ECHT REU

An online math REU organized by the Electronic Computational Homotopy Theory Research Community

June 12 - July 21, 2023
Fully Virtual
Minimal Prerequisites

Homotopical Combinatorics: Model Structures and Transfer Systems on Lattices

Program Details: s.wayne.edu/echt/reu-2023/
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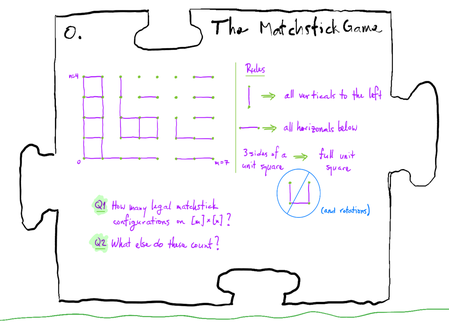
Kyle Ormsby | UW Combinatorics Seminar
12.I.23

Homotopical Combinatorics

0. A puzzle
1. What is homotopy theory?
2. Weak factorization systems
3. Model structures
4. An invitation

0. The Matchstick Game



What is a homotopical context?

Category $\mathcal{C}$

$W, F, C \subseteq \operatorname{Mor}\mathcal{C}$ (weak equivalences, fibrations, cofibrations)

- Want to understand $\mathcal{C}[W^{-1}]$
- F, C + axioms provide control over localization

E.g.
- $\mathcal{C} = \mathrm{Top}$ or $\Delta^{op}\mathrm{Set}$, $W = \pi_*$-isos
- $\mathcal{C} = \mathrm{Ch}_{\geq 0}(R\text{-Mod})$, $W = H_*$-isos

$(\mathcal{C}, W, F, C)$ model category (model structure on $\mathcal{C}$)

And why?

- Fibrant/cofibrant replacement mimics injective/projective resolution
- Bousfield localization: invert more morphisms, retain cofibrations
- Quillen adjunctions/equivalences compare model categories

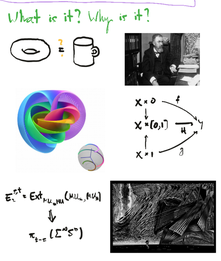
E.g. (Quillen, Sullivan, Bousfield-Gugenheim 1967-77)

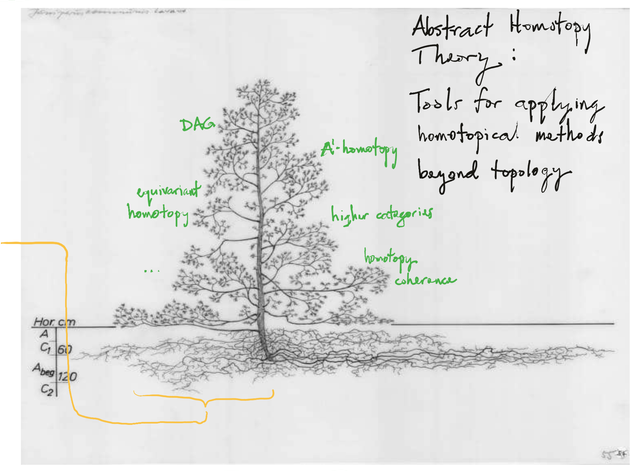
$$\begin{array}{ccc} \mathrm{Top} \simeq \Delta^{op}\mathrm{Set} & \rightleftarrows & \mathrm{dgca}^{\geq 0}_{\mathbb{Q}} \\ \cup\!\!\mid & & \cup\!\!\mid \\ \mathrm{Top}^{nft} & \underset{L_{\mathbb{Q}}}{\rightleftarrows} & \mathrm{Top}^{nft}_{\mathbb{Q}} \end{array}$$

nft = nilpotent finite type

1. Homotopy Theory

What is it? Why is it?



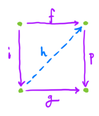


GOAL

Classify and count model structures on lattice posets.

- Poset $(P, \leq)$ is a set P with a transitive, reflexive, anti-symmetric relation $\leq$, corresponding category P has objects P, morphisms $P(x,y) = \begin{cases} * & \text{if } x \leq y \\ \emptyset & \text{o/w.} \end{cases}$
- P is a lattice if it admits all meets $\wedge$ = limits and joins $\vee$ = colimits

2. Weak Factorization Systems



Say i has the left lifting property wrt p (and p has the right lifting property wrt i) and write $i \boxtimes p$ when for $\forall f, g$ s.t. $\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "i"'] & \cdot \arrow[d, "p"] \\ \cdot \arrow[r, "g"'] \arrow[ur, dashed, "h"] & \cdot \end{tikzcd}$ commutes, $\exists h$ such that $f = hi$, $g = ph$.

For $M \subseteq \operatorname{Mor}\mathcal{C}$, write ${}^{\boxtimes}M := \{i \mid i \boxtimes p \ \forall p \in M\}$, $M^{\boxtimes} := \{p \mid i \boxtimes p \ \forall i \in M\}$.

A weak factorization system on $\mathcal{C}$ consists of $(L, R) \subseteq \operatorname{Mor}\mathcal{C} \times \operatorname{Mor}\mathcal{C}$ such that
- WF1) $\operatorname{Mor}\mathcal{C} = R \circ L$ i.e. $\begin{tikzcd} \cdot \arrow[rr, "f"] \arrow[dr, "i"'] & & \cdot \\ & \cdot \arrow[ur, "p"'] & \end{tikzcd}$ with $i \in L, p \in R$,
- WF2) $L \boxtimes R$ i.e. $L \subseteq {}^{\boxtimes}R$ and $R \subseteq L^{\boxtimes}$, and
- WF3) L, R closed under retracts.

Thm (Franchere-O-Osorno-Qin-Waugh)
If P is the category associated with a lattice, then R is the right set of a WFS on P iff
- R is a wide subcategory of P closed under pullbacks in P

or equivalently
- R is a subposet of P satisfying $x \to y \in R,\ z \in P \implies x \wedge z \to y \wedge z \in R$. (meets computed in P)

We have $\mathrm{WFS}(P) = \{({}^{\boxtimes}R, R) \mid R \text{ as above}\}$.

Subposet: same set, potentially fewer relations (e.g. $(\mathbb{Z}_+, |) \subseteq (\mathbb{Z}_+, \leq)$).

Ordering $\mathrm{WFS}(P)$ by relation inclusion results in a lattice.

Let $P = [n] = \{0 < 1 < \cdots < n\}$.

Thm (Balchin-Barnes-Roitzheim)
There is a lattice isomorphism
$\mathrm{WFS}([n]) \cong$ Tamari lattice $\mathbb{A}_n$ of planar full binary trees with $n+2$ leaves (under rotation $\curlyvee \to \curlyvee$).

As such, $|\mathrm{WFS}([n])| = \underbrace{\mathrm{Cat}(n+1)}_{(n+1)\text{-st Catalan number}} = \frac{1}{n+2}\binom{2n+2}{n+1}$.

3. Model structures

A premodel structure (Barton) on $\mathcal{C}$ is an interval of weak factorization systems on $\mathcal{C}$

$$\begin{array}{ccc} (\mathcal{C}, & \mathrm{AF}) & \text{cofibrations, anodyne fibrations } = {}^{\boxtimes}\mathrm{AF} \\ \cup\!\!\mid & \cap\!\!\mid & \\ (\mathrm{AC}, & \mathrm{F}) & \text{anodyne cofibrations, fibrations } = {}^{\boxtimes}\mathrm{F} \end{array}$$

$\mathrm{Pre}(\mathcal{C}) = \mathrm{Int}(\mathrm{WFS}(\mathcal{C}))$ is the lattice of premodel structures on $\mathcal{C}$.

For finite chains we now have

$$\mathrm{Pre}([n]) = \mathrm{Int}(\mathrm{WFS}([n])) \cong \mathrm{Int}(\mathbb{A}_{n+1}).$$

Thm (Chapoton) $|\mathrm{Int}\,\mathbb{A}_n| = \frac{2}{n(n+1)}\binom{4n+1}{n-1}$

Cor (Balchin-O-Osorno-Roitzheim)
$$|\mathrm{Pre}[n]| = \frac{2}{(n+1)(n+2)}\binom{4n+5}{n}$$

Thm (Joyal-Tierney) For a bicomplete category $\mathcal{C}$, $\{((\mathcal{C}, \mathrm{AF}), (\mathrm{AC}, \mathrm{F})) \in \mathrm{Pre}(\mathcal{C}) \mid \mathrm{AF} \circ \mathrm{AC} \text{ satisfies } 2 \Rightarrow 3\}$ is in bijection with model structures on $\mathcal{C}$.

$2 \Rightarrow 3$: If two of f, g, gf belong to $\mathrm{AF} \circ \mathrm{AC}$, then so does the third.

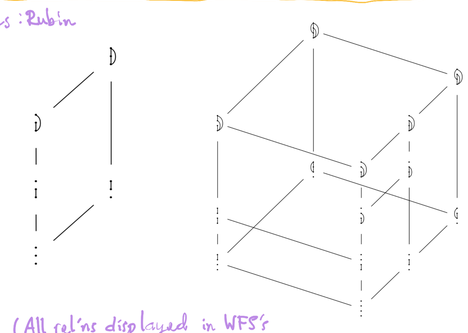
The bijection is given by

$$\begin{array}{ccc} (\mathcal{C}, \mathrm{AF}) & & \\ \cup\!\!\mid \ \cap\!\!\mid & \longmapsto & (\mathrm{AF} \circ \mathrm{AC}, \mathrm{F}, \mathcal{C}) \\ (\mathrm{AF}, \mathrm{F}) & & \end{array}$$

$$\begin{array}{ccc} (\mathcal{C}, W \cap \mathrm{F}) & & \\ \cup\!\!\mid \ \cap\!\!\mid & \longleftarrow & (W, \mathrm{F}, \mathcal{C}) \\ (W \cap \mathrm{F}, \mathrm{F}) & & \end{array}$$

Note Quillen's original definition of model structure used (redundant!) axioms MC1 - MC5 on W, F, C.

Images: Rubin



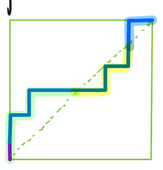
(All rel'ns displayed in WFS's, not just covering relations.)

Thm (BOOR) The number of model structures on $[n]$ is $|\mathcal{Q}([n])| = \binom{2n+1}{n}$, and for $0 \leq k \leq n$, $\frac{2(k+1)}{n+k+2}\binom{2n+1}{n-k}$ of these have homotopy category $\cong [k]$.

Pf Idea Specify k s.t. $[k] \cong \mathrm{Ho}([n])$, then specify $k+1$ weak equivalence classes, then count choices of contractible model structures:

$$|\mathcal{Q}([n])| = \sum_{k=0}^{n} \underbrace{\sum_{\substack{i_0 + \cdots + i_k \\ = n+1}} \prod_{j=0}^{k+1} \mathrm{Cat}(i_j)}_{\text{Shapiro (1976): } \frac{2(k+1)}{n+k+2}\binom{2n+1}{n-k}} = \binom{2n+1}{n}.$$

Final equality by lattice paths $(0,0)$ to $(n+1, n+1)$ with initial step north:
- $(i_0, \ldots, i_k)$ give diagonal x-ings
- Dyck paths have Catalan count

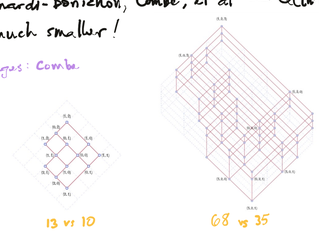


3'. Composition closed premodel structures

Fill the gap between $\mathrm{Pre}(P)$ & $\mathcal{Q}(P)$.

$\mathrm{Int}(\mathbb{A}_n)$ is well-studied by Chapoton, Bernardi-Bonichon, Combe, et al — $\mathcal{Q}([n-1])$ is much smaller!

Images: Combe



Call a premodel structure $\begin{array}{ccc} (\mathcal{C}, & \mathrm{AF}) \\ \cup\!\!\mid & \cap\!\!\mid \\ (\mathrm{AC}, & \mathrm{F}) \end{array}$ on $\mathcal{C}$ composition closed when $W := \mathrm{AF} \circ \mathrm{AC}$ is closed under composition. (Need $2 \Rightarrow 3$ for model str.)

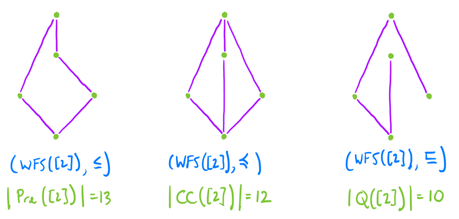
For $C = P$ a lattice, write $R \leq R'$ for a premodel str / interval of transfer systems.

Thm (Balchin-MacBrough-O) For P a complete lattice, $\exists$ partial order $\lesssim$ on $\mathrm{WFS}(P)$ refining $\leq$ and such that $R \lesssim R'$ if and only if $R \circ {}^{\boxtimes}R'$ is closed under composition. Moreover, $R \lesssim R'$ iff every square $\begin{tikzcd} x \arrow[r] \arrow[d, "R"'] & z \arrow[d, "R'"] \\ y \arrow[r] & w \end{tikzcd}$ has a splitting $\begin{tikzcd} x \arrow[r] \arrow[d, "R"'] & z' \arrow[r] \arrow[d, "R"'] & z \arrow[d, "R'"] \\ y \arrow[r] & w' \arrow[r, "R'"'] & w \end{tikzcd}$.

Thm (ct'd) If P is finite, then $(\mathrm{WFS}(P), \lesssim)$ is a finite lattice. Thus $\underbrace{\mathrm{CC}(P)}_{\text{composition closed premodel structures}} = \mathrm{Int}(\mathrm{WFS}(P), \lesssim)$ is a finite lattice.

Note There is also an ordering $\sqsubseteq$ on $\mathrm{WFS}(P)$ such that $R \sqsubseteq R'$ iff the pair forms a model structure, but $(\mathrm{WFS}(P), \sqsubseteq)$ is not a lattice.

E.g.



Specializing to $P = [n]$ again, $\lesssim$ recovers the Kreweras order on noncrossing partitions.

For $R \in \mathrm{Tr}[n]$, define $\pi_R : [n] \to [n]$, $i \mapsto \max\{j \mid iRj\}$.

$[n] = B_1 \sqcup \cdots \sqcup B_k$ is NC when $a, b \in B_i$, $x, y \in B_j$, and $a < x < b < y \implies i = j$.

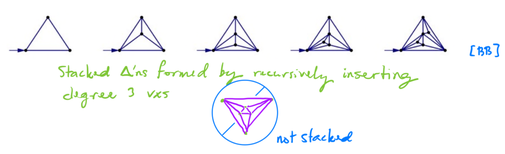
Prop (FOOQW) Write NC_{n+1} for the set of noncrossing partitions of $\{0, 1, \ldots, n\}$. Then $\mathrm{Tr}[n] \to \mathrm{NC}_{n+1}$, $R \mapsto \{\pi_R^{-1}\{i\} \mid 0 \leq i \leq n,\ \pi_R^{-1}\{i\} \neq \emptyset\}$ is a bijection.

Thm (BMO) Pulling back the Kreweras (refinement) order on NC_{n+1} to $\mathrm{Tr}[n]$ gives the ordering $\lesssim$.

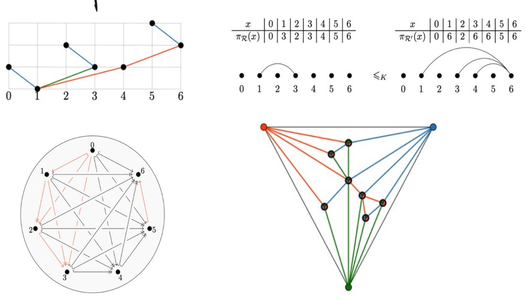
Cor $\mathrm{CC}([n]) \cong \mathrm{Int}(\mathrm{NC}_{n+1})$ and thus
$$|\mathrm{CC}([n])| = \underbrace{\frac{1}{2n+3}\binom{3n+3}{n+1}}_{\text{Kreweras: } = |\mathrm{Int}(\mathrm{NC}_{n+1})|}.$$

The following structures are equinumerous:
- Composition closed premodel structures on $[n]$
  - BMO
- Kreweras intervals in NC_{n+1}
  - Bernardi-Bonichon
- (admissably ordered) ternary trees on $n+1$ nodes
  - BB
- stacked triangulations

(identify model str's among these)



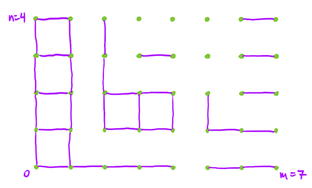
E.g. Via our bijections, the following structures all correspond to the same model structure



Has anyone solved the matchstick problem?

Legal matchstick config'ns on $[m] \times [n]$ correspond to saturated transfer systems and to model structures in which all morphisms are fibrations.

Thm (Hafeez-Marcus-O-Osorno) The number $s(m,n)$ of such model structures on $[m] \times [n]$ is
$$s(m,n) = \frac{1}{2}\sum_{j=2}^{m+2} (-1)^{m-j} \left\{ {m+1 \atop j-1} \right\} j! \cdot j^n$$
with exponential generating function
$$\sum_{m,n \geq 0} \frac{s(m,n)}{m!\,n!} x^m y^n = \frac{\exp(2x+2y)}{(\exp(x) + \exp(y) - \exp(x+y))^3}.$$



n \ m	0	1	2	3
0	1	2	4	8
1	2	7	23	78
2	4	23	115	533
3	8	73	533	3451

$s(7,4) = 58\ 718\ 873$

Thm (BMO) On a finite self-dual lattice P, the following structures are in bijective correspondence:
- saturated transfer systems,
- model structures with $F = \mathrm{All}$,
- closure operators,
- closure systems,
- submonoids of $(P, \wedge)$,
- monads on P.

(R. Liu) For $P = [m] \times [n]$, this is half the number of $\lrcorner$-diagrams of shape $[m] \times [n]$.

(Also interpretable in terms of acyclic orientations of a complete bipartite graph, hyperplane arrangements, etc.)

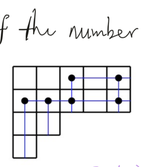
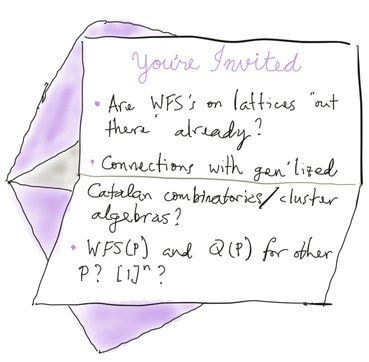


Image: Postnikov

4. An Invitation



And more: iterative constructions, power series, eDAG, etc.

References:

- Self-duality of the lattice of transfer systems via weak factorization systems, Franchere-O-Osorno-Qin-Waugh
- Model structures on finite total orders, Balchin-O-Osorno-Roitzheim
- Saturated and linear isometric transfer systems for cyclic groups of order $p^n q^n$, Hafeez-Marcus-O-Osorno
- Composition closed premodel structures and the Kreweras lattice, Balchin-MacBrough-O
- Lifting N_∞ operads from conjugacy data, Balchin-MacBrough-O
- The combinatorics of N_∞ operads for C_{p^n} and D_{p^n}, Balchin-MacBrough-O

- Access at kyleormsby.github.io/research/.