J. K-Theory **7** (2011), 573–596
doi:10.1017/is011003012jkt150

Convergence of the Motivic Adams Spectral Sequence

by

P.Hu, I.Kriz and K.Ormsby *

Abstract

We prove convergence of the motivic Adams spectral sequence to completions at p and η under suitable conditions. We also discuss further conditions under which η can be removed from the statement.

Key Words: motivic Adams spectral sequence, motivic homotopy theory.
Mathematics Subject Classification 2010: 14F42, 55T15.

1. Introduction

In [9], we proved convergence of the motivic Adams spectral sequence for spectra X of finite type (for definition, see Section 2 below) in the (bi)stable homotopy category over $Spec(F)$ where F is an algebraically closed field of characteristic 0 to the homotopy groups of the 2-completion of X. Since that time, a number of people asked how far the argument can be generalized using the same method. Notably, the question came up in discussion at the Conference on Motivic Homotopy Theory in Münster, 2009. The purpose of this note is to answer this question. Specifically, recall that $K_M(F)$ denotes Milnor K-theory of F, and $K_{MW}(F)$ is the Milnor-Witt ring [13], which is canonically isomorphic to $\pi_{0+*\alpha}(S)$. This is the "0-slice" of the motivic stable homotopy groups of the sphere; recall that we write $\pi_{k+\ell\alpha} = \pi_{(k+\ell,\ell)}$. Recall also that there is an element $\eta \in K^1_{MW} = \pi_\alpha(S)$ such that

$$K_{MW}/\eta \cong K_M. \tag{1}$$

Then, in this note, we prove convergence of the motivic Adams spectral sequence for motivic cell spectra X of finite type to the homotopy groups of the completion of X at p and η for any field of characteristic 0, and p a prime number. We further prove that completion at p and η in the above statement can be replaced by completion at p under suitable conditions.

Throughout this paper (except where specified explicitly), we shall work in the (bi)stable motivic homotopy category (cf. [13]), i.e. $S^1 = S^{(1,0)}$ and $S^\alpha = S^{(1,1)}$

* Hu was supported by NSF grant DMS 0503814. Kriz was supported by NSA grant H 98230-09-1-0045. Ormsby was supported by NSF RTG grant DMS 0602191.

have inverses with respect to the smash product. Denote by $H\mathbb{Z}/p$ the motivic homology spectrum with coefficients in $\mathbb{Z}/p$, and by X^{Ad} the realization of the semi-cosimplicial object

$$(X^{Ad})_n = X \wedge \underbrace{H\mathbb{Z}/p \wedge \ldots \wedge H\mathbb{Z}/p}_{n}$$

where co-faces are given by the unit $S \to H\mathbb{Z}/p$. X^{Ad} is what is referred to as the *nilpotent completion* of X with respect to $H\mathbb{Z}/p$ by Bousfield [5].

Denote by $X^\wedge_p$ the Bousfield localization $L_{M\mathbb{Z}/p}X$ [5] of X at the (pushforward of the) Moore spectrum $M\mathbb{Z}/p$. In a very general context, including the present situation, this is equivalent to

$$\operatorname*{holim}_{\overleftarrow{n}} X/p^n,$$

so there is the usual exact sequence

$$0 \to \mathrm{Ext}^1(\mathbb{Z}/p^\infty, \pi_* X) \to \pi_*(X^\wedge_p) \to \mathrm{Hom}(\mathbb{Z}/p^\infty, \pi_{*-1}X) \to 0. \tag{2}$$

In the present motivic context, this is treated explicitly in [15, 9]. Denote similarly

$$X^\wedge_{p,\eta} = \operatorname*{holim}_{\overleftarrow{n}} X/(p^n, \eta^n).$$

There are canonical maps $X \to X^\wedge_p$, $X \to X^\wedge_{p,\eta}$, which we will refer to as *completion at* p resp. *completion at* p,η.

The main result we prove here is the following

Theorem 1 *Let F be a field of characteristic 0. Let X be a cell spectrum of finite type over $Spec(F)$ (see Section 2 below for definition). Then the natural map*

$$X \to X^{Ad} \tag{3}$$

is a completion at p,η. When $p > 2$ and $cd_p(F) < \infty$ and $-1 \in F$ is a sum of squares, or $p = 2$ and $cd_2(F[i]) < \infty$, then (3) is a completion at p.

For the definition of cd_p, see [17]. At $p = 2$, one has the following result, which follows in a trivial way from the Gysin sequence in Galois cohomology associated with a quadratic field extension

$$\ldots H^n(F;\mathbb{Z}/2) \longrightarrow H^n(F[i];\mathbb{Z}/2) \longrightarrow H^n(F;\mathbb{Z}/2) \xrightarrow{[-1]} H^{n+1}(F;\mathbb{Z}/2)\ldots$$

(cf. [2] — here $[-1]$ denotes multiplication by $[-1] \in F^\times/(F^\times)^2 = H^1(F,\mathbb{Z}/2)$.).

Proposition 2 *For any field F of characteristic 0, $cd_2(F[i]) < \infty$ if and only if*

$$\begin{aligned}&\textit{There exists a constant } r \textit{ such that } [-1] : H^n(F,\mathbb{Z}/2) \to \\ &H^{n+1}(F,\mathbb{Z}/2) \textit{ is an isomorphism for } n \geq r.\end{aligned} \tag{4}$$

Obviously, fields of finite transcendence degree, local fields, number fields and $\mathbb{R}$ are covered for $p = 2$ by Theorem 1. These cases of our present convergence result already have been used in the papers [8, 16].

Corollary 3 *Under the assumptions of Theorem 1, there exists a convergent spectral sequence ("motivic Adams spectral sequence") with*

$$E_2 = Cotor_{(H\mathbb{Z}/p_*, H\mathbb{Z}/p_* H\mathbb{Z}/p)}(H\mathbb{Z}/p_* X, H\mathbb{Z}/p_*)$$

convergent strongly to the homotopy groups of the respective completion of X.

Proof: The existence and convergence of the Adams spectral sequence is a formal consequence of Theorem 1 by the "Mittag-Leffler convergence Lemma" of Bousfield and Kan (Lemma 5.6, p. 264 of [3]). The point is that the tower involved in the Adams spectral sequence is actually

$$(X^{(n)})$$

where $X^{(n)}$ is the homotopy fiber of the canonical map from the relevant completion of X as in Theorem 1 to the n'th cosimplicial co-skeleton of X^{Ad}. Thus, by Theorem 1, we have

$$\operatorname*{holim}_{\overleftarrow{n}} X^{(n)} = *,$$

which implies that both that

$$\lim_{\leftarrow} \pi_* X^{(n)} = \lim_{\leftarrow}{}^1 \pi_* X^{(n)} = 0.$$

The identification of the E_2-term goes back to Adams [1]. □

Comments:

1. It is well known ([13, 14]) that in dimension $n\alpha$, $n > 0$, $\pi_{n\alpha}(S)$ is isomorphic to the Witt ring W and that the effect of completion at $2,\eta$ in these dimensions is the completion of W at its augmentation ideal. It is easy to show that for general fields, this does not coincide with completion at 2 (one example mentioned in [10] is the field $F = \mathbb{Q}(x_1,...,x_m,...)[i]$). Therefore, the second statement of Theorem 1 would be false if we omit the assumption (4).

For $p > 2$, this issue is even sharper. Completion of $\pi_{*\alpha}(S^0)$ at p is isomorphic to

$$W_p^\wedge \oplus K_M(F)_p^\wedge. \tag{5}$$

To see this, recall that by Morel's structure theorem [14, 13],

$$\pi_{n\alpha}(S^0) = \begin{array}{ll} W & \text{for } n > 0 \\ I^{-n} \times_{I^{-n}/I^{-n+1}} (K_M)_{-n}(F) & \text{for } n \leq 0. \end{array} \tag{6}$$

where I is the augmentation ideal of W. By induction on n, W/I^n is 2-torsion (since $2 = [-1] \in I$), so when completing at p odd, (6) becomes (5). The map η induces an isomorphism on the first summand and 0 on the second. Therefore, the completion at p is η-complete if and only if

$$W_p^\wedge = 0. \tag{7}$$

Referring to [11], if the field F has a real ordering (which is equivalent to -1 not being a sum of squares), then there exists an epimorphism

$$W \to \mathbb{Z},$$

which means that (7) cannot hold. Therefore, completion at η is necessary for convergence of the Adams spectral sequence at $p > 2$ for any such field.

2. In the following sections, by the statement that "the motivic Adams spectral sequence converges for X" we shall simply mean that $X \to X^{Ad}$ is an equivalence. For essentially formal reasons, the Adams spectral sequence always converges to the homotopy groups of X^{Ad}, [5], Section 6, also [6].

3. We have, by work of Voevodsky [19],

$$H\mathbb{Z}/2_* = K_M/2[\theta]$$

where θ is an element of dimension $1-\alpha$ (the "Tate twist"). For $p > 2$, the element θ exists only when F has p'th roots of unity, but for any prime p, we have by the Bloch-Kato conjecture proved by Voevodsky [18],

$$\mathbb{HZ}/p_{k+\ell\alpha} = \begin{array}{ll} H^{-\ell-k}(F,\mathbb{Z}/p(\ell)) & \text{for } k \geq 0 \\ 0 & \text{otherwise.} \end{array}$$

The structure of the dual Steenrod algebra is determined by Voevodsky [20], with a gap filled in [21] (see also [9]). One has ([20], Theorem 12.6):

$$A_* = H\mathbb{Z}/2[\tau_i,\xi_i]/(\tau_i^2 = \theta\xi_{i+1} + [-1]\tau_{i+1}) \tag{8}$$

for $p = 2$ and

$$A_* = H\mathbb{Z}/p[\tau_i,\xi_i]/(\tau_i^2 = 0) \tag{9}$$

for $p > 2$, where the elements ξ_i, $i > 0$ resp. τ_i, $i \geq 0$ have dimensions $(p^i - 1)(1 + \alpha)$ resp. $(p^i - 1)(1 + \alpha) + 1$. Actually, as shown in [20], $(H\mathbb{Z}/2_*, A_*)$ is not a

Hopf algebra but a Hopf algebroid; in (8), θ is identified with $\eta_L\theta$. If one uses $\eta_R\theta$ instead, one gets an additional term, which is why the formula in Theorem 12.6 of [20] looks slightly different. This structure theorem will be needed in the remaining sections.

4. We stress however that the Adams filtration on homotopy groups can be quite bad. Let, for example, $p = 2$, $F = \mathbb{Q}$ and consider the spectrum Y which is the homotopy cofiber of the sequence

$$S^{-\alpha} \vee S \xrightarrow{[3]\vee 2} S.$$

Let $X = Y/(4,\eta)$. Then one can see that the kernel of the canonical onto map

$$\pi_{*\alpha}X \to H\mathbb{Z}/2_{*\alpha}X$$

is not finitely presented as a $K_{MW}(\mathbb{Q})$-module.

Acknowledgments: We would like to thank Mike Hopkins and Fabien Morel for conversations between the years 2008 and 2010, in which they conjectured the convergence of the motivic Adams spectral sequence for cell spectra of finite type to the completion at (p,η) for general fields of characteristic 0. Although no method of proof was suggested, there is no doubt that those conversations contributed to the present paper. We are also thankful to Paul Arne Østvær for comments on an earlier (less general) version of this note, and to Alexander Merkurjev for a reference on Proposition 2.

2. Cell spectra of finite type

For most of our purposes, it will be convenient to work directly in the motivic (bi)stable homotopy category $\mathcal{SH}_F$ over F. From this point of view, *attaching cells to a spectrum* X means simply to form a homotopy cofiber of a map (in $\mathcal{SH}_F$) of the form

$$\bigvee_i S^{k_i+\ell_i\alpha} \to X.$$

The expressions $1 + k_i + \ell_i\alpha$ will be referred to as the *dimensions* of the cells. Starting with a point and iterating this construction, we obtain the notion of a (motivic) *cell spectrum*. A priori we could iterate the construction transfinitely, passing to homotopy colimits at limit ordinals, but commutation of homotopy groups with infinite wedges shows that one can construct any cell spectrum by applying at most ω steps, and taking homotopy direct limit once. It can also be shown (by formal arguments) that a map between motivic cell spectra is an equivalence if and only if it induces isomorphism on all homotopy groups $\pi_{k+\ell\alpha}$ (for general motivic spectra, such map will be called a *very weak equivalence* in [9]). Also by formal arguments, for any motivic spectrum X, there exists a cell spectrum X' and a very weak equivalence $X' \to X$ (cf. [7]). We will call a cell spectrum X *k-connective* if $\pi_{m+n\alpha}X = 0$ for $m < k$. For a map of cell spectra $f : X \to Y$, we will call f a *k-equivalence* if its homotopy cofiber is $k+1$-connective.

Definition: A *cell spectrum of finite (resp. bounded) type* is a cell spectrum X where there exists a $k \in \mathbb{Z}$ such that X has no cells in dimension $m + \ell\alpha$, $m < k$, and at most finitely many cells in dimension $m + \ell\alpha$ for any $m \in \mathbb{Z}$ (resp. no cells of dimension $m + \ell\alpha$ for $\ell < N_m$ where $N_m \in \mathbb{Z} \cup \{+\infty\}$ depend only on m). In the bounded case, we will refer to the numbers N_m as *bounds* for X; we require that there exists a $k \in \mathbb{Z}$ such that $N_m = +\infty$ for $m < k$.

Analogously, we will say that the *homotopy groups π_*X are of bounded type* if X is k-connective for some $k \in \mathbb{Z}$ and there exist $N_m \in \mathbb{Z} \cup \{\infty\}$ for all $m \in \mathbb{Z}$ such that $\pi_{m+\ell\alpha}(X) = 0$ for $\ell < N_m$.

A priori, in addition to the notion of finite type, we also have another weaker notion of *cell spectrum of weakly finite type*, which is a homotopy colimit of spectra of the form X_m where we have $X_{-1} = *$, and we have cofiber sequences of the form

$$Y_m \to X_m \to X_{m+1}$$

where Y_m is a wedge summand of a cell spectrum Z_m, and the wedge $\bigvee Z_m$ is a cell spectrum of finite type. Proposition 15 and comments in the Appendix of [9] imply that $H\mathbb{Z}/p$ is a cell spectrum of weakly finite type (while [9] focuses on the case of an algebraically closed field and $p = 2$, the same discussion applies in the general case).

Now, however, we have the following

Lemma 4 *A wedge summand X of a cell spectrum X_0 of finite type is equivalent to a cell spectrum of finite type X' such that for every cell of X' of dimension $k + \ell\alpha$, there exists a $k' < k$ and a cell of X_0 of dimension $k' + \ell\alpha$.*

Proof: A classic Eilenberg swindle. Let

$$X_0 \simeq X \vee Y.$$

Let X_1 be the cofiber of the map

$$X_0 \to X_0$$

which is Id on Y and 0 on X. Then

$$X_1 \simeq X \vee \Sigma X.$$

Let X_2 be the cofiber of the map

$$\Sigma X_0 \to X_1$$

which is Id on ΣX and 0 on ΣY. Then

$$X_2 \simeq X \vee \Sigma^2 Y.$$

Iterating this procedure, we obtain a spectrum of finite type equivalent to X, satisfying the dimensional condition stated. □

Corollary 5 *A cell spectrum of weakly finite type is equivalent to a cell spectrum of finite type.*

Proof: Use Lemma 4 successively on the spectra Y_m in the definition of a cell spectrum of weakly finite type. □

Thus, we have

Lemma 6 *The spectrum $H\mathbb{Z}/p$ is equivalent to a spectrum of finite type.*

□

Now suppose we have a cell spectrum X and a number $k \in \mathbb{Z}$ such that X is k-connective. Then it is easy to see that there exists an equivalent cell spectrum X' with no cells of dimension $m + \ell\alpha$, $m < k$. In fact, X' can be obtained in ω steps where in i'th step, $i \in \omega$, we attach cells in dimension $m + i + *\alpha$ - simply attach to X all cells necessary to cancel homotopy in that dimension, and in the end take homotopy fiber of the canonical map from X - here we are using the fact [13] that

$$\pi_{m+\ell\alpha} S = 0 \text{ for } m < 0. \tag{10}$$

We must be substantially more careful to prove an analogous statement for cell spectra of finite type. To this end, we will need a process called "cell cancellation".

Lemma 7 *Let X be a k-connective motivic cell spectrum of finite (resp. bounded) type. Then there exists an equivalent cell spectrum of finite (resp. bounded) type X' such that X' has no cells in dimension $m + \ell\alpha$, $m < k$.*

Proof: Let us first consider the finite type case. As remarked above, we may assume X is constructed in ω steps. Let i be the first step in which one of the cells e attached has dimension $m+\ell\alpha$, $m<k$ (if no such i occurs, we are done). By (10), the attaching map of e is homotopic to 0. Since we already remarked that attaching cells can be interpreted as an operation in the motivic stable homotopy category, we may assume (without changing the dimensions of the cells constituting X) that the attaching map of e is actually 0. This gives us a map

$$S^{m+\ell\alpha} \to X. \tag{11}$$

Let X_1 be the homotopy cofiber of the map (11). Note carefully that X_1 can be constructed by following the construction of X while "omitting" the cell e, i.e. that X_1 is equivalent to a cell spectrum of finite type with cells of the exact same dimensions as the cells of X with the exception of the cell e, which is omitted.

Now note however that by our assumption about X, the map (11) must be trivial, i.e. we must have

$$X_1 \simeq X \vee S^{m+1+\ell\alpha}. \tag{12}$$

We see then that by attaching a single cell of dimension

$$m+2+\ell\alpha$$

to X_1 whose attaching map is the injection to the second wedge summand (12), we obtain a cell spectrum equivalent to X with cells in the exact same dimensions except the cell e, which is replaced by a cell of dimension greater by 2.

By iterating this procedure, we clearly eventually eliminate all cells of dimension $m+\ell\alpha$, $m<k$.

In the bounded type case, the argument is identical. Limit steps, which are required in this case, are filled in by taking direct limits. □

Lemma 8 *Let X be a cell spectrum such that for every n, there exists a cell spectrum X_n of finite type and an n-equivalence*

$$X_n \xrightarrow{\phi_n} X\ . \tag{13}$$

Then X is equivalent to a cell spectrum of finite type.

Proof: By an analogue of Whitehead's theorem (which is true for formal reasons in the present situation, cf. [7]), we get homotopy commutative diagrams of the form

$$\begin{array}{ccc} X_n & \xrightarrow{\phi_n} & X \\ {\scriptstyle k_n}\downarrow & \nearrow{\scriptstyle \phi_{n+1}} & \\ X_{n+1}. & & \end{array}$$

It follows that an induced map

$$\underset{\rightarrow}{\operatorname{holim}} X_n \to X$$

is an equivalence. Further, the homotopy cofiber of k_n must be n-connective by the assumption on ϕ_n. By Lemma 7, X_{n+1} can be obtained from X_n by attaching finitely many cells in each dimension $m + *\alpha$, $m \geq n$. We deduce that

$$\underset{\rightarrow}{\operatorname{holim}} X_n$$

is a cell spectrum of finite type. □

Comment: There is no harm in turning around the arrows in the assumption (13): If we assume instead we have an n-equivalence

$$X \xrightarrow{f} X_n, \tag{14}$$

by the Whitehead theorem, if we denote by X'_{n-1} the cell spectrum of finite type obtained from X_n by omitting any cells of dimension $(\geq n) + *\alpha$ and by

$$\iota : X'_{n-1} \to X_n$$

the canonical map, there exists a map (in the homotopy category) $g : X'_{n-1} \to X$ such that

$$fg \simeq \iota.$$

Further, ι is an $n-1$-equivalence, and hence so is g.

For completeness, we also note

Lemma 9 *A homotopy cofiber of a map of k-equivalences is a k-equivalence, and a homotopy direct limit of a sequence of k-equivalences is a k-equivalence.*

Proof: For the first statement, consider a diagram of cofibration sequences

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & Z \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' \\ \downarrow & & \downarrow & & \downarrow \\ C & \longrightarrow & D & \longrightarrow & E. \end{array}$$

Then α and β are k-equivalences if and only if C, D are $(k+1)$-connective. But then E is $(k+1)$-connective by the long exact sequence of homotopy groups. For the second statement, simply note that isomorphisms and surjections are preserved by direct limits of sequences. □

3. Convergence

Lemma 10 *1. The canonical map*

$$S \to H\mathbb{Z}/p \tag{15}$$

is a 0*-equivalence.*

2. For $p = 2$*, the canonical map*

$$S/(2,\eta) \to H\mathbb{Z}/2 \tag{16}$$

is a 1*-equivalence. For* $p > 2$*, the canonical map*

$$\iota : S/(p,\eta) \to H\mathbb{Z}/(p,\eta) \simeq H\mathbb{Z}/p \vee \Sigma H\mathbb{Z}/p \tag{17}$$

is a 1*-equivalence.*

Proof: To prove the first statement, the map (15) realizes in homotopy groups the projection

$$K_{MW}(F) \to K_M(F)/p$$

([13, 14]), which is of course onto.

To prove the second statement, for $p = 2$, we have

$$\pi_{*\alpha}(S/(2,\eta)) = K_{MW}(F)/(2,\eta) = K_M(F)/2 = H\mathbb{Z}/2_{*\alpha},$$

and the isomorphism is induced by (16). Using the multiplicative structure of the Milnor-Witt ring, to prove the onto part of the statement, it suffices to prove that the generator

$$\theta \in \pi_{1-\alpha} H\mathbb{Z}/2 \tag{18}$$

lifts to

$$\theta' \in \pi_{1-\alpha}(S/(2,\eta)) \cong \mathbb{Z}/2. \tag{19}$$

Now using the pushforward, it suffices to produce θ' for the field $F = \mathbb{Q}$. In that case, we have

$$2[-1] = 0 \in K_M(\mathbb{Q}),$$

so

$$0 \neq [-1] \in Im(\beta : \pi_{1-\alpha} H\mathbb{Z}/2 \to \pi_{-\alpha} H\mathbb{Z}).$$

So we must have

$$[-1] = \beta\theta.$$

Now note that in K_{MW},

$$[-1](2 + [-1]\eta) = 0,$$

so

$$2[-1] = 0 \in \pi_{-\alpha}(S/\eta).$$

Thus, again, $[-1] \in \pi_{-\alpha}(S/\eta)$ satisfies

$$[-1] \in Im(\beta : \pi_{1-\alpha}(S/(2,\eta) \to \pi_{-\alpha}S/\eta),$$

and we can let θ' be any element of $\pi_{1-\alpha}(S/(2,\eta))$ such that $\beta\theta' = [-1]$.

For $p > 2$, this method works when F contains p'th roots of unity, but otherwise the Tate twist device is not available. However, instead, we may proceed as follows. Consider the diagram

$$\begin{array}{ccc} S/(p,\eta) \wedge H\mathbb{Z}/p & \xrightarrow{\iota \wedge H\mathbb{Z}/p} & H\mathbb{Z}/(p,\eta) \wedge H\mathbb{Z}/p \\ {\scriptstyle Id \wedge proj.}\uparrow & & \uparrow{\scriptstyle Id \wedge proj.} \\ S/(p,\eta) \wedge H\mathbb{Z} & \xrightarrow[\iota \wedge H\mathbb{Z}]{} & H\mathbb{Z}/(p,\eta) \wedge H\mathbb{Z}. \end{array} \tag{20}$$

By (9), the top row is a 1-equivalence, but by the fact that p annihilates the bottom row, the verticals are homotopy retracts, so the bottom row is also a 1-equivalence. Now the cofiber $C\iota$ is a cell spectrum of finite type such that

$$0 = \eta : C\iota \to C\iota.$$

Suppose $C\iota$ is k-connective but not $k+1$-connective for some k. We have

$$\pi_{k+*\alpha}C\iota = (\pi_{k+*\alpha}C\iota) \otimes_{\mathbb{Z}[\eta]} \mathbb{Z} = (\pi_{k+*\alpha}C\iota) \otimes_{K_{MW}} K_{MW}/\eta = H\mathbb{Z}_{k+\ell\alpha}C\iota. \tag{21}$$

Note that this is a "Künneth-like" argument, but we don't have to discuss the Künneth spectral sequence here; the $k + *\alpha$-homotopy groups of a smash product of a k-connective spectrum X and a 0-connective spectrum Y is always to

$$\pi_{k+*\alpha}X \otimes_{K_{MW}} \pi_{0+*\alpha}Y$$

by the long exact sequence in homotopy groups.

Thus, by the fact that the bottom row of (20) is a 1-equivalence, we have $k \geq 2$, as claimed. □

Lemma 11 *For any k-connective motivic cell spectrum X (not necessarily of finite type), X^{Ad} is k-connective.*

Proof: By Lemma 10 1. and Lemma 9,

$$X \to X \wedge H\mathbb{Z}/p \tag{22}$$

is a k-equivalence for any k-connective cell spectrum X, and hence the homotopy fiber X_1 of (22) is k-connective. Iterating this construction by forming a fiber sequence

$$X_{n+1} \to X_n \to X_n \wedge H\mathbb{Z}/p,$$

we obtain by induction that $\operatorname{holim}_{\overleftarrow{n}} X_n$ is k-connective, and hence X^{Ad}, which is the homotopy cofiber of the canonical map

$$\operatorname{holim}_{\overleftarrow{n}} X_n \to X,$$

is k-connective. □

The following lemma is not strictly needed in full generality as a part of our proof, but it is nice to note that we can prove it at this point.

Lemma 12 *If X is a motivic cell spectrum of finite type, then the motivic Adams spectral sequence converges for $X \wedge H\mathbb{Z}/p$.*

Proof: The statement is true for $X = S$ by calculation, so it is true for a cell spectrum with finitely many cells. Now let $X_{(n)}$ be the cell spectrum obtained by attaching the first n cells of X. Then by induction, the motivic Adams spectral sequence converges for $X_{(n)} \wedge H\mathbb{Z}/p$. Consider the diagram

$$\begin{array}{ccccc}
 & & X_{(n)} \wedge H\mathbb{Z}/p & \xrightarrow{\simeq} & (X_{(n)} \wedge H\mathbb{Z}/p)^{Ad} \\
 & & \downarrow & & \downarrow \\
F & \longrightarrow & X \wedge H\mathbb{Z}/p & \longrightarrow & (X \wedge H\mathbb{Z}/p)^{Ad} \\
\simeq\downarrow & & \downarrow & & \downarrow \\
F^{(n)} & \longrightarrow & X^{(n)} \wedge H\mathbb{Z}/p & \longrightarrow & (X^{(n)} \wedge H\mathbb{Z}/p)^{Ad}.
\end{array}$$

All the lines are cofiber sequences. By considering the bottom row and Lemma 11, the connectivity of $F^{(n)}$ goes to ∞ with n, while it does not depend on n by the middle row. Thus, $F = F^{(n)} \simeq *$, as claimed. □

Lemma 13 *The spectrum $(S/(p,\eta))^{Ad}$ is very weakly equivalent to a cell spectrum of finite type.*

Proof: The proof mimics the analogous statement in topology. Let us first consider $p = 2$. In topology, this spectrum is known as the first Brown-Gitler spectrum $B(1)$, and its Adams resolution is studied in [4], where it is shown that its connectivity (in total degree) is increasing. In the motivic setting, a similar argument can be made as follows.

Set

$$B_* := H\mathbb{Z}/2_*(S/(2,\eta)).$$

Then $(H\mathbb{Z}/2_*, B_*)$ is a sub-coalgebroid (in particular a bi-comodule) of $(H\mathbb{Z}/2_*, A_*)$ via the map

$$B_* \to A_* \tag{23}$$

induced by the canonical map $S/(2,\eta) \to H\mathbb{Z}/2$. Define now $J_0 = A_*$, and define inductively short exact sequences of right A_*-comodules

$$0 \to J_m \square_{A_*} B_* \to J_m \to J_{m+1} \to 0$$

where the first map is the inclusion.
Claim: $\varinjlim J_m = 0$, and

$$J_m \square_{A_*} B_* \tag{24}$$

is isomorphic to a sum of copies of B_*.

The proof is the same as in topology: For example, filter $(H\mathbb{Z}/2_*, A_*)$ by powers of the augmentation ideal. Then the associated graded object $(H\mathbb{Z}/2_*, E_0 A_*)$ is a (commutative) Hopf algebra, and isomorphic, as a coalgebra, to the tensor product of $(H\mathbb{Z}/2_*, E_0 B_*)$ with another coalgebra (where $E_0 B_*$ is the associated graded object of B_* with respect to the filtration by powers of its augmentation ideal). Therefore, the statement is true for the associated graded objects and hence also for the original objects (one shows by induction on m that $J_m(E_0 A_*) = E_0 J_m$).

Noticing further that J_1 is concentrated in degrees $(\geq 2) + *\alpha$, the Claim implies by induction that there exists a resolution

$$B_* \longrightarrow F_0 \xrightarrow{d_0} F_1 \xrightarrow{d_1} F_2 \xrightarrow{d_2} \ldots \tag{25}$$

where F_i is a sum of copies of A_* on a finite set S_i of generators s in dimensions

$$k_s = j + *\alpha,\ j \geq 2i. \tag{26}$$

As usual, to a Hopf algebroid extended/free resolution of the type (25), we may assign a tower of fibrations

$$X_{i+1} \to X_i \to K_i \tag{27}$$

where

$$K_i \simeq \Sigma^{-i} \bigvee_{s\in S_i} \Sigma^{k_s} H\mathbb{Z}/2, \tag{28}$$

and

$$H\mathbb{Z}/2_* X_i = Ker(d_i),\ X_0 = S/(2,\eta).$$

Further, by standard arguments, up to equivalence,

$$\operatorname*{holim}_{\overleftarrow{n}} X_n \tag{29}$$

does not depend on the choice of the resolution (25). In particular, for a suitable choice of resolution, (29) becomes $(S/(2,\eta))^{Ad}$, and hence we expressed $(S/(2,\eta))^{Ad}$ as a homotopy limit of a tower obtained by successively taking fibers of maps into K_i, so by (26), (28), Lemma 8 and Lemma 12 applied to

$$X = \Sigma^{-i} \bigvee_{s\in S^i} S^{k_s},$$

$(S/(2,\eta))^{Ad}$ is very weakly equivalent to a cell spectrum of finite type.

Now consider $p > 2$. In this case, the above argument actually applies with $B(1)$ replaced by $M\mathbb{Z}/p$ (since ξ_1 is in dimension $(\geq 2) + \ell\alpha$). Thus, $M\mathbb{Z}/p^{Ad}$ is very weakly equivalent to a cell spectrum of finite type, hence the same is true with $M\mathbb{Z}/p$ replaced by $S/(p,\eta)$, as $(?)^{Ad}$ preserves cofibration sequences. □

Remark: Note that we have not proved (yet) that the motivic Adams spectral sequence converges for $S/(p,\eta)$. However, we have the following

Lemma 14 *For any* 0*-connective cell spectrum* X *of finite type,*

1. *The canonical map* $X/(p,\eta) \to (X/(p,\eta))^{Ad}$ *is a* 1*-equivalence.*
2. $(X/(p,\eta))^{Ad}$ *is very weakly equivalent to a cell spectrum of finite type.*
3. *The canonical map*

$$(X/(p,\eta))^{Ad} \to ((X/(p,\eta))^{Ad})^{Ad}$$

is a very weak equivalence.

Proof: 1. For $X = S$, the characterization of the homotopy type of $(S/(p,\eta))^{Ad}$ as a homotopy inverse limit of maps into the K_i's in the proof of Lemma 13, implies that the canonical map

$$H\mathbb{Z}/2 \to (S/(2,\eta))^{Ad}$$

for $p = 2$, and

$$H\mathbb{Z}/p \to (M\mathbb{Z}/p)^{Ad}$$

for $p > 2$ is a 1-equivalence. Therefore, the statement follows from Lemma 10. Thus, by Lemma 9, the statement follows for the cell spectrum $X_{(n)}$ obtained by attaching the first n cells of X. Now consider the diagram

$$\begin{array}{ccc} \Sigma^{-1}X^{(n)} & \longrightarrow & \Sigma^{-1}(X^{(n)})^{Ad} \\ \downarrow & & \downarrow \\ X_{(n)} & \longrightarrow & (X_{(n)})^{Ad} \\ \downarrow & & \downarrow \\ X & \longrightarrow & X^{Ad} \end{array} \tag{30}$$

where the columns are cofibration sequences. By Lemma 11, for $n \gg 0$, the first row is a map of 2-connective spectra, and hence is a 1-equivalence. We just proved that the middle row is a 1-equivalence. Thus, the bottom row is a 1-equivalence by Lemma 9.

2. Consider the right hand column of (30). The middle term is very weakly equivalent to a cell spectrum of finite type by Lemma 13, while the connectivity of the top term goes to ∞ with n by Lemma 11. Thus, our statement follows from Lemma 8.

3. Using Lemma 11 again, it suffices to prove the statement for $X = S$. we go back to the model

$$(S/(p,\eta))^{Ad} = \underset{\overleftarrow{n}}{\operatorname{holim}}\, Y_n$$

where Y_n are the cofibers

$$X_n \to S/(p,\eta) \to Y_n$$

(see (27)). The Y_n's are obtained by taking successive fibers of maps into the spectra K_i of increasing connectivity. Since the motivic Adams spectral sequence converges for K_i by (a special case of) Lemma 12, it converges for Y_n. Since the connectivity of the X_n's goes to ∞ with n, our statement follows from Lemma 11. $\square$

Remark: Note that since $X/(p^M,\eta^N)$ may be represented by a motivic spectrum with a finite filtration where the associated graded pieces are $X/(p,\eta)$, all the statements of Lemma 14 remain valid with $X/(p,\eta)$ replaced by $X/(p^M,\eta^N)$.

Lemma 15 *Let X be a cell spectrum of finite type and suppose there exists an N such that*

$$p^N = \eta^N = 0 : X \to X. \tag{31}$$

Then the motivic Adams spectral sequence converges for X.

Proof: Suppose without loss of generality that X is 0-connective. Using (31), there exists an M_1, N_1 such that

$$X/(p^{M_1},\eta^{N_1}) \simeq X \vee \Sigma X \vee \Sigma^{1+\alpha} X \vee \Sigma^{2+\alpha} X. \tag{32}$$

(Choosing $M_1 \geq N$ makes $X/p^M \simeq X \vee \Sigma X$. By choosing another, possibly larger, number N_1, $0 = \eta^{N_1} : X/(p^{M_1}) \to X/(p^{M_1})$.) Then by Lemma 14 1., the canonical map

$$X/(p^{M_1},\eta^{N_1}) \to (X/(p^{M_1},\eta^{N_1}))^{Ad} = X^{Ad}/(p^{M_1},\eta^{N_1}) \tag{33}$$

is a 1-equivalence, while by Lemma 14 2., the right hand side is very weakly equivalent to a cell spectrum of finite type, for which, further, the motivic Adams spectral sequence converges by Lemma 14, 3. Thus, the fiber X_1 of (33) is 1-connective, and satisfies again the conditions of Lemma 15.

Therefore, by iterating this procedure, for any chosen k, we may construct a wedge X' of X and some suspensions of the form $\Sigma^{(\geq 1)+\ell\alpha} X$ such that

$$X' \to (X')^{Ad} \tag{34}$$

is a k-equivalence. But a wedge summand of a k-equivalence is a k-equivalence, so

$$X \to X^{Ad}$$

is also a k-equivalence for any k, and hence a very weak equivalence. □

Our next step is proving an analogue of Lemma 23 of [9]. Because we are no longer working over an algebraically closed field, it is more involved, and we need some preliminary steps.

Lemma 16 *Suppose X is a motivic spectrum over F, and*

$$X = \operatorname{holim}_{\leftarrow} X_s$$

where each motivic spectrum X_s is of bounded type with the same bounds $(M_m)_{m\in\mathbb{Z}}$. Assume one of the following conditions hold:

1. *There exists a prime $p > 2$ and a number $N \in \mathbb{N}$ such that*

$$0 = p^N : X_s \to X_s$$

for each s. Further, -1 is a sum of squares in F, and $cd_p(F) < \infty$.

2. There exists an $N \in \mathbb{N}$ such that

$$0 = 2^N = [-1]^N : X_s \to X_s$$

For each s. Further, $cd_2(F[i]) < \infty$.

Then there exists a very weak equivalence

$$Y \to X$$

where Y is a cell motivic spectrum of bounded type.

Proof: By induction on i, we will construct an i-equivalence

$$Y_i \to X \tag{35}$$

where Y_i is a cell spectrum of bounded type and there exists a number $N_i \in \mathbb{N}$ such that

$$0 = p^{N_i} : Y_i \to Y_i \tag{36}$$

in Case (1) and

$$0 = 2^{N_i} = [-1]^{N_i} : Y_i \to Y_i \tag{37}$$

in Case (2).

If k is such that $M_m = \infty$ for $m \leq k$, we may put $Y_k := *$. Now suppose (35) has been constructed for a given i. Let X_{i+1} be the homotopy cofiber of (35). Then

$$\pi_{m+\ell\alpha} X_{i+1} = 0 \text{ for } m \leq i.$$

Furthermore,

$$X_{i+1} = \operatorname{holim}_{\overleftarrow{n}} X_{i+1,s}$$

where the motivic spectra $X_{i+1,s}$ are $(i+1)$-connective, bounded below with the same bounds, and annihilated by the same power of p in Case (1), and the same power of 2, $[-1]$ in Case (2).

Now note however that the condition (1) implies that for each $k \in \mathbb{N}$, there exists an $M \in \mathbb{Z}$ such that

$$\pi_{\ell\alpha} M\mathbb{Z}/(p^k) = 0 \text{ for } \ell < M \tag{38}$$

and condition (2) implies that for each $k \in \mathbb{N}$ there exists an $M \in \mathbb{Z}$ such that

$$\pi_{\ell\alpha} S/(2^k, [-1]^k) = 0 \text{ for } \ell < M. \tag{39}$$

By Lemma 7, then, there is an $M \in \mathbb{Z}$ such that

$$\pi_{i+1+\ell\alpha} X_{i+1,s} = 0 \text{ for } \ell < M.$$

Hence,

$$\pi_{i+1+\ell\alpha} X_{i+1} = 0 \text{ for } \ell < M.$$

Thus, by attaching cells of dimension $i+1+\ell\alpha$, $\ell \geq M$, to X_{i+1}, we may kill $\pi_{i+1+*\alpha} X_{i+1}$. Further, the attaching map will factor through the smash product with $M\mathbb{Z}/p^K$ for some K in Case (1), and with $S/(2^{K_1}, [-1]^{K_2})$ with some $K_2 > K_1 > 0$ (since smashing X_{i+1} with such spectra will produce a wedge sum of copies of X_{i+1}).

Composing these attaching maps with the connecting map $\Sigma^{-1} X_{i+1} \to Y_i$ of (35) and taking homotopy cofiber produces Y_{i+1}. □

Now consider the homotopy category $\mathcal{C}$ of all motivic cell spectra (not necessarily of finite type) where equivalence is weak equivalence. This is a triangulated category. Further, we have a t-structure on $\mathcal{C}$ associated with the "homology theory" $\pi_{*+*\alpha}$: For a cell spectrum X, we construct a map

$$X \to \tau_{\leq k} X \tag{40}$$

by attaching cells inductively to kill all homotopy groups of dimensions $\ell + m\alpha$, $\ell > k$. The fiber of (40) is denoted by $\tau_{>k} X$. By obstruction theory,

$$[\tau_{>k} X, \tau_{\leq k} Y] = 0 \text{ for any } X, Y,$$

which is the key property required for getting a t-structure. (Note that this is substantially more restricted than the t-structure on the entire motivic stable homotopy category, constructed by Morel [14].) Now also for formal reasons, the heart of our t-structure can be identified with the subcategory of $\mathcal{C}$ consisting of cell spectra X such that $\pi_{k+\ell\alpha} X = 0$ for $k \neq 0$.

Proposition 17 *The heart $\mathcal{C}_0$ of the t-structure defined above is equivalent to the category of graded $K_{MW}(F)$-modules.*

Proof: Let X be an object in the heart. Then clearly $\pi_* X = \pi_{0+*\alpha} X$ is a graded $K_{MW}(F)$-module. Let

$$F_1 \xrightarrow{i} F_0 \longrightarrow \pi_* X \longrightarrow 0 \tag{41}$$

be a presentation of $\pi_* X$ where F_0, F_1 are free $K_{MW}(F)$-modules. Clearly, we may realize i as a 2-stage cell spectrum X', which is the cofiber of

$$\bigvee_j S^{m_j \alpha} \xrightarrow{I} \bigvee_i S^{p_i \alpha}. \tag{42}$$

Also automatically, we obtain a map

$$X' \to X$$

inducing isomorphism in $\pi_{0+*\alpha}$, so clearly we have an equivalence

$$\tau_{\leq 0} X' \simeq X.$$

Since X' only depends on $\pi_* X$, (and since also every $K_{MW}(F)$-module can be realized in this way), we have shown that there is precisely one object in the heart for each K_{MW}-module. By a similar argument, clearly any morphism of $K_{MW}(F)$-modules is realized by a morphism in the heart and any morphism which induces 0 on $\pi_{*\alpha}$ factors through a cell spectrum with homotopy groups in dimensions $k + \ell\alpha$ for $k > 0$ only, and hence is 0. □

Lemma 18 *Let $M \in Obj\, \mathcal{C}_0$ and suppose that either*

$$\begin{aligned} &p > 2,\ -1 \textit{ is a sum of squares in } F,\ cd_p(F) < \infty \\ &\textit{and } 0 = p^N : M \to M \end{aligned} \tag{43}$$

or

$$cd_2(F) < \infty \textit{ and } 0 = 2^N = [-1]^N : M \to M. \tag{44}$$

Then M is equivalent to a cell spectrum of bounded type.

Proof: The point is we may form a "resolution" of the form

$$F_i \to M_i \to M_{i+1} \tag{45}$$

where $M_0 = M$, $F_i, M_i \in Obj(\Sigma^i \mathcal{C}_0)$, the first map in (45) is surjective on homotopy groups and such that $\Sigma^{-i} F_i$ is a direct sum of copies of

$$K_{MW}/p^N \tag{46}$$

in Case (43) and of

$$K_{MW}/(2^N, [-1]^N) \tag{47}$$

in Case (44). Thus, it suffices to prove the statement for (46) in Case (43) and (47) in Case (44).

The idea now is to apply Lemma 16 to X of the form (46) resp. (47), $X_s = X/\eta^s$. The spectra X_s can be constructed can be constructed by taking finitely many cofibers of (suspensions of) copies of $K_{MW}/(\eta, p) = K_M/p$.

Thus, it suffices to show that K_M/p is of bounded type. For $p = 2$, we have a cofibration

$$\Sigma^{1-\alpha} H\mathbb{Z}/2 \to H\mathbb{Z}/2 \to K_M/2$$

(the first map being the Tate twist), so this follows from $H\mathbb{Z}/2$ being of finite type. For $p > 2$, $cd_p(F) < \infty$ and the Bloch-Kato conjecture [18] imply that there exists an $N \in \mathbb{Z}$ such that $H\mathbb{Z}/p_{m+n\alpha} = 0$ for $m + n < N$. It follows that constructing K_M/p from $H\mathbb{Z}/p$ by killing off homotopy groups $\pi_{k+*\alpha}$, $k \geq 1$ by attaching cells in the category of (rigid) $H\mathbb{Z}/p$-modules produces a finite type $H\mathbb{Z}/p$-module and hence a finite type (motivic) spectrum. □

Lemma 19 *Let X be a motivic spectrum of bounded type over F, and suppose one of the following conditions holds:*

$$\begin{array}{l} p > 2,\ -1 \text{ is a sum of squares in } F,\ cd_p(F) < \infty \\ \text{and } 0 = p^N : X \to X \end{array} \tag{48}$$

or

$$cd_2(F) < \infty \text{ and } 0 = 2^N = [-1]^N : X \to X. \tag{49}$$

Then $\pi_* X$ is of bounded type.

Proof: The strategy is to consider the canonical (co)fibration sequences

$$\tau_{\geq i+1} X \to \tau_{\geq i} X \to Y_i$$

and notice that by induction on i, $\tau_{\geq i} X$ satisfy the assumptions of this Lemma, while Y_i satisfy the assumptions of Lemma 18. (Start with $i = k$ where X is k-connective.)

Next, notice that by our assumptions, the fact that $\tau_{\geq i} X$ is of bounded type and by Lemma 7, for each i there exists an N_i such that $\pi_{i+\ell\alpha} \tau_{\geq i} X = 0$ for $\ell < N_i$. □

Lemma 20 *When $p > 2$, -1 is a sum of squares in F and $cd_p(F) < \infty$, or $p = 2$ and (4) holds, the canonical map*

$$\Psi : M\mathbb{Z}/p \to \underset{\overleftarrow{n}}{\mathrm{holim}}(M\mathbb{Z}/p)/\eta^n \tag{50}$$

is a very weak equivalence.

Proof: Similarly as in [9], the fiber $F\Psi$ of Ψ is the homotopy inverse limit of

$$\ldots \longrightarrow \Sigma^{(k+1)\alpha} M\mathbb{Z}/p \xrightarrow{\eta} \Sigma^{k\alpha} M\mathbb{Z}/p \longrightarrow \ldots . \tag{51}$$

Let us first assume that $p > 2$. Then, by [18], there exists a constant N such that

$$(K_M(F)/p)_n = 0 \text{ for } n > N. \tag{52}$$

Then we will prove that for all $q \in \mathbb{Z}$ there exists an $\ell \geq 0$ such that

$$\pi_q \Sigma^{k\alpha} M\mathbb{Z}/p = 0 \text{ for } k \geq \ell, \tag{53}$$

hence proving our statement, since this implies

$$\lim_{\leftarrow} \pi_q \Sigma^{k\alpha} M\mathbb{Z}/p = \lim_{\leftarrow}{}^1 \pi_q \Sigma^{k\alpha} M\mathbb{Z}/p = 0.$$

In effect, by Lemma 19, $\pi_* M\mathbb{Z}/p$ is of bounded type, which implies (53).

To treat the case when $p = 2$ and (4) is satisfied, first observe that by Lemma 19 applied to $X = S/(p,[-1])$

$$\pi_{*+*\alpha}(F\Psi/[-1]) = 0.$$

Therefore,

$$[-1] : \pi_{*+*\alpha}(F\Psi) \to \pi_{*+*\alpha}(F\Psi)$$

is an isomorphism. Since an isomorphism of these groups is also induced by η, the same is true for $[-1]\eta^2$, but $[-1]\eta^2 = 0 \in K_{MW}(F)/2$. Thus, $\pi_{*+*\alpha} F\Psi = 0$, as claimed. □

Comment: It may seem that in the case $p = 2$, condition (4) could be replaced by a weaker condition, replacing $[-1]$ by any element (or sequence of elements) which, when multiplied by η, become nilpotent in $K_{MW}/2$. Note however that by Morel's structure theorem (6) [14, 13],

$$K_{MW}/2 = \begin{array}{ll} W/2 & \text{in dimensions } (\geq 0)\alpha \\ I^n/2 & \text{in dimensions } (\leq 0)\alpha \end{array}$$

where I is the augmentation ideal of the Witt ring. (In (6), $I^{-n}/I^{-n+1} \cong (K_M)_n/2$, so modulo 2, the right hand leg of the pullback becomes an isomorphism.) Thus, an element α annihilates η in $K_{MW}/2$ if and only if α represents an element of W which is divisible by 2. But $2 = [-1] \in W$, so this means that α is divisible by $[-1]$, showing that essentially no further generalization is meaningful.

Lemma 21 *Let X be a motivic cell spectrum of finite type, and let either $p > 2$, $cd_p(F) < \infty$ and -1 be a sum of squares in F or $p = 2$ and F satisfy the condition (4). Then for any k, the canonical map*

$$X/p^k \to \operatorname*{holim}_{\overleftarrow{n}} X/(p^k, \eta^n)$$

is a very weak equivalence.

Proof: Let, again, $X_{(m)}$ be the spectrum obtained by attaching the first m cells of X. Consider the diagram of cofibration sequences

$$\begin{array}{ccccc}
 & & X_{(m)}/p^k & \xrightarrow{\simeq} & \operatorname{holim}_{\overleftarrow{n}} X_{(m)}/(p^k,\eta^n) \\
 & & \downarrow & & \downarrow \\
F & \longrightarrow & X/p^k & \longrightarrow & \operatorname{holim}_{\overleftarrow{n}} X/(p^k,\eta^n) \\
\downarrow \simeq & & \downarrow & & \downarrow \\
F^m & \longrightarrow & X^{(m)}/p^k & \longrightarrow & \operatorname{holim}_{\overleftarrow{n}} X^{(m)}/(p^k,\eta^n).
\end{array}$$

The top row is an equivalence by Lemma 20. The connectivity of F^m goes to ∞ with m by considering the bottom row, but does not depend on m by considering the middle row. Thus, $F = F^m \simeq *$, as claimed. □

Proof of Theorem 1: The first statement follows directly from Lemma 15. The second statement follows then from Lemma 21. □

REFERENCES

1. J.F. Adams: *Stable homotopy and generalised homology*, Chicago Lectures in Mathematics, University of Chicago Press, 1973
2. J. Arason: Cohomologische invarianten quadratischer Formen, *J. Algebra* **36** (1975) 448-491
3. A.K. Bousfield, D.M. Kan: *Homotopy limits, completions and localizations*, Lecture Notes in Mathematics **304**, Springer Verlag, 1972
4. E.H. Brown, R.L. Cohen: The Adams spectral sequence of $\Omega^2 S^3$ and Brown-Gitler spectra, in: *Algebraic Topology and Algebraic K-theory*, W.Browder, ed., Annals of Math. Studies **113**, Princeton Univ. Press, Princeton, NJ, 1987, pp. 101-125
5. A.K. Bousfield: The localization of spectra with respect to homology, *Topology* **18** (1979) 257-281
6. D. Dugger, D. Isaksen: The motivic Adams spectral sequence, *Geom. Topol.* **14**/2 (2010) 967-1014
7. D. Dugger, D. Isaksen: Motivic cell structures, *Algebr. Geom. Topol.* **5** (2005) 615-652
8. M. Hill: Ext and the motivic Steenrod algebra over $\mathbb{R}$, *J. Pure and Appl. Alg.* **215**/5 (2011) 715-727

9. P. Hu, I. Kriz, K. Ormsby: Remarks on motivic homotopy theory over algebraically closed fields, *J. K-Theory* **7** (2011) 55-89

10. P. Hu, I. Kriz, K. Ormsby: Equivariant and real motivic stable homotopy theory, K-theory archive 952

11. T. Y. Lam. *Introduction to quadratic forms over fields*, *Graduate Studies in Mathematics* **67**, American Mathematical Society, Providence, RI, 2005.

12. J. Milnor: Algebraic K-theory and quadratic forms, *Invent. Math.* **9** (1969/1970) 318-344

13. F. Morel: On the motivic stable π_0 of the sphere spectrum, *Axiomatic, Enriched and Motivic Homotopy Theory*, pp. 219-260, 2004 Kluwer Academic Publishers

14. F. Morel: An introduction to $\mathbb{A}^1$-homotopy theory, *Contemporary developments in algebraic K-theory*, ICTP Lect. Notes, XV (2004) 357-441

15. P.A. Østvær, O. Röndigs: Rigidity in motivic homotopy theory, *Math. Ann.* **341** (2008) 651-675

16. K. Ormsby: Motivic invariants of p-adic fields, *J. K-Theory* **7** (2011) 597-618

17. J.P. Serre: Galois Cohomology, *Springer Monographs in Mathematics*, Springer-Verlag, Berlin, 2002

18. V. Voevodsky: On Motivic Cohomology with $\mathbb{Z}/\ell$ coefficients, arXiv:0805.4430

19. V. Voevodsky: Motivic cohomology with $\mathbf{Z}/2$-coefficients, *Publ. Math. Inst. Hautes Études Sci.* **98** (2003) 59-104

20. V. Voevodsky: Reduced power operations in motivic cohomology, *Publ. Math. Inst. Hautes Études Sci.* **98** (2003) 1-57

21. V. Voevodsky: Motivic Eilberg-Maclane spaces, *Publ. Math. IHES* **112** (2010) 1-99

PO HU `po@math.wayne.edu`

Department of Mathematics
Wayne State University
U.S.A.

IGOR KRIZ `ikriz@umich.edu`

Department of Mathematics
University of Michigan
U.S.A.

KYLE ORMSBY ormsby@math.mit.edu

Department of Mathematics
MIT
U.S.A.

Received: April 15, 2010