continuous 1-1map from this quotientspace onto the space $\operatorname{Spec} \Omega(G)$. To show, that this map is a homeomorphism, one has to prove, that $t : S(G) \times \operatorname{Spec} \mathbb{Z} \to \operatorname{Spec} \Omega(G) : (U,p) \mapsto p(U,p)$ is identifying, i.e. that for $p_o \in \mathcal{O} \subseteq \operatorname{Spec} \Omega(G)$ and $t^{-1}(\mathcal{O}) = \{(U,p) \in S(G) \times \operatorname{Spec} \mathbb{Z} \mid p(U,p) \in \mathcal{O}\}$ open one can find an element $x \in \Omega(G)$ with $x \notin p_o$ and $\{q \in \operatorname{Spec} \Omega(G) \mid x \notin q\} \subseteq \mathcal{O}$. This is a little bit tiring and may wery well be skipped. At first one may observe, that for any $(U,p) \notin t^{-1}(\mathcal{O})$ one has

$p(U,p) \nsubseteq p_o$ (otherwise - since definitely $p(U,p) \neq p_o - \mathbf{p} = 0$ and $p(U,p_o) = p_o$ with $p_o = \text{char}\ \Omega(G)/p_o$, thus $(U,p_o) \in t^{-1}(\mathbf{0}) \Rightarrow (U,o) \in t^{-1}(\mathbf{0})$, a contradiction). Thus one can find an element $x_{(U,p)} \in \Omega(G)$ with $x_{(U,p)} \in p(U,p)$, $x_{(U,p)} \notin p_o$. Moreover one can find an open normal subgroup $N \overset{g}{<} G$ with $x_{(U,p)} \in \Omega(G/N) \subseteq \Omega(G)$. Thus for any fixed p the set $\{V \in S(G) \mid \varphi_V(x_{(U,p)}) \equiv 0\ (p)\}$ is open in $S(G)$. Since moreover $\{U \in S(G) \mid p(U,p) \notin \mathbf{0}\}$ is compact, we can find finitely many $U_1,\dots,U_n$ with $\{U \in S(G) \mid p(U,p) \notin \mathbf{0}\} \subseteq \bigcup_{i=1}^{n} \{V \in S(G) \mid \varphi_V(x_{(U_i,p)}) \equiv 0(p)\}$.

Thus with $x_p = \prod_{i=1}^{n} x_{(U_i,p)}$ we have at least an element in $\Omega(G)$ with $x_p \notin p_o$, but $x_p \in p(U,p)$ for all $(U,p) \notin t^{-1}(\mathbf{0})$.

Now assume $p_o = p(V,q)$ for some $(V,q) \in S(G) \times \text{Spec}\ \mathbb{Z}$ and take an open and closed subset $X_o \subseteq S(G)$ with $V \in X_o \subseteq \{U \in S(G) \mid (U,o) \in t^{-1}(\mathbf{0})\}$.

Since X_o is open, one may even assume $x_o \in p(U,o)$ for all $U \in S(G)$, $U \notin X_o$

Since X_o is closed, thus compact and totally disconnected, we may identify $X_o \times \text{Spez}\ \mathbb{Z}$ with $\text{Spec}\ A_{X_o}$, A_{X_o} the ring of all continuous maps $X_o \longrightarrow \mathbb{Z}$. Thus the open set $(X_o \times \text{Spec}\ \mathbb{Z}) \cap t^{-1}(\mathbf{0})$ is given by an ideal $\mathfrak{a} \subseteq A_{X_o}$ in the form $\mathfrak{a} \subseteq p \Longleftrightarrow p \notin (X_o \times \text{Spec}\ \mathbb{Z}) \cap t^{-1}(\mathbf{0})$. Since $X_o \times o \subseteq t^{-1}(\mathbf{0})$ we have finite residue-class-characteristic for any $p \supseteq \mathfrak{a}$. But this implies $n \in \mathfrak{a}$ for some $n \in \mathbb{N}$, i.e. there are only a finite number of primes, say $p_1,\dots,p_r$ with $(U,p_i) \notin t^{-1}(\mathbf{0})$ for some $U \in X_o$. Now with $x = x_o \cdot \prod_{i=1}^{r} x_{p_i}$ one may verify the above statements. (There should be a more simple proof, but unfortunately I couldn't find any better proof during a 6 hours faculty meeting).

To close this section, let us state some more corollaries:

At first one has as in [11] for the case of finite groups:

Corollary 2.2: Spec $\Omega(G)$ is connected, if and only if G is prosolvable.

Proof: For any $N \overset{g}{\trianglelefteq} G$ the epimorphism $G \to G/N$ defines a monomorphism $\Omega(G/N) \to \Omega(G)$, which we use to identify $\Omega(G/N)$ with a subring of $\Omega(G)$. Because $G = \varprojlim G/N$ (N open normal in G) we have: $\Omega(G) = \bigcup_{N \overset{g}{\trianglelefteq} G} \Omega(G/N)$. Thus: Spec $\Omega(G)$ connected $\Longleftrightarrow$ the only idempotents in $\Omega(G)$ are o and 1 $\Longleftrightarrow$ the only idempotents in $\Omega(G/N)$ are 0 and 1 for all $N \overset{g}{\trianglelefteq} G \Longleftrightarrow G/N$ is solvable for all $N \overset{g}{\trianglelefteq} G \Longleftrightarrow G$ is prosolvable.

Corrollary 2.3: For a prime p we have: a) G is p-free (i.e. $(p, (G:N)) = 1$ for all $N \overset{g}{\trianglelefteq} G \Longleftrightarrow$ Spec $\Omega_p(G) = S_c(G) \times$ Spec $\mathbb{Z}_p \Longleftrightarrow \Omega_p(G)$ is the ring of continuous (i.e. locally constant) maps from $S_c(G)$ into $\mathbb{Z}_p$. (With $\mathbb{Z}_p = \{\frac{n}{m} \in \mathbb{Q} \mid (m,p) = 1\}$, $\Omega_p(G) = \mathbb{Z}_p \otimes \Omega(G)$).

b) G is a pro-p-group $\Longleftrightarrow \Omega(G)$ contains exactly one primeideal with residue class characteristic p.

Proof: Again take the canonical representation of G as limit of its finite quotients and use the validity of Corollary 2.3 for finite groups.

Corollary 2.4: Let G be a pro-p-group and R be a factorring of $\Omega(G)$. Then the following statements are equivalent:

(i) Any torsion in R is p-torsion

(ii) The residueclassfield-characteristic for a minimal prime ideal is either p or 0.

Under these conditions we have further the following alternatives for R: If there exists a minimal prime ideal $\mathfrak{p}$ with residueclass-characteristic p, then $\mathfrak{p}$ is the only primeideal in R (and of course R itsself is a p-torsion group). If there exists a minimal primeideal $\mathfrak{p}$ with residueclass characteristic o, then all minimal primeideals have residueclass characteristic 0 and their intersection (the nilradical) is as well the radical and the torsionsubgroup of R.

Proof: Again we may assume G to be a finite p-group. Let q be a prime $\neq$ p and $\mathbb{Z}_q = \{\frac{n}{m} \in \mathbb{Q} \mid (m,q) = 1\}$. Then we have: There is no q-torsion in R $\longleftrightarrow$ there is no q-torsion in $R_q = \mathbb{Z}_q \otimes R \longleftrightarrow R_q$ is (as a factorring of $\mathbb{Z}_q \otimes \Omega(G) \cong \prod_{S_c(G)} \mathbb{Z}_q$ isomorphic to a product of a finite number of copies of $\mathbb{Z}_q \longleftrightarrow$ No primeideal in R (or R_q) with residueclass characteristic q is minimal.

Moreover $\Omega(G)$ has exactly one primeideal $\mathfrak{p}$ with char $\Omega(G)/\mathfrak{p} = p$, which contains all primeideals $\mathfrak{p}(U,o)$ $(U \leqslant G)$ with residueclass characteristic 0. Thus R has at most one primeideal $\mathfrak{p}$ with char $R/\mathfrak{p} = p$ and if this exists and is minimal, R cannot contain any primeideal with residueclass-characteristic 0, and because of (ii) no other primeideal at all. The rest follows easily from the next Corollary, which of course is true for factorrings of any inductive limit of rings Ω, which are finite over $\mathbb{Z}$ and with $\mathbb{Q} \otimes \Omega$ semisimple:

<u>Corollary 2.5</u>: For any factorring R of $\Omega(G)$ (G arbitrary profinite) we have:

Radical R = Nilradical R $\subseteq$ Tor R with equality if and only if any minimal primeideal in R has residue-class characteristic 0.

I omit the purely ringtheoretic and easy proof.

Finally let U be an open subgroup of G. Then one has a canonical additive map $i^*_{U,G} = i^* : \Omega(U) \to \Omega(G)$, defined on the $\mathbb{Z}$-basis of transitive U-sets by $i^*(U/V) = G/V$. If G is the Galoisgroup of an algebraic Galoisextension E/K and $L = E^U$ the fixfield of U, then $i^*: \Omega(U) = \Omega(L,E) \to \Omega(G) = \Omega(K,E)$ can be defined by considering L-algebras as K-algebras.

$i^*(\Omega(U))$ is easily seen to be an ideal in $\Omega(G)$. We want to give necessary and sufficient conditions for a primeideal $p(W,p) \subseteq \Omega(G)$ to contain $i^*(\Omega(U))$. For this purpose consider once more $W^{(p)} = \bigcap N$, where N runs through all $N \overset{g}{\trianglelefteq} W$ with W/N a p-group, and $N_G(W^{(p)}) \supseteq W$. In $N_G(W^{(p)})/W^{(p)}$ take a p-Sylowsubgroup (which may contain $W/W^{(p)}$, if one likes) and let $W_{(p)}$ be its preimage in $N_G(W^{(p)})$.

Then we have:

<u>Corollary 2.6</u>: $i^*(\Omega(U)) \subseteq p(W,p) \Longleftrightarrow W_{(p)}$ is not conjugate to a subgroup of U in G ($W_{(p)} \not\lesssim U$). In other words:

(i) $i^*(\Omega(U)) \subseteq p(W,o) \Longleftrightarrow W \not\lesssim U$,

(ii) $i^*(\Omega(U)) + p(W,p) = \Omega(G) \Longleftrightarrow W_{(p)} \lesssim U$ $(p \neq 0)$.

Proof: $\Longleftarrow$: $W_{(p)} \not\lesssim U$ implies $\varphi_{W_{(p)}}(G/U) = 0$ and as well $\varphi_{W_{(p)}}(G/V) = 0$ for $V \leq U$, thus $i^*(\Omega(U)) \subseteq p(W_{(p)},0) \subseteq p(W_{(p)},p) = p(W^{(p)},p) = p(W,p)$.

$\Longrightarrow$: At first represent G as limit of finite groups: $G = \varprojlim F$ with surjective maps $\mu_\alpha : G \to F_\alpha$. With arguments based on Lemma 2.2 (as above) one can show, that in any F_α one can choose $W_{\alpha'(p)}$, related to $W_\alpha = \mu_\alpha(W) \leqslant F$ in the same way as $W_{(p)}$ to $W \leqslant G$ (preimage in $N_{F_\alpha}(W_\alpha^{(p)})$ of a p-Sylowsubgroup in $N_{F_\alpha}(W_\alpha^{(p)})/W_\alpha^{(p)}$),

such that $\varphi_{\alpha,\beta} : F_\alpha \to F_\beta$ maps $W_{\alpha,(p)}$ in $W_{\beta,(p)}, \mu_\alpha(W_{(p)}) \subseteq W_{\alpha,(p)}$

and $W_{(p)} = \varprojlim W_{\alpha,(p)}$, i.e. $W_{(p)} = \bigcap_\alpha \mu_\alpha^{-1}(W_{\alpha,(p)})$. Now assume $W_{(p)} \lesssim U$, thus w.l.o.g. $W_{(p)} \leqslant U$.

Because U is open in G, i.e. $G-U$ is compact, and $G-U \subseteq G-W_{(p)}$ $\subseteq \bigcup_\alpha G - \mu_\alpha^{-1}(W_{\alpha,(p)})$ there exists α with $\mu_\alpha^{-1}(W_{\alpha,(p)}) \subseteq U$.

But (see [10], p.35 - 41)

$\varphi_W(G/\mu_\alpha^{-1}(W_{\alpha,(p)})) = \varphi_{W_\alpha}(F_\alpha/W_{\alpha,(p)}) \not\equiv 0\ (p)$, i.e. $i^*(\Omega(U)) \notin p(W,p)$.

3. Now let K be a field with char $K \neq 2$ and let $W(K)$ be the Wittring of K (for definition see [3]). Any finite separable extension L over K defines an element $\underline{L} \in W(K)$ as described in the introduction. Because the isomorphism classes of finite separable extentions form a free $\mathbb{Z}$-basis of $\Omega(K)$ this gives an additive map $Sc: \Omega(K) \to W(K)$.

<u>Theorem 3.1</u>: a) Sc is a ringhomomorphism

b) Sc is surjective. More precisely: Let $K_2 = K(\sqrt{a} | a \in K)$ be the minimal fieldextension of K, which contains square roots of all elements $a \in K$. Then $\Omega(K,K_2) \subseteq \Omega(K)$ and Sc, restricted to $\Omega(K,K_2)$, is already surjective.

For the proof let us first recall some properties of the trace-map:

<u>Lemma 3.1:</u> Let R be a commutative ring with $1 \in R$ and let P be a finitely generated, projective (f.g.p.) R-module. Then the inverse of the natural isomorphism: $\mathrm{Hom}_R(P,R) \otimes P \to \mathrm{End}_R(P)$:

$s \otimes x \mapsto (y \mapsto s(y)x)$ $(s \in Hom_R(P,R) = P', \ x, \ y \in P)$ composed with the evaluation map: $ev: \ P' \otimes P \to R: \ s \otimes x \mapsto s(x)$ $(s \in P', \ x \in P)$, defines the trace map $t_P^R = t_P : End_R(P) \to R$.

This map has the following properties:

(i) $t_P(\alpha\beta) = t_P(\beta\alpha)$ $(\alpha,\beta \in End_R(P))$

(ii) $t_{P \otimes Q}(\alpha \otimes \beta) = t_P(\alpha) \ t_Q(\beta)$ with P and Q f.g.p. R-modules, $\alpha \in End_R(P)$, $\beta \in End_R(Q)$.

(iii) If $\rho : R \to A$ is a ringhomomorphism into a ring A and P, α as above, then $t^T_{T \otimes P}(Id_T \otimes \alpha) = \rho \ t_P^R(\alpha)$.

<u>Lemma 3.2</u>: Let R be a commutative ring with $1 \in R$ and A a commutative R-algebra, which is f.g.p. as R-module, then the imbedding $A \to End_R(A)$, defined by multiplication $a \mapsto (x \mapsto ax)$ $(a,x \in A)$, composed with $t_A : End_R(A) \to R$ defines the trace $t_{A/R} : A \to R$, which has the following properties (see e.g. [13]):

(i) A is a separabel R-algebra if and only if $t' : A \to A' = Hom_R(A,R)$ defined by $x \mapsto (y \mapsto t_{A/R}(xy))$ $(x,y \in A)$, is an isomorphism, i.e. $(x,y) \mapsto t_{A/R}(xy)$ defines a nondegenerate R-bilinear form on A with discriminant a unit in R.

(ii) Let M be f.g.p. A-module and $\alpha \in End_A(M)$, then M can be considered as f.g.p. R-Module and $\alpha \in End_R(M)$ and we have $t_M^R(\alpha) = t_{A/R}(t_M^A(\alpha))$, especially if B is a commutative A-algebra, f.g.p. as A-Module, then $t_{B/R} = t_{A/R} \cdot t_{B/A} : B \to R$.

(iii) If A and B are two commutative R-algebras, f.g.p. as R-modules, then $t_{A \otimes B/R}(a \otimes b) = t_{A/R}(a) \ t_{B/R}(b)$.

Lemma 3.2, (iii) of course immediately implies Theorem 3.1, a.
To proof the second part, it is enough to show:
$\langle a\rangle \in Sc(\Omega(K,K_2))$ for any $a \in K^{\times}$ with $\langle a\rangle$ the bilinear form $K \times K \to K : (x,y) \mapsto a \cdot x \cdot y$.

But $\langle 1\rangle = \underline{K} = Sc(1) \in Sc(\Omega(K,K_2))$ and for $a \in K^{\times}$ not a square we have $\underline{K(\sqrt{a})} = \langle 2\rangle \perp \langle 2a\rangle$ (orthogonal sum). Now either 2 is a square and thus $\underline{K(\sqrt{a})} = \langle 1\rangle \perp \langle a\rangle$ and $\langle a\rangle = \underline{K(\sqrt{a})} - \underline{K} \in Sc(\Omega(K,K_2))$ or 2 is not a square and then $\underline{K(\sqrt{2})} = \langle 2\rangle \perp \langle 4\rangle = \langle 2\rangle \perp \langle 1\rangle$, $\langle 2\rangle = \underline{K(\sqrt{2})} - \underline{K} \in Sc(\Omega(K,K_2)) \longrightarrow \langle 2\rangle \otimes \underline{K(\sqrt{a})} - \underline{K} = \langle a\rangle \in Sc(\Omega(K,K_2))$.
As a corollary we could get the results on the torsion in $W(K)$, using the fact, that the Galoisgroup $Gal(K_2/K)$ is a pro-2-group (more precisely an "pro-elementary-abelian-2-group"), the results of Leicht-Lorenz on primeideals in $W(K)$ and Corollary 2.4.

On the other hand, the Theorem implies immediately property c) from the introduction for Wittrings and thus explains the analogy between the arithmetic structure of Wittrings and Burnsiderings.

One gets also, that $Spec(\mathbb{Q} \otimes W(K)) \subseteq Spec(\mathbb{Q} \otimes \Omega(K))$ is a totally disconnected compact Hausdorfspace and $\mathbb{Q} \otimes W(K)$ the ring of continuous i.e. locally constant functions of $Spec(\mathbb{Q} \otimes W(K))$ into $\mathbb{Q}$, if K is formally real, i.e. $\mathbb{Q} \otimes W(K) \neq 0$, (see [2]), using the analogous result for $\Omega(K)$ (Theorem 2.3). As another application we get a canonical, filtered system of subrings $\{W(K,E) = Sc(\Omega(K,E)) \mid E$ a finite Galoisextension of $K\}$ of $W(K)$, which are finite over $\mathbb{Z}$ and cover all of $W(K)$. It may be interesting to look for a description of this filtration more in terms of quadratic forms, and also to characterize those E, for which $W(K,E)$ is torsionfree.

Finally, considering Theorem 2.1 and 2.2, one may ask for a representation of the preimage $Sc^{-1}(p) \subseteq \Omega(K)$ of any primeideal $\mathbf{p} \subseteq W(K)$ in the form $p(U,p)$. Using the explicit description of primeideals in $W(K)$, given in [3], one gets:

If m is the unique primeideal (The "Pfisterideal") in $W(K)$ with $W(K)/m \cong \mathbb{F}_2$, then $Sc^{-1}(m) = p(U,2)$ with U an arbitrary pro-2-subgroup of the full Galoisgroup G of K.

If $p \neq m$ and p is defined as the kernel of the Sylvester-inertia-index-map modulo $p(p = \text{char } W(K)/p \neq 2)$ with respect to some ordering "$\leqslant$" of K, then $Sc^{-1}(p) = p(U,p)$ with U the Galoisgroup of a real closure of K with respect to "$\leqslant$".

All these U are conjugate and of order 2. On the other hand, any subgroup $U \leqslant G$ of G of order 2 has a really closed field as fixed field and thus defines an ordering of K. Thus Spec $\mathbb{Q} \otimes W(K)$ is naturally isomorphic to the subspace of conjugate classes of subgroups of order 2 in $S_c(G)$. [1])

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[1]) Conjugate classes of subgroups of fixed finite order do not necessarily form a closed subset of $S_c(G)$, G an arbitrary profinite group. This is a special property of full Galois-groups.

Moreover if E is a finite Galoisextension of K with Galoisgroup $H = \mathrm{Gal}(E/K)$ and α an ordering of K, the map $\Omega(H) \cong \Omega(K,E) \to W(K,E) \xrightarrow{I_\alpha} \mathbb{Z}$ with the last arrow defined by the inertia index I_α with respect to α equals the map $\varphi_U : \Omega(H) \to \mathbb{Z}$, where $U \leq H$ is the fixgroup of a maximal subfield F of E, to which α extends. This gives a possibility to compute $I_\alpha(\underline{L})$ for any finite separable extension L of K: imbed L into a finite Galoisextension E and choose a maximal subfield F in E, to which α extends. Let $H = \mathrm{Gal}(E/K)$, $U = \mathrm{Gal}(E/F)$, $V = \mathrm{Gal}(E/L)$.
Then $I_\alpha(\underline{L}) = \varphi_U(H/V) = |\{ hV \in H/V \mid UhV = hV \}|$.

Especially if L is itsself a Galoisextension, the inertiaindex of $\underline{L}$ with respect to α is $\mathrm{Dim}_K L$ or 0, depending on whether α extends to L or not.

Of course, these last statements can also be verified directly in a rather simple way.

One can also describe $\mathbb{Q} \otimes W(K,E)$ as the subspace of those functions on $\mathrm{Spec}(\mathbb{Q} \otimes W(K))$, the set of orderings of K, which are constant on E-equivalent orderings, where two orderings α and β of K are called E-equivalent if the maximal subfields L_α and L_β of E, to which α, resp. β extends, are conjugate. (By the way, the topology on $\mathrm{Spec}(\mathbb{Q} \otimes W(K))$ can be described as the coarsest topology, for which all E-equivalence-classes are open, where E runs through all finite Galoisextensions of K).
Finally I want to apply Cor.2.6. Let L/K be a finite separable extension. As noticed by Scharlau [6], any nonzero K-linear map $s: L \to K$ defines an additive map: $s^*: W(L) \to W(K)$.

It is easy to see, that if s is choosen to be the trace $t : L \longrightarrow K$, one gets a commutative diagramm:

$$\begin{array}{ccc} \Omega(L) & \xrightarrow{i^*} & \Omega(K) \\ \downarrow{\scriptstyle Sc} & & \downarrow{\scriptstyle Sc} \\ W(L) & \xrightarrow{t^*} & W(K) \end{array}$$

where i^* is the map defined in § 2 by considering L-algebras as K-algebras. The image $t^*(W(L)) =_{Df} W(L)^K$ is an ideal in $W(K)$ as well as $i^*(\Omega(L)) =_{Df} \Omega(L)^K$ in $\Omega(K)$.

We want to use Cor. 2.6. to prove:

Theorem 3.2: If $L_1,\dots,L_n$ is a family of finite separable extensions of K, then the following three statements are equivalent:

(i) $2^k \cdot 1_{W(K)} \in \sum_1^n W(L_i)^K$ for some $k \in \mathbb{N}$;

(ii) The only primeideal in $W(K)$, which may contain $\Sigma\, W(L_i)^K$ is the "Pfisterideal" m;

(iii) Any ordering of K can be extended to at least one of the L_i.

Corollary 3.1: (Scharlau [6], [7]):

(a) If $a, b \in K^\times$ and $L_1 = K(\sqrt{a})$, $L_2 = K(\sqrt{b})$, $L_3 = K(\sqrt{ab})$, then $2^k \cdot 1_{W(K)} \in \sum_1^3 W(L_i)^K$ for some $k \in \mathbb{N}$.

(b) If $(L : K) \equiv 1\ (2)$, then $W(K) = W(L)^K$.

Proof: (a) Any ordering of K can be extended to at least one of the $L_i (i = 1,2,3)$, because a,b and ab cannot be all together negative.

(b) Because any ordering is extendable to L, one has $W(L)^K \subseteq m$ or $W(L)^K = W(K)$. But $\dim_K L \equiv 1(2)$, thus $\underline{L} \notin m$ and $W(L)^K = W(K)$.

Proof of Theorem 3.2: Because m is the only primeideal in $W(K)$ with $\mathrm{char}(W(K)/m) = 2$ one has easily (i) $\Longleftrightarrow$ (ii), using only the theory of commutative rings (if $2^k \cdot 1_{W(K)} \notin \Sigma\, W(L_i)^K$ for all $k \in \mathbb{N}$, then there exists a primeideal p with $\Sigma\, W(L_i)^K \subseteq p$, $2^k \cdot 1_{W(k)} \notin p$ for all k and vice versa).

(ii) $\Longrightarrow$ (iii) follows from

Lemma 3.3: Let L/K be a finite extension and α an ordering of K. Then α is extendable to L or $W(L)^K \subseteq \mathrm{Ke}(I_\alpha : W(K) \to \mathbb{Z})$.

Proof:
Assume α not to be extendable to L. $W(L)^K$ is generated by elements of the form $\underline{F}$ with F a finite extension of L and thus α not extendable to F. Let U be a subgroup of order 2 in the full Galoisgroup G of K, corresponding to α and let H be the fixgroup of F. Because α is not extendable to F, we have $U \nsubseteq H$ and thus:

$I_\alpha(\underline{F}) = \varphi_U(G/H) = 0$, q.e.d.

(iii) $\Longrightarrow$ (ii) Here we need:

<u>Lemma 3.4:</u> Let G be the full Galoisgroup of K and U a subgroup of order 2. Then $N_G(U) = U$.

Proof (cf. [1], Proof of Hilfssatz 2.3): Let R be the really closed fixfield of U and F be the fixfield of $N_G(U)$. We have to show that any automorphism of R/F is trivial. But on the one hand any automorphism of R is compatible with the ordering of R (squares go on squares...), on the other hand any $r \in R$ has only finitely many conjugates over F, which are permutated by any automorphism of R/F, thus have to stay fixed, q.e.d.

Now assume $\Sigma\, W(L_i)^K$ is contained in some primeideal, different from m. Thus (cf. [3]) there exists an ordering α and a prime $p \neq 2$ with $I_\alpha(\Sigma\, W(L_i)^K) \equiv 0(p)$, i.e. there exists a subgroup U of order 2 in the full Galoisgroup G of K with $\Sigma\, \Omega(L_i)^K \subseteq p(U,p)$. But by Cor. 2.6. this implies $U_{(p)} \nsubseteq H_i$ (with H_i the fixgroup of L_i in G) for all $i = 1,\dots,n$. But $U_{(p)} = U$ for $p \neq 2$ by Lemma 3.4 and $U \nsubseteq H_i$ of course implies that α cannot be extended to L_i, $i = 1,\dots, n$, q.e.d.

Remark: On the other hand one can use Cor. 2.6 and Scharlau's results stated in Cor. 3.1, to give another proof of Lemma 3.4. This observation was the starting point of Theorem 3.2.

4. For a finite group G and a finite abelian group A one can define the Grothendieckring $\Omega(G, A)$ of A-monomial permutation representations of G. (see [12]). Of course, also this definition can easily be generalized to profinite G: a (G,A)-set S is a finite set, on which the direct product $G \times A$ acts continuously from the left by permutations, such that $A \subseteq G \times A$ acts freely. For S a (G, A)-set let $\bar{S}$ be the G-set of A-orbits in S. If $s \in S$, $\bar{s} = A \cdot s$ and $G_s = \{g \in G \mid g\bar{s} = \bar{s}\}$, then one has a homomorphism $\psi_{\bar{s}} : G_{\bar{s}} \to A$, defined by $g \cdot s = \psi_s(g) \cdot s$. If S is a transitive $G \times A$-set ($\Longleftrightarrow \bar{S}$ is a transitive G-set), then $S \cong G \times A/H$ with $H = \{(g, \psi_s(g)) \in G \times A \mid g \in G_{\bar{s}}\}$ for any $\bar{s} \in \bar{S}$. Any (G,A)-set is in a unique way the disjoint union of transitive (G, A)-sets.

For S, T two (G, A)-sets define $S + T$ to be the disjoint union (with an obvious (G, A)-set-structure) and $S \underset{A}{\otimes} T$ the set of A-orbits in $S \times T$ with respect to the following A-action: $\alpha \cdot (s,t) = (\alpha x, \alpha^{-1} t)$ $(\alpha \in A,\ s \in S,\ t \in T)$ and with the following welldefined $G \times A$-action
$(g,\alpha)\ (s \otimes t) = g\ \alpha\ s \otimes gt = gs \otimes g\ \alpha\ t$.

Again isomorphismclasses of (G, A)-sets form a commutative semiring with respect to $+$ and $\underset{A}{\otimes}$. Let $\Omega(G, A)$ be the associated Grothendieckring.

Now assume A to be a subgroup of $\mathbb{C}^{\times}$ (thus to be cyclic). For any closed subgroup $U \subseteq G$ and any element $u \in U$ one can define a homomorphism $\varphi_{U,u} : \Omega(G,A) \longrightarrow \mathbb{Z}\,[A] \subseteq \mathbb{C} : S \mapsto \sum_{\bar{s} \in \bar{S}^U} \varphi_s(u)$. Again one can prove that any homomorphism $\Omega(G,A) \to R$ into an integral domain R factors through some $\varphi_{U,u}$ and also give necessary and sufficient conditions for equality of $\varphi_{U,u}$ and $\varphi_{V,v}$ mod p, p some primeideal in $\mathbb{Z}[A]$.

I want to give a fieldtheoretic interpretation of $\Omega(G,A)$ in case G is the Galoisgroup of a Galoisextension E/K, char $K \neq 2$ and $A = \{\pm 1\} \cong Z_2$.
For this purpose I consider (finite, commutative, separable) K-algebras A, such that $A \otimes_K E$ is a direct product of a finite number of copies of E (E-split), together with an involutory K-automorphism i, such thus $\mathrm{Dim}_K A^+ = \mathrm{Dim}_K A^-$ with $A^+ = \{a \in A \mid ia = a\}$, $A^- = \{a \in A \mid ia = -a\}$.
Char $K \neq 2$ implies $A = A^+ \oplus A^-$. Generally one has $\mathrm{Dim}_K A^+ \geq \mathrm{Dim}_K A^-$ for involutory automorphisms i.

For any (G,Z_2)-set S the set $A = A_S = \mathrm{Hom}_G(S,E)$ of G-equivariant maps $f : S \to E$ can be considered as such an algebra: Sum, product and K-action are defined, using the structure of E, and i is definded by $(if)(s) = f(-s)$, thus $A^+ = \mathrm{Hom}_G(\overline{S},E)$, $A^- = \mathrm{Hom}_{G \times Z_2}(S,E)$ (where Z_2 acts on E by multiplication with ± 1). Moreover, Z_2 acting free on S implies $\mathrm{Dim}_K A^+ = |\overline{S}| = \frac{1}{2}|S| = \frac{1}{2}\mathrm{Dim}_K A$, thus $\mathrm{Dim}_K A^+ = \mathrm{Dim}_K A^-$.

On the other hand for such a K-algebra A (let them be called E-split (K,Z_2)-algebras) the set $S = S_A = \mathrm{Hom}_K(A,E)$ of K-algebra-homomorphisms $f : A \to E$ defines a (G,Z_2)-set, where the G-action on S is defined using the G-action on E and $(-f)(x) =_{Df} f(ix)$. Using $\mathrm{Dim}_K A^+ = \mathrm{Dim}_K A^-$ one can prove $f \neq -f$ for all $f \in S$, thus Z_2 acts freely on S.

This establishes a (contravariant) 1-1 correspondance between (G,Z_2)-sets and E-split (K,Z_2)-algebras. Moreover, $A_{S+T} \cong A_S \times A_T$ and $A_{S \otimes_{Z_2} T} = A_S \overset{Z_2}{\underset{K}{\otimes}} A_T$, where $A \overset{Z_2}{\underset{K}{\otimes}} B$ for two (K,Z_2)-algebras is defined by $(A^+ \otimes_K B^+) \oplus (A^- \otimes_K B^-)$, the first summand being $(A \overset{Z_2}{\underset{K}{\otimes}} B)^+$, the second being $(A \overset{Z_2}{\underset{K}{\otimes}} B)^-$.

Thus the Grothendieckring $\Omega(G,Z_2)$ of (G,Z_2)-sets can as well be interpreted as Grothendieckring $\Omega(K,E;Z_2)$ of E-split (K,Z_2)-algebras. Again for E a separable closure of K we write $\Omega(K,Z_2)$ instead of $\Omega(K,E;Z_2)$.

I want to construct a canonical (surjective) ringhomomorphism:

Sc_2: $\Omega(K,Z_2) \to W(K)$.

For this let A be a separable (K,Z_2)-algebra and let $t_{A^+/K} : A^+ \to K$ be the trace.

Define $\underline{A}$ to be the K-vectorspace A^- together with the bilinear map $(a,b) \to t_{A^+/K}(ab)$ $(a,b \in A^- \Rightarrow ab \in A^+ !)$.

Again using Lemma 3.2 one can prove that $A \to \underline{A}$ defines a ringhomomorphism Sc_2: $\Omega(K,Z_2) \to W(K)$. In several ways this map is more convenient than Sc. E.g. the surjectivity of Sc_2 is even more obvious then in the case of Sc: For $a \in K^\times$ one defines $K(\sqrt{a}) = K[x]/(x^2-a)$ with $ix = -x$, - whether a is a square or not. Then $\underline{K(\sqrt{a})} = \langle a \rangle$, the bilinear form $K \times K \to K : (b,c) \mapsto abc$.

More precisely the (G,Z_2)-sets S with $|S| = 2$ (or the separable (K,Z_2)-algebras A with $\mathrm{Dim}_K A = 2$) form a multiplicative group with respect to $\overset{\otimes}{Z_2}$, isomorphic to the group $G' = \mathrm{Hom}(G,Z_2) \cong K^\times/K^{\times 2}$ of homomorphisms $G \to Z_2$, thus the grouping $\mathbb{Z}[G']$ can be embedded into $\Omega(G,Z_2)$ and the above argument just repeats the fundamental fact that the canonical map $\mathbb{Z}[G'] \cong \mathbb{Z}[K^\times/K^{\times 2}] \to W(K)$ is surjective.

Now let α be an ordering of K and $U \leq G$ the fixgroup of a maximal subextension of E, to which α extends. Then either U is the trivial subgroup of G or U is of order 2, in any way $U = \langle u \rangle$ for some unique $u \in G$. One can prove: $I_\alpha(A_S) = \varphi_{U,u}(S)$ for any (G,Z_2)-set S.

Especially, if L is some finite separable extension of K and $a \in L^\times$, then the inertia index of the bilinear form $\underline{L(\sqrt{a})} : L \times L \to K : (b,c) \to t_{L/K}(abc)$ with respect to α can be computed by mapping $L(\sqrt{a})$ into a Galoisextension E/K with Galoisgroup G, such that $L \to E^H$, $L(\sqrt{a}) \to E^F (F \trianglelefteq H \leq G$ and $(H : F) = 1$ or 2 , whether $a \in L^{\times 2}$ or not), forming

$S = G \times Z_2/\{(g,\psi(g)) \mid g \in H\}$ with $\psi: H \to Z_2$ the map with kernel F and then taking $\varphi_{U,u}(S)$ with $U = <u>$ the fixgroup of a maximal subextension of E, to which α extends.

In other words:

$\underline{I_\alpha(L(\sqrt{a}))} = |\{gH \in G/H \mid u \in g F g^{-1}\}| - |\{gH \in G/H \mid u \in g(H - F)g^{-1}\}|$

The fact, proved by Scharlau [6], that there exists always an element $a \in L^\times$ with $\underline{L(\sqrt{a})} = 1_{W(K)}$ if $(L : K)$ odd has thus a curious grouptheoretic interpretation: If G is the full Galoisgroup of a formally real field, then in any open subgroup $H \leq G$ of odd index there exists a subgroup F of index 2 with $|G/H^U| + 1 = |G/F^U|$ for all subgroups $U \leq G$ of order two.

More precisely one can show: if the fixfield L of H is generated by x and if $f(x)$ is the irreducible polynomial of x over K, then one can choose F to be the fixgroup of $L(\sqrt{a})$ with $a = \frac{N_{L:K}(x)}{x \cdot f'(x)}$.

Finally I want to give some kind of permutation group theoretic interpretation of a remark of Scharlau on induction theorems (cf [7], §4).

At first for any open subgroup U of a profinite group G one has (as in § 2) an additive map

$$i^*_{U,G} : \Omega(U,A) \to \Omega(G,A)$$

defined on the $\mathbb{Z}$-basis of transitive (U,A)-sets by

$$i^*_{U,G}(U \times Z_n/W) = G \times Z_n/W.$$

The image $\Omega(U,A)^G$ is again an ideal in $\Omega(G,A)$.

Now to avoid technical difficulties assume A to be cyclic: $A = Z_n \subseteq \mathbb{C}$ and also G to be finite. Let Y be an ideal in $\Omega(G,Z_n)$, such that for any pair $U \trianglelefteq V \leq G$ with V/U abelian of exponent $d|n$ there exists a unit $e_{V,U}$ in $\Omega(G,Z_n)/Y$ with

(*) $G \times Z_n/U \times 1 \equiv e_{V,U} \cdot \sum_{\varphi: V/U \to Z_n} G \times Z_n/V_\varphi \mod Y$

and $V_\varphi = \{(v,\varphi(v)) \in G \times Z_n \mid v \in V\}$

Let $N \trianglelefteq V$ with V/N nilpotent and $\mathcal{U} = \{U \leq V \mid N \leq U,\ U/N \text{ cyclic}\}$.

Then one can show, using Scharlau's technique:

$$n^k \cdot \Omega(V,Z_n)^G \subseteq Y + \sum_{U \in \mathcal{U}} \Omega(U,Z_n)^G \quad \text{for some power } n^k \text{ of } n.$$

If $n = 2$, $a,b \in K^\times$, $G = \mathrm{Gal}(K(\sqrt{a},\sqrt{b})/K)$ of order 4 and $Y = \mathrm{Ke}(\Omega(G,Z_2) \subseteq \Omega(K,Z_2) \to W(K))$, Scharlau uses this result to prove Corollary 3.1, (a).

If G is nilpotent, $n = |G|$ and Y the kernel of the canonical map

$$\Omega(G,Z_n) \to X(G,\mathbb{C}),$$

the characterring of G defined by

$$S \mapsto X_S \in X(G,\mathbb{C}) \quad \text{with} \quad X_S(g) = \varphi_{<g>,\ g}(S),$$

then - as pointed out by Scharlau - the result can be used to prove Artin's and Brauer's inductiontheorem in this special case. It would be rather desirable to state formulas similar to (*), which give special relations between the characters X_S of various (G,Z_n)-sets S and can be used to prove Brauer's inductiontheorem in the general case or at least (and this should be possible) for hyperelementary groups, - desirable not so much, to give still another proof for Brauer's Theorem, but because such formulas must contain rather valuable information on the relations between the characters of monomial representations and irreducible characters of a finite group.

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