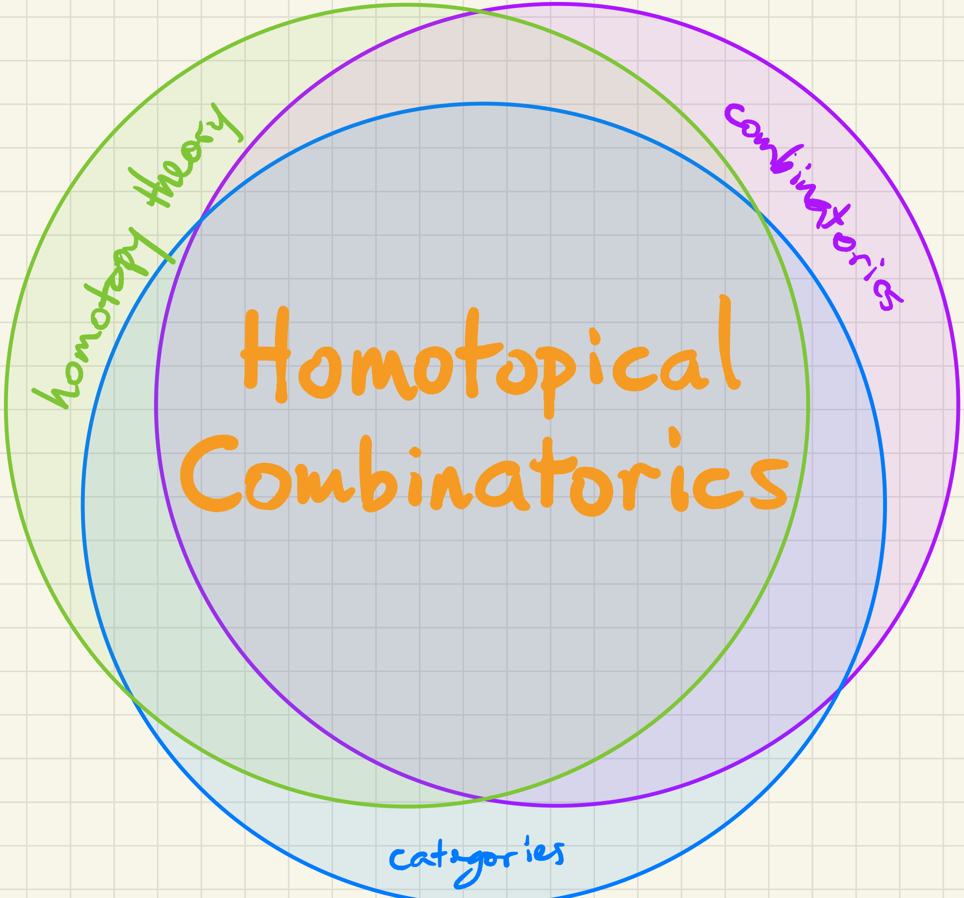
Homotopical Combinatorics



Kyle Ormsby

25. X. 22

Joint with

Scott Balchin
MPIM

Evan Franchere
UKy [Reed]

Usman Hafeez
[Reed]

Ethan MacBrough
Reed

Peter Marcus
Tulane [Reed]

Angélica Osorno
Reed

Weihang Qin
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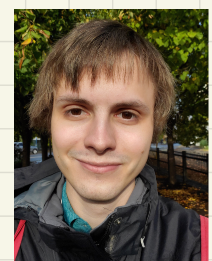
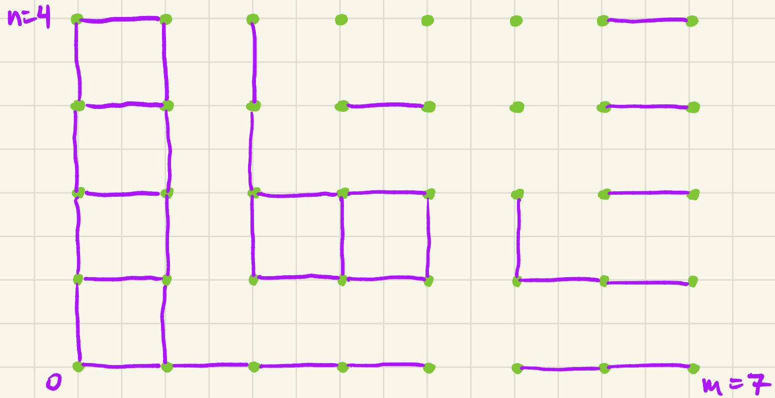
Constanze Roitzheim
Kent



Riley Waugh
[Reed]

Warmup

The Matchstick Game

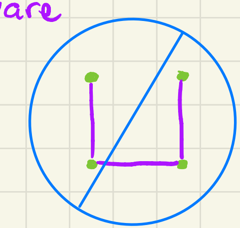


Rules

 $\Longrightarrow$ all verticals to the left

 $\Longrightarrow$ all horizonals below

3 sides of a unit square $\Longrightarrow$ full unit square



(and rotations)

Q1 How many legal matchstick configurations on $[m] \times [n]$?

Q2 What else do these count?

Quillen model structures

A model structure on a category $\mathcal{C}$ consists of three classes of morphisms:

W = weak equivalences

F = fibrations

C = cofibrations

($AF = W \cap F$ = acyclic fibrations

$AC = W \cap C$ = acyclic cofibrations)

satisfying five axioms:

MC1) $\mathcal{C}$ is complete & cocomplete

MC2) W satisfies $2 \Rightarrow 3$: two of $f, g, fg \in W \Rightarrow$ all in W

MC3) W, F, C are closed under retracts (in the arrow category)

MC4) $C \boxslash F$ (lifting condition defined later)

MC5) $AF \circ C = \mathrm{Mor}(\mathcal{C}) = F \circ AC$ (factorization)

Theorems

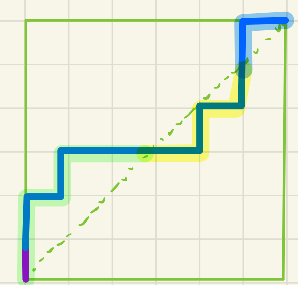
Goal Given a (co)complete category C, classify and enumerate all model structures on C.

Thm (Folklore/Goodwillie/Barthel-Antolín Camarena) There are precisely nine model structures on Set.

Thm (Balchin-O-Osorno-Roitzheim) There are precisely

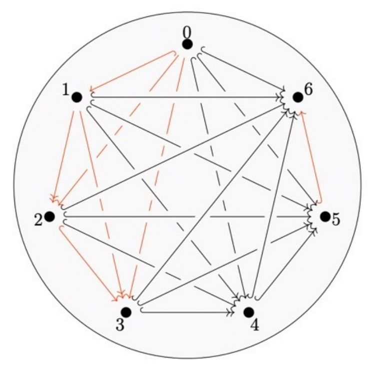
$\binom{2n+1}{n}$ model structures on $[n] := \langle 0 \to 1 \to 2 \to \cdots \to n \rangle$.

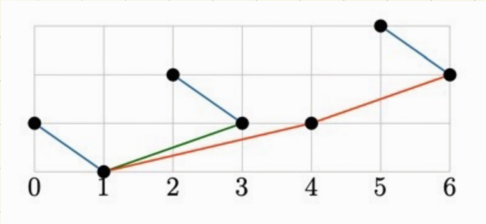
For $0 \leq k \leq n$, precisely $\frac{2(k+1)}{n+k+2}\binom{2n+1}{n-k}$ of these have homotopy category $\simeq [k]$.



Theorems

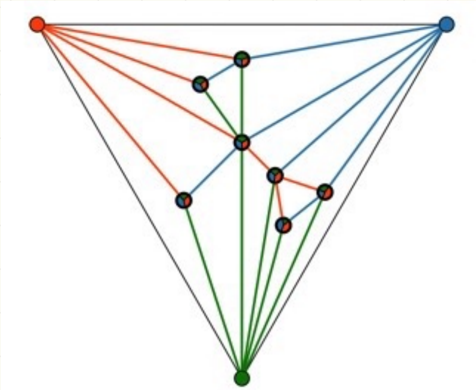
<u>Thm</u> (Balchin-O-MacBrough) Moreover, model structures on $[n]$ are in bijection with (a) model triangulations, (b) model tricolored trees, and (c) model intervals in the Kreweras lattice of noncrossing partitions.





x	0	1	2	3	4	5	6
$\pi_{\mathcal{R}}(x)$	0	3	2	3	4	5	6

x	0	1	2	3	4	5	6
$\pi_{\mathcal{R}'}(x)$	0	6	2	6	6	5	6



Lifting + Weak Factorization Systems

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[d, "i"'] & \bullet \arrow[d, "p"] \\ \bullet \arrow[r, "g"'] \arrow[ur, dashed, "h"] & \bullet \end{tikzcd}$$

Say i has the left lifting property wrt p (and p has the right lifting property wrt i) and write $i \boxslash p$ when for $\forall f, g$ s.t. $\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[d, "i"'] & \bullet \arrow[d, "p"] \\ \bullet \arrow[r, "g"'] & \bullet \end{tikzcd}$ commutes, $\exists$ $\begin{tikzcd} & \bullet \\ \bullet \arrow[ur, dashed, "h"] \end{tikzcd}$ such that $f = hi$, $g = ph$.

For $M \subseteq \mathrm{Mor}\, C$, write
$$^{\boxslash}M := \{ i \mid i \boxslash p \;\; \forall p \in M \}$$
$$M^{\boxslash} := \{ p \mid i \boxslash p \;\; \forall i \in M \}.$$

A weak factorization system on C consists of $(L, R) \subseteq \mathrm{Mor}\, C \times \mathrm{Mor}\, C$ such that

WF1) $\mathrm{Mor}\, C = R \circ L$ i.e. $\begin{tikzcd} \bullet \arrow[rr, "f"] \arrow[dr, dashed, "i"'] & & \bullet \\ & \bullet \arrow[ur, dashed, "p"'] & \end{tikzcd}$ with $i \in L$, $p \in R$,

WF2) $L \boxslash R$ i.e. $L \subseteq {}^{\boxslash}R$ and $R \subseteq L^{\boxslash}$, and

WF3) L, R closed under retracts.

Premodel Structures

<u>Defn</u> (Barton) A premodel structure on a (co)complete category $\mathcal{C}$ is a pair of weak factorization systems

$$\begin{array}{ccc} (C, & AF) \\ \text{\rotatebox{90}{$\supseteq$}} & \text{\rotatebox{90}{$\subseteq$}} \\ (AC, & F) \end{array}$$

with $C = {}^{\boxslash}AF$, $AC = {}^{\boxslash}F$.

Call AF =: anodyne fibrations,

AC =: anodyne cofibrations.

Every model structure (W, F, C) induces a premodel structure

$$\begin{array}{ccc} (C, & W \cap F) \\ \text{\rotatebox{90}{$\supseteq$}} & \text{\rotatebox{90}{$\subseteq$}} \\ (W \cap C, & F), \end{array}$$

In this scenario, $W = AF \circ AC$.

Joyal–Tierney presentation of model structures

Thm (Joyal–Tierney) For a (co)complete category $\mathcal{C}$,

$$\left\{ \begin{matrix} (C, AF) \\ \rotatebox{90}{$\supseteq$} \quad \rotatebox{90}{$\subseteq$} \\ (AC, F) \end{matrix} \text{ premodel structure on } \mathcal{C} \;\middle|\; \begin{matrix} AF \circ AC \text{ satisfies} \\ 2 \Rightarrow 3 \end{matrix} \right\} \overset{\cong}{\longleftrightarrow} \left\{ \begin{matrix} (W, F, C) \text{ model} \\ \text{structure on } \mathcal{C} \end{matrix} \right\}$$

Upshot Premodel structures on $\mathcal{C}$ are intervals in the poset $WFS(\mathcal{C})$ (ordered by $(L,R) \le (L',R') \Leftrightarrow R \subseteq R'$ $[\Leftrightarrow L' \subseteq L]$).

These are model structures iff $R \circ L'$ has $2 \Rightarrow 3$.

$$\begin{matrix} (C, AF) \\ \rotatebox{90}{$\supseteq$} \quad \rotatebox{90}{$\subseteq$} \\ (AC, F) \end{matrix} \longmapsto (AF \circ AC, F, C)$$

$$\begin{matrix} (C, W \cap F) \\ \rotatebox{90}{$\supseteq$} \quad \rotatebox{90}{$\subseteq$} \\ (W \cap C, F) \end{matrix} \longleftarrow\!\!\shortmid (W, F, C).$$

Lattices

Producing all WFS's on a general $\mathcal{C}$ is hard!

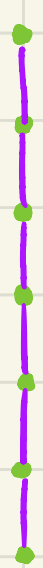
A **complete lattice** is a poset admitting all

meets ($\wedge$ = infimum = limit)

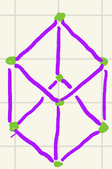
& joins ($\vee$ = supremum = colimit).

<u>E.g.</u> Chain $[n] = \{0 < 1 < \cdots < n\}$, Boolean lattice $[1]^n$, subgroup lattice $\mathrm{Sub}(G)$, divisibility lattice, Tamari lattice, Kreweras lattice, ...

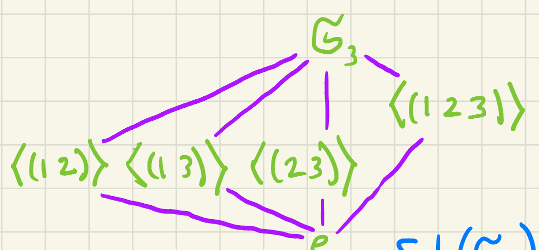
(rooted planar binary trees under rotation) (noncrossing partitions under refinement)



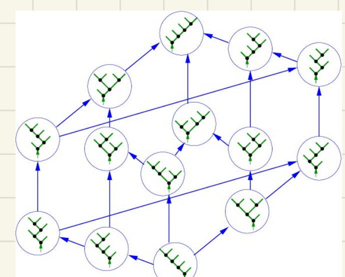
$[6]$



$[1]^3$



$\mathrm{Sub}(\mathfrak{S}_3)$



Tamari

[Bernardi-Bonichon]

Lattice Categories

Henceforth, lattice = finite lattice = category induced thereby:

$$(P,\leq) \rightsquigarrow \quad \mathrm{Ob}\, P := P$$

$$P(x,y) = \begin{cases} \{x \xrightarrow{\exists!} y\} & \text{if } x \leq y \\ \varnothing & \text{otherwise.} \end{cases}$$

The category P is (co)complete iff P is a complete lattice.

Refined Goal Enumerate and determine the structure of

$$\mathrm{WFS}(P),\quad \mathrm{Pre}(P) \cong \mathrm{Int}(\mathrm{WFS}(P)),\quad Q(P).$$

weak factorization systems on P

premodel structures on P

intervals

Quillen model structures on P: subset of $\mathrm{Pre}(P)$ with $W = AF \circ AC$ satisfying $2 \Rightarrow 3$.

$\mathrm{Int}(L) = \{(x,y) \in L^2 \mid x \leq y\}$ with

$(x,y) \leq (x',y')$:iff $x \leq x'$ and $y \leq y'$.

If L is a lattice, so is $\mathrm{Int}(L)$.

Transfer systems

Blumberg–Hill: equivariant DAG

To determine WFS(P) we get an assist from ... ¡¿N_∞ operads?!

<u>Thm</u> (Rubin et al) $\mathrm{Ho}(G\text{-}N_\infty \text{ operads}) \simeq \underbrace{\mathrm{Tr}}_{\text{transfer systems}}(\mathrm{Sub}(G))$.

A transfer system on a lattice $(P, \leq)$ is a transitive relation R on P refining $\leq$ (so $aRb \Rightarrow a \leq b$) that is closed under restriction:

$$\begin{tikzcd} & x \arrow[d, "R"] \\ y \arrow[r, "\leq"'] & z \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} x \wedge y \arrow[r, "\leq"] \arrow[d, "R"'] & x \arrow[d, "R"] \\ y \arrow[r, "\leq"'] & z \end{tikzcd}.$$

(If $P = \mathrm{Sub}(G)$, also require R to be closed under conjugation.)

$\mathrm{Tr} \cong \mathrm{WFS}$

Thm (Franchere-O-Osorno-Qin-Waugh) For P a lattice, we have a lattice isomorphism

$$\begin{array}{ccc} \mathrm{Tr}(P) & \xrightarrow{\cong} & \mathrm{WFS}(P). \\ R & \longmapsto & ({}^{\boxslash}R, R) \end{array}$$

R: {subrelation of $\leq$}

R: {subset of $\mathrm{Mor}\,P$}

In case $P = [n]$, we get some immediate progress:

Thm (Balchin-Barnes-Roitzheim) $\mathrm{Tr}[n] \cong \mathcal{A}_{n+1}$, the Tamari lattice of planar full binary trees with $n+2$ leaves. As such,

$$|\mathrm{Tr}[n]| = \mathrm{Cat}(n+1) = \frac{1}{n+2}\binom{2n+2}{n+1}.$$

Catalan #s

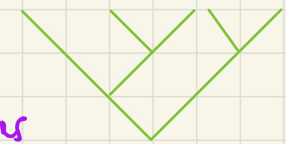
The **Catalan numbers** $\mathrm{Cat}(n)$, $n \geq 0$, are the sequence $1, 1, 2, 5, 14, 42, 132, \dots$

n: 0 1 2 3 4 5 6

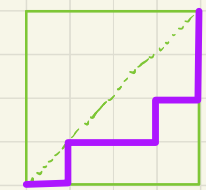
satisfying $\quad \mathrm{Cat}(n+1) = \sum_{i=0}^{n} \mathrm{Cat}(i)\,\mathrm{Cat}(n-i)$.

They enumerate

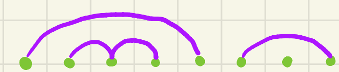
- planar full binary trees with $n+1$ leaves



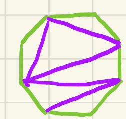
- Dyck paths from $(0,0)$ to (n,n)



- noncrossing partitions of an n-element set

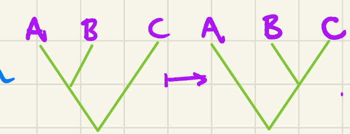


- triangulations of a convex $(n+2)$-gon by chords



- much more!

The **Tamari order** on $\mathcal{A}_n$ is generated by **tree rotation**



The **Kreweras order** on $\underbrace{NC_n}_{\text{noncrossing partitions}}$ is given by refinement.

Model Structures on $[n]$

Amalgamating results, $\mathrm{Pre}\,[n] \cong \mathrm{Int}(\mathrm{Tr}\,[n]) \cong \mathrm{Int}\,\mathcal{A}_{n+1}$.

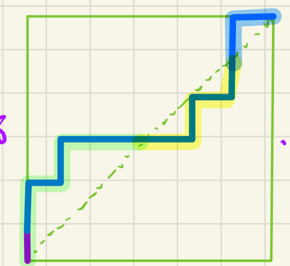
<u>Thm</u> (Chapoton) $\quad |\mathrm{Int}\,\mathcal{A}_n| = \dfrac{2}{n(n+1)}\dbinom{4n+1}{n-1}$.

<u>Cor</u> (BOOR) $\quad |\mathrm{Pre}\,[n]| = \dfrac{2}{(n+1)(n+2)}\dbinom{4n+5}{n}$.

<u>Thm</u> (BOOR) $\quad |Q([n])| = \dbinom{2n+1}{n}$, and for $0 \leq k \leq n$, precisely $\dfrac{2(k+1)}{n+k+2}\dbinom{2n+1}{n-k}$ of these have homotopy category $\cong [k]$.

<u>Pf Idea</u> Specify k s.t. $[k] \cong \mathrm{Ho}([n])$, then specify $k+1$ weak equiv classes, then count choices of contractible model structures:

$$|Q([n])| = \sum_{k=0}^{n} \underbrace{\sum_{\substack{i_0+\cdots+i_k \\ = n+1}} \prod_{j=0}^{k+1} \mathrm{Cat}(i_j)}_{\text{Shapiro (1976): } \frac{2(k+1)}{n+k+2}\binom{2n+1}{n-k}} = \binom{2n+1}{n} \text{ by lattice paths}$$



$\square$

CC Premodel Structures

Fill the gap between premodel & model structures.

Call a premodel structure $\begin{array}{c}(C, \; AF) \\ \rotatebox{90}{$\subseteq$} \quad \rotatebox{-90}{$\subseteq$} \\ (AC, \; F)\end{array}$ on C composition closed when

$W := AF \circ AC$ is closed under composition. (Need 2⇒3 for model str.)

For $C = P$ a lattice, write $R \leq R'$ for a premodel str / interval of transfer systems.

<u>Thm</u> (Balchin-MacBrough-O) For P a complete lattice, ∃ partial order $\preccurlyeq$ on $WFS(P)$ refining $\leq$ and such that $R \preccurlyeq R'$ if and only if $R \circ {}^{\boxslash}R'$ is closed under composition. Moreover, $R \preccurlyeq R'$ iff every square

$$\begin{tikzcd} x \arrow[r] \arrow[d, "R"'] & z \arrow[d, "R'"] \\ y \arrow[r] & w \end{tikzcd}$$

has a splitting

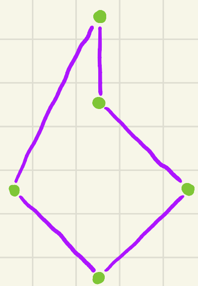
$$\begin{tikzcd} x \arrow[r] \arrow[d, "R"'] & z' \arrow[r] \arrow[d, "R"] & z \arrow[d, "R'"] \\ y \arrow[r] & w' \arrow[r, "R'"'] & w \end{tikzcd}.$$

CC Premodel Structures

Thm (ct'd) If P is finite, then $(WFS(P), \precsim)$ is a finite lattice. Thus $\underbrace{CC(P)}_{\text{composition closed premodel structures}} = Int(WFS(P), \precsim)$ is a finite lattice.

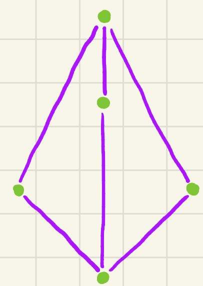
Note There is also an ordering $\sqsubseteq$ on $WFS(P)$ such that $R \sqsubseteq R'$ iff the pair forms a model structure, but $(WFS(P), \sqsubseteq)$ is not a lattice.

E.g.



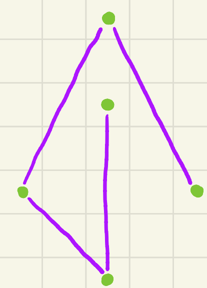
$(WFS([2]), \leq)$

$|Pre([2])| = 13$



$(WFS([2]), \precsim)$

$|CC([2])| = 12$



$(WFS([2]), \sqsubseteq)$

$|Q([2])| = 10$

Kreweras Intervals

$[n] = B_1 \sqcup \cdots \sqcup B_k$ is NC when $a, b \in B_i$, $x, y \in B_j$, and $a < x < b < y \Rightarrow i = j$.

For $R \in \mathrm{Tr}\,[n]$, define $\pi_R : [n] \longrightarrow [n]$

$$i \longmapsto \max\{ j \mid i R j \}.$$

Prop (FOOQW) Write NC_{n+1} for the set of noncrossing partitions of $\{0, 1, \ldots, n\}$. Then

$$\mathrm{Tr}\,[n] \longrightarrow NC_{n+1}$$

$$R \longmapsto \{ \pi_R^{-1}\{i\} \mid 0 \le i \le n,\ \pi_R^{-1}\{i\} \neq \emptyset \}$$

is a bijection.

Thm (BMO) Pulling back the Kreweras (refinement) order on NC_{n+1} to $\mathrm{Tr}\,[n]$ gives the ordering $\precsim$.

Cor $\mathrm{CC}([n]) \cong \mathrm{Int}(NC_{n+1})$ and thus

$$|\mathrm{CC}([n])| = \underbrace{\frac{1}{2n+3}\binom{3n+3}{n+1}}_{\text{Kreweras: } = |\mathrm{Int}(NC_{n+1})|}.$$

Kreweras Intervals

The following structures are equinumerous:

- Composition closed premodel structures on $[n]$

  $\updownarrow$ BMO

- Kreweras intervals in NC_{n+1}

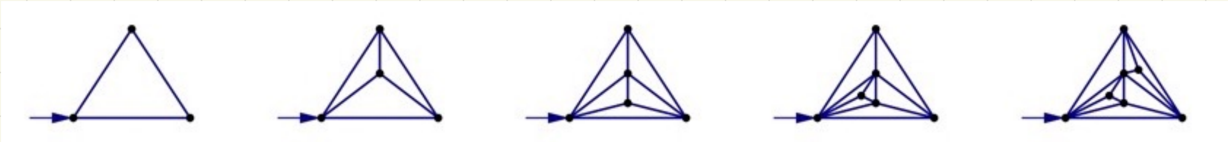
  $\updownarrow$ Bernardi-Bonichon

- (admissably ordered) ternary trees on $n+1$ nodes

  $\updownarrow$ BB

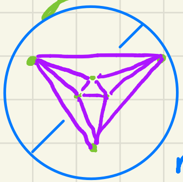
- stacked triangulations

identify model str's among these



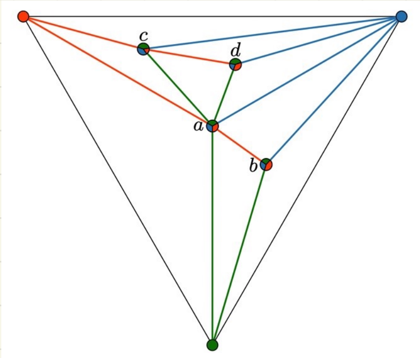
[BB]

Stacked Δ'ns formed by recursively inserting degree 3 vxs

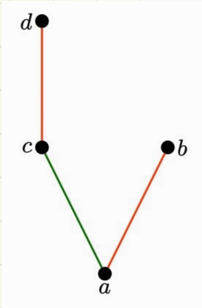


not stacked

$\Delta'_n \to$ trees $\to NC \to CC$



$\rightsquigarrow$



- color node's opposite side according to incoming edge
- color edges with parent node's color

- if w subdivides Δxyz, connect w to highest of x, y, z on tree (so $d - c$, not $d - a$)

Now sort the tricolored tree left-to-right such that

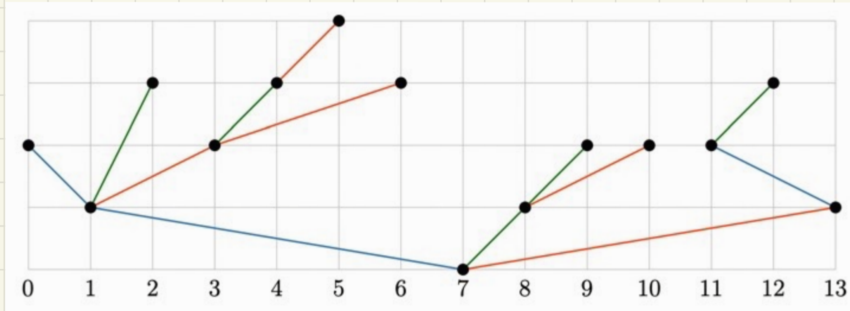
(1) $x \to y$ blue $\Rightarrow$ y & above left of x

(2) $x \to y$ green $\Rightarrow$ y & above right of x

(3) $x \to y$ red $\Rightarrow$ y & above right of x + right of all in green branch of x

$\Delta'_n \to$ trees $\to NC \to CC$

E.g. An admissably ordered tricolored tree :



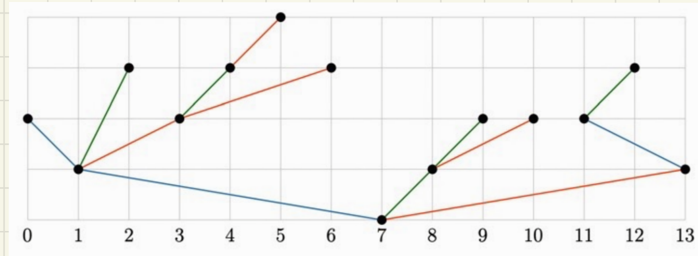
- Write $x \sim y$ when the path from x to y has no blue (just red/green), $x \to y$ when y is descended from x.
- Define $\pi_{R'}(x) := \max\{y \mid x \sim y\}$

$$\pi_R(x) := \max\left\{ y \,\middle|\, \begin{array}{l} x \sim y,\ x \to y,\ \text{\& either } x = y \text{ or the path} \\ \text{from } x \text{ to } y \text{ begins with green} \end{array} \right\}$$

E.g.

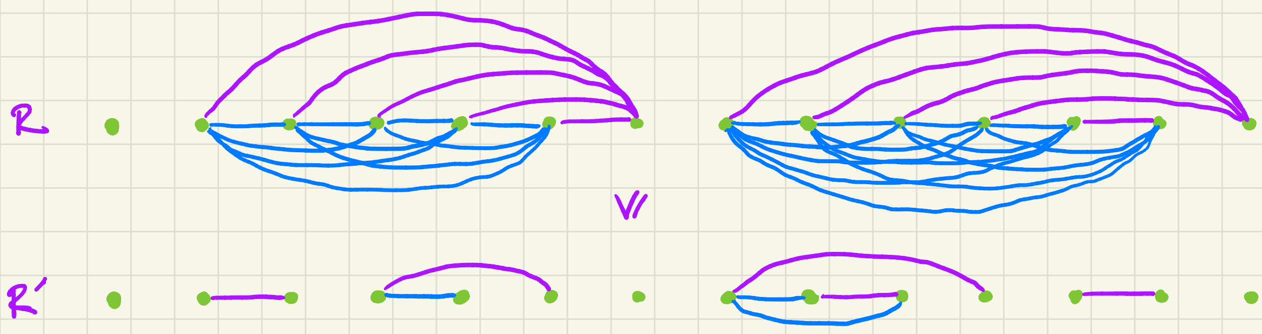
x	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$\pi_{R'}(x)$	0	6	6	6	6	6	6	13	13	13	13	12	12	13
$\pi_R(x)$	0	2	2	5	4	5	6	10	9	9	10	12	12	13

$\Delta'n \to \text{trees} \to NC \to CC$



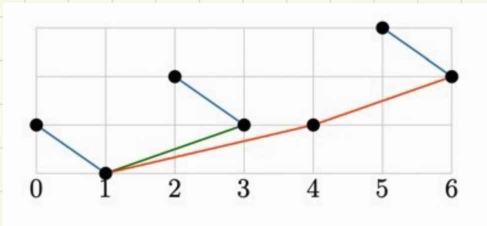
x	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$\pi_{R'}(x)$	0	6	6	6	6	6	6	13	13	13	13	12	12	13
$\pi_R(x)$	0	2	2	5	4	5	6	10	9	9	10	12	12	13

(— = max'l rel'ns ⇝ NC part'n, — = gen'd rel'ns)



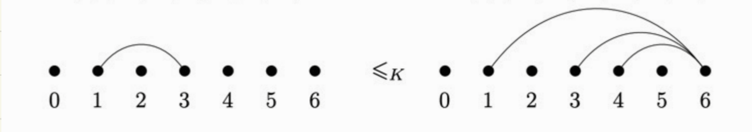
Model trees

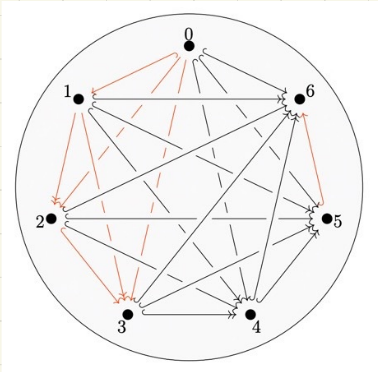
<u>Thm</u> (BMO) Via these bijections, model structures on $[n]$ correspond to "blue-green trees growing from a red field." This recovers the enumeration of $Q([n])$ from BOOR.

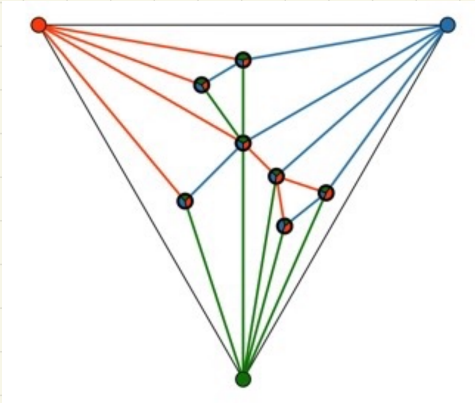


x	0	1	2	3	4	5	6
$\pi_{\mathcal{R}}(x)$	0	3	2	3	4	5	6

x	0	1	2	3	4	5	6
$\pi_{\mathcal{R}'}(x)$	0	6	2	6	6	5	6







Saturated Transfer Systems

A transfer system is saturated when it satisfies $2 \Rightarrow 3$.

Thm (Rubin) Transfer systems on $\mathrm{Sub}(G)$ induced by G-linear isometries operads are saturated.

Thm (BMO) For a finite self-dual lattice P, the following structures are in bijective correspondence:

- saturated transfer systems,
- model str's in which all morphisms are fibrations,
- closure operators on P,
- submonoids of $(P, \wedge)$, &
- monads on P.

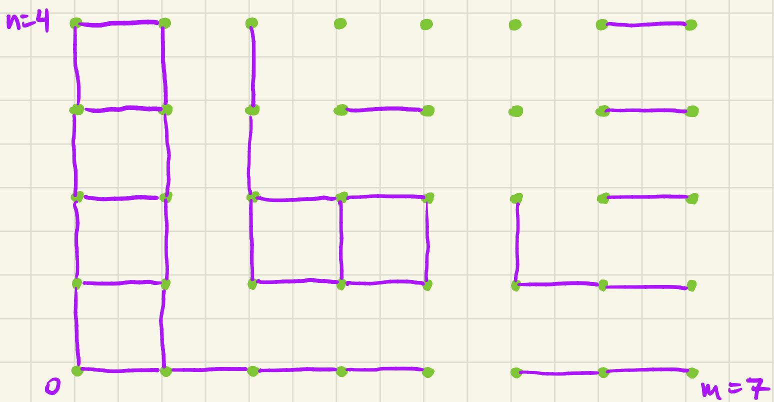
Thm (Hafeez-Marcus-O-Osorno) Saturated transfer systems are generated by covering relations.

Matchsticks Again

Thm (HM00) $\mathrm{Tr}^{sat}([m]\times[n]) \cong \{\text{legal matchstick config'ns on } [m]\times[n]\}$.

There are precisely $s(m,n) := \boxed{\frac{1}{2}\sum_{j=2}^{m+2} (-1)^{m-j} \left\{ {m+1 \atop j-1} \right\} j!\, j^n}$ of these, and the exponential generating function for $s(m,n)$ takes the form

$$\sum_{m,n\geq 0} \frac{s(m,n)}{m!\,n!} x^m y^n = \frac{\exp(2x+2y)}{(\exp(x)+\exp(y)-\exp(x+y))^3}.$$



$n \backslash m$	0	1	2	3
0	1	2	4	8
1	2	7	23	78
2	4	23	115	533
3	8	73	533	3451

$\left\{ {k \atop \ell} \right\}$ = Stirling number of the second kind = # ℓ-block partitions of a k-element set

$s(7,4) = 58\,718\,873$

Summary + Q's

On a finite lattice P,

- $\mathrm{WFS}(P) \cong \mathrm{Tr}(P)$
- $\mathrm{Pre}(P) \cong \mathrm{Int}(\mathrm{Tr}(P), \leq)$
- $\mathrm{CC}(P) \cong \mathrm{Int}(\mathrm{Tr}(P), \preccurlyeq)$
- $|\mathrm{WFS}(P)| = \mathrm{Cat}(n+1)$
- $|\mathrm{Pre}[n]| = \frac{2}{(n+1)(n+2)} \binom{4n+5}{n}$
- $|\mathrm{CC}([n])| = \frac{1}{2n+3} \binom{3n+3}{n+1}$
- $|Q([n])| = \binom{2n+1}{n}$
- Kreweras intervals, stacked Δ'ns, & tricolored trees for $\mathrm{CC}([n])$
- $\mathrm{Tr}^{sat}(P) \cong \{(W, \mathrm{All}, c) \in Q(P)\} \cong \{\text{closure operators on } P\} \cong \{\text{submonoids of } (P, \wedge)\}$
- $|\mathrm{Tr}^{sat}([m] \times [n])| = s(m,n) = |\{\text{legal matchstick config'ns}\}|$

<u>Questions</u>

- Are transfer systems on lattices already "out there"?
- Connections to generalized Catalan combinatorics / associahedra / cluster algebras / representation theory?
- $Q(P)$ for other families of lattices P? $P = [1]^n$?

Thank you!

- Self-duality of the lattice of transfer systems via weak factorization systems, Franchere-O-Osorno-Qin-Waugh
- Model structures on finite total orders, Balchin-O-Osorno-Roitzheim
- Saturated and linear isometric transfer systems for cyclic groups of order $p^m q^n$, Hafeez-Marcus-O-Osorno
- Composition closed premodel structures and the Kreweras lattice, Balchin-MacBrough-O
- Lifting N_∞ operads from conjugacy data, Balchin-MacBrough-O
- The combinatorics of N_∞ operads for C_{qp^n} and D_{p^n}, Balchin-MacBrough-O

- Access at kyleormsby.github.io/research/.