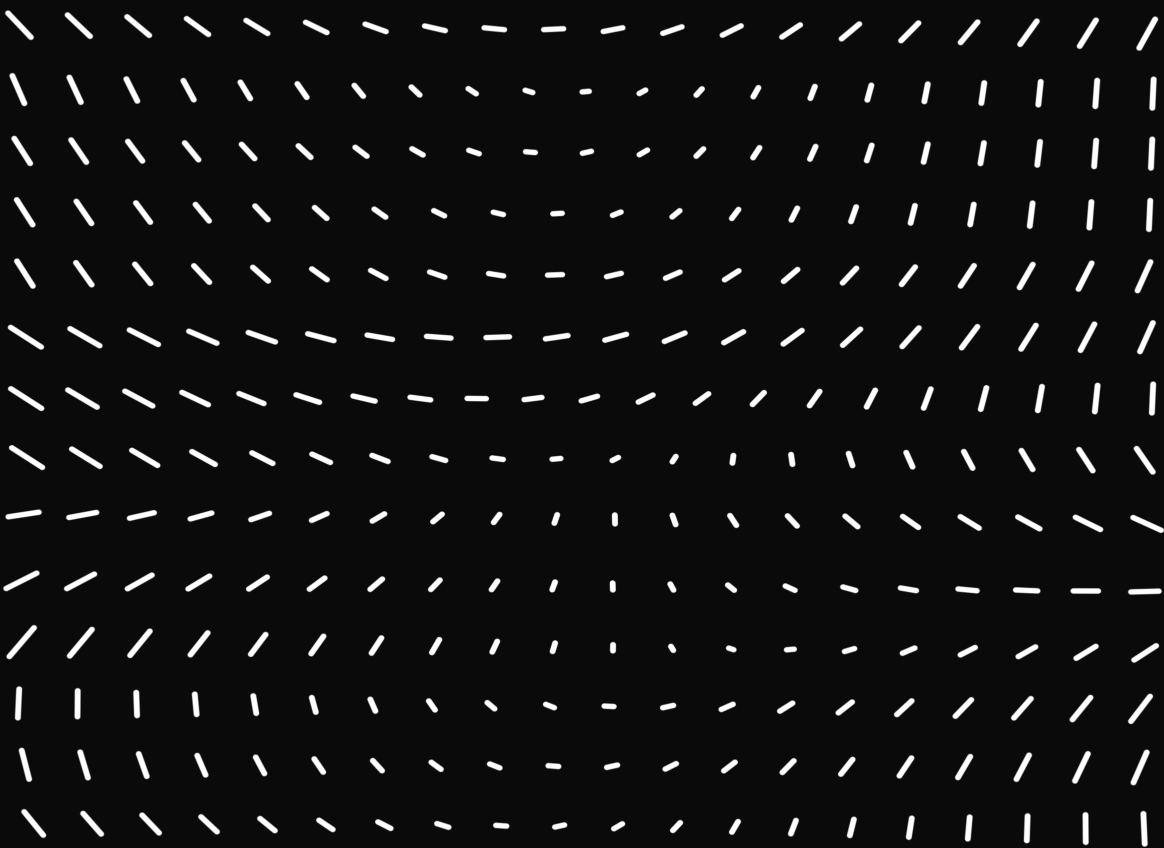
PCMI UFP 2021

Milnor forms and algebraic singularities



2 Aug 2021

The second derivative test meets singularities and Milnor numbers

Thm (2nd derivative test) Suppose $f: \mathbb{R}^2 \to \mathbb{R}$ has continuous 2nd partial derivatives and that $p = (a, b)$ is a critical point of f.

$\nabla f(p) = 0$

Let

$$Hf(p) = \left.\begin{pmatrix} \partial^2 f/\partial x^2 & \partial^2 f/\partial x \partial y \\ \partial^2 f/\partial y \partial x & \partial^2 f/\partial y^2 \end{pmatrix}\right|_p$$

$$= \begin{pmatrix} \alpha & \beta \\ \beta & \delta \end{pmatrix}.$$

Then

- $\alpha > 0$, $\alpha\delta - \beta^2 > 0 \implies f(p)$ local min
- $\alpha < 0$, $\alpha\delta - \beta^2 > 0 \implies f(p)$ local max
- $\alpha\delta - \beta^2 < 0 \implies f(p)$ saddle point.

$\square$

Quadratic form

$$Qf_p : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$(x,y) \longmapsto (x \;\; y)\, Hf(p) \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \alpha x^2 + 2\beta xy + \gamma y^2$$

By Taylor approx $\exists\, c \in [0,1]$ s.t. for h,k small

$$f(p+(h,k)) = f(p) + \frac{1}{2} Qf_{p+c(h,k)}(h,k)$$

When Qf_p is nondegenerate

$\alpha\gamma - \beta^2 \neq 0$ —— the shape of

is the same for small h,k

$\Longrightarrow$ 2nd deriv test $\square$

Note (1) This works for $f: \mathbb{R}^n \to \mathbb{R}$ as well.

$$Hf(p) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)$$ is symmetric and

$\exists$ invertible matrix S s.t. SHS^T is diagonal w/ entries $0, +1, -1$ on diag.

Triple (n_0, n_+, n_-) of # of such entries is Sylvester type of Qf_p

The form is nondegen iff $n_0 = 0$.

$(0, \underbrace{\cancel{p}, \cancel{q}}_{n_+,\ n_-})$ $\qquad \underbrace{\cancel{p-q}}_{n_+ - n_-}$ =: signature of form Qf_p

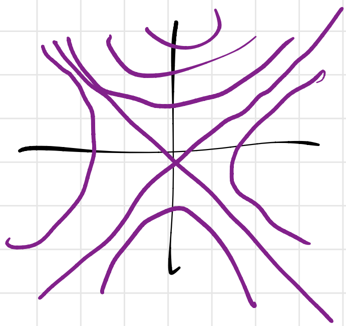
Dimn + signature classifies Qf_p

$\Longrightarrow$ classifies crit pt in nondegen case.

(2) $\underline{n=2}$ Look at level curves of f:

$$\{(x,y) \mid f(x,y) = d\}$$

crit pts of f corr to singularities of level sets



(3) Other fields: $k = \mathbb{C}, \mathbb{Q}, \mathbb{F}_p$

Local min/max — makes less sense

But: level curves or hypersurfaces

If f is polynomial, get algebraic varieties. Want to understand "shape" of singularities.

(4) Need: algebraic thy of quad forms

Milnor form $= \mathbb{A}^1$-Milnor number

is a quad form over k will tell us about this shape.

rank Milnor form = Milnor number

Milnor numbers

$f \in \mathbb{C}[x_1, x_2, \dots, x_n]$ polynomial

$n=2$ is good!

$$V(f) := \{x \in \mathbb{C}^n \mid f(x) = 0\} \subseteq \mathbb{C}^n$$

A singular point of $V(f)$ is $p \in \mathbb{C}^n$ s.t.

$$f(p) = 0, \quad \nabla f(p) = 0.$$

$\mathrm{Sing}(f) :=$ set of singular pts

$V(f) \setminus \mathrm{Sing}(f)$ are regular points.

<u>Goal</u> local topology of $V(f)$ near a singular point.

$p \in \mathrm{Sing}(f)$ which is isolated: $\exists$ nbhd of p in $V(f)$ with no other singular pts.

Look at "slices" of $V(f)$ near p.

Let $\varepsilon > 0$, $S^{2n-1}_{\varepsilon}(p) = \{x \in \mathbb{C}^n \mid \|x-p\| = \varepsilon\}$.

$$K_{p,\varepsilon}(f) := S^{2n-1}_{\varepsilon}(p) \cap V(f)$$

since p is isolated singularity, $K_{p,\varepsilon}(f)$ contains only regular pts for small ε.

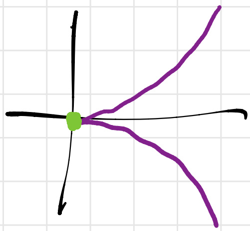
<u>$n=2$</u> real dim'n of $S^{2n-1}_{\varepsilon}(p)$ is 3

" $V(f)$ is 2

intersection $K_{p,\varepsilon}(f)$ has real dimn 1 object — a ~~knot~~ link in $S^3_\varepsilon(p)$.

Eg: $f(x,y) = x^3 - y^2$.

The real points of $V(f)$ are a cusp



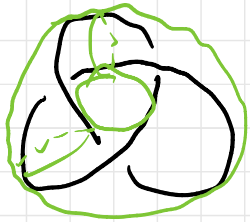
$$\mathrm{Sing}(x^3-y^2) = \{(a,b) \mid (3a^2, -2b) = (0,0)\}$$

$$= \{(0,0)\}$$

$K_{0,\varepsilon}(f)$ lies on a copy of $S^1 \times S^1$:

$$\{(x,y) \mid \|x\| = \varepsilon, \|y\| = \eta\}$$

This is in fact a $(2,3)$ torus knot.



For $x^m - y^n$ for m, n relatively prime:

$$K_{0,\varepsilon}(x^m - y^n) = (m,n)\text{-torus knot.}$$

Milnor proves $K_{p,\varepsilon}(f)$ is independent of ε for ε small.

Let $D^{2n}_\varepsilon(p)$ = closed disc of radius ε

$$D^{2n}_\varepsilon(p) \cap V(f) \cong C(K_{p,\varepsilon}(f))$$

Milnor map

$$M_f : S^{2n-1}_\varepsilon(p) \setminus K_{p,\varepsilon}(f) \longrightarrow S^1 \subseteq \mathbb{C}$$

$$x \longmapsto \frac{f(x)}{\|f(x)\|}$$

M_f is a fiber bundle with fibers

$$F_\Theta := M_f^{-1}\{e^{i\Theta}\}$$

diffeomorphic smooth parallelizable $(2n-2)$-dim'l manifolds.

$\overline{F_\Theta} = F_\Theta \cup K_{p,\varepsilon}(f)$ is a mfld w/ boundary $K_{p,\varepsilon}(f)$.

Thm. $\overline{F_\Theta}$ is homotopy equiv to a bouquet of $(n-1)$-diml spheres

$$\overline{F_\Theta} \simeq \underbrace{S^{n-1} \vee S^{n-1} \vee \cdots \vee S^{n-1}}_{\mu}$$

The middle homology $H_{n-1}(\overline{F_\Theta}; \mathbb{Z})$ is free of rank μ.

Defn The Milnor number of f at p

if $\mu_p(f) := \mu$.

μ measures degeneracy of our singularity

$$Hf(p) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p) \right)_{i,j}$$

crit pt p is nondegenerate iff

$Hf(p)$ is nonsingular

Claim This happens iff the multiplicity of ∇f at p is 1:

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right)$$

$$V(\nabla f) = V\left(\frac{\partial f}{\partial x_1}\right) \cap \cdots \cap V\left(\frac{\partial f}{\partial x_n}\right)$$

i.e. $V(\nabla f)$ is smooth at p

iff Jacobian of ∇f is nonsingular at p

iff $Hf(p)$ is nonsingular.

Upshot Multiplicity of int'n of $\frac{\partial f}{\partial x_i}$ measures degeneracy.

Local int'n number of $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$

is $\dim_{\mathbb{C}} \underbrace{\mathbb{C}[x_1, \ldots, x_n]_p}_{\substack{\text{localization at} \\ (x_1 - p_1,\, x_2 - p_2,\, \ldots,\, x_n - p_n) \\ = \text{subring of } \mathbb{C}(x_1, \ldots, x_n) \\ \text{w/ denominator doesn't vanish} \\ \text{at } p.}} \Big/ \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right)$

Note p is isolated iff this int'n # is finite

Thm This multiplicity $= \mu_p(f)$.

Other faces of μ:

Thm $\chi(\bar{F}_\theta) = 1 + (-1)^{n-1}\mu_p(f)$

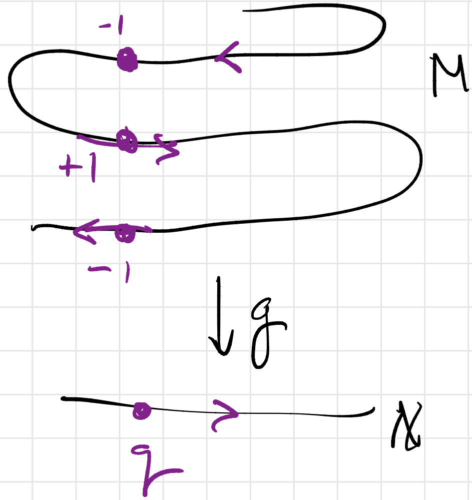
Can also access μ via degree:

$g : M \longrightarrow N$ b/w smooth orientable mflds of same dimn: Give M, N charts w/ compatible orientable orientations

Given $p \in M$ at which g is regular define local degree of g at p to be $\deg_p(g) := \text{sign}(\det(dg)_p)$
$\in \{\pm 1\}$.

If $q \in N$ is a regular value,

$$\deg(g) = \sum_{p \in g^{-1}\{q\}} \deg_p(g)$$



$\deg(g) = -1+1-1$
$= -1$

In our case $\nabla f : \mathbb{C}^n \longrightarrow \mathbb{C}^n$

⚠ ∇f is not smooth at degenerate crit pts!

Look at

$\dfrac{\nabla f}{\|\nabla f\|}$ on a small sphere $S^{2n-1}_{\varepsilon}(p) \to S^{2n-1}$

Take degree of $\left.\dfrac{\nabla f}{\|\nabla f\|}\right|_{S^{2n-1}_{\varepsilon}(p)} =: \deg_p(\nabla f)$

Thm $\mu_p(f) = \deg_p(\nabla f)$

$\mu_p(f)$: topologically — middle homology of F_θ
— $\chi(F_\theta)$
— $\boxed{\text{local degree of } \nabla f}$

algebraically / geometrically : $\boxed{\dim \mathbb{C}[x_1, \dots, x_n]_p \,/\, (\nabla f)}$

3 Aug 2021

Quadratic forms and the Grothendieck-Witt ring

Fix k a field of char $\neq 2$.

<u>Defn</u> A symmetric bilinear form over k is a k-vs V and function $b: V \times V \longrightarrow k$ s.t.

(1) $b(v,w) = b(w,v) \quad \forall v, w \in V$

(2) linear in each variable

Given k-sbf b, define $q = q_b : V \longrightarrow k$
$$v \longmapsto b(v,v).$$

Then (1) q is homogeneous of degree 2
$$q(\lambda v) = \lambda^2 q(v) \quad \forall \lambda \in k, v \in V$$

(2) the polarization of q recovers b
$$V \times V \longrightarrow k$$
$$(v,w) \longmapsto \tfrac{1}{2}\big(q(v+w) - q(v) - q(w)\big)$$

A function $q: V \to k$ homogeneous of degree 2 with bilinear polarization is a quadratic form.

Choose a basis $e_1, \dots, e_n$ of V. The Gram matrix of b (or q) is

$$G = \big(b(e_i, e_j) \big)_{1 \le i,j \le n} \in \mathrm{Sym}_{n \times n}(k)$$

Fact $b(v,w) = [v]^T G [w]$ for $[v]$ = col vector rep'n of v wrt $e_1, \dots, e_n$.

Given k-sbf's b on V, b' on V', call b, b' isometric when $\exists\, \phi : V \longrightarrow V'$ a linear isomorphism s.t. $b'(\phi v, \phi w) = b(v,w)$ $\forall v, w \in V$

Fact k-sbf's on V are isometric iff their Gram matrices are G, G' congruent:

$$G' = A^T G A \quad \text{for some } A \in GL_n(k)$$

Note Quadratic forms are homogeneous degree 2 polynomials: Gram matrix (b_{ij}) corresponds to

$$q(x_1, \dots, x_n) = (x_1 \cdots x_n)(b_{ij})\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$= \sum_{1 \le i \le j \le n} a_{ij} x_i x_j$$

for $$a_{ij} = \begin{cases} b_{ii} & \text{if } i = j \\ 2b_{ij} & \text{if } i < j \\ \cancel{2b_{ji}} & \cancel{\text{if } i > j} \end{cases}$$

E.g. $ax^2 \cancel{+ 0y^2 + 0z^2}$ $\quad (a)$

$x^2 + y^2 \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$x^2 - y^2 \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$xy \quad \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$

isometric

$x^2 - y^2 = (x+y)(x-y)$

An sbf is nondegenerate (or regular) when the function

$$\begin{aligned} V &\longrightarrow V^* := \mathrm{Hom}_k(V,k) \\ v &\longmapsto \left(\begin{matrix} w \\ \downarrow \\ b(v,w) \end{matrix} \right) \end{aligned}$$

is an isomorphism (iff $\det(\mathrm{Gram}) \neq 0$).

$\cancel{\text{iff } \varphi_b(v) = 0 \Rightarrow v = 0}$

Thm Every sbf is isometric to a diagonal form

$$\langle a_1, \dots, a_n \rangle := a_1 x_1^2 + \cdots + a_n x_n^2 .$$

Pf Idea Complete the square + induction. $\square$

⚠ Diagonalizations are not unique.

Operations b_1, b_2 k-sbfs on V_1, V_2

Orthogonal sum

$$b_1 \perp b_2 : (V_1 \oplus V_2) \times (V_1 \oplus V_2) \longrightarrow k$$

$$((v_1, v_2), (w_1, w_2)) \longmapsto b_1(v_1, w_1) + b_2(v_2, w_2)$$

Kronecker

Tensor product

$$b_1 \otimes b_2 : V_1 \otimes V_2 \times V_1 \otimes V_2 \longrightarrow k$$

$$(v_1\otimes v_2, w_1\otimes w_2) \longmapsto b_1(v_1,w_1)\, b_2(v_2,w_2)$$

On Gram matrices

$$G_{b_1\perp b_2} = \left(\begin{array}{c|c} G_{b_1} & 0 \\ \hline 0 & G_{b_2}\end{array}\right)$$

Note $b_1 \cong b_1'$
$\Rightarrow b_1 \perp b_2 \cong b_1' \perp b_2$,
$b_1 \otimes b_2 \cong b_1' \otimes b_2$

$$G_{b_1\otimes b_2} = \begin{pmatrix} a_{11}G_{b_2} & \cdots & a_{1n}G_{b_2} \\ \vdots & \ddots & \vdots \\ a_{n1}G_{b_2} & \cdots & a_{nn}G_{b_2}\end{pmatrix}$$

for $G_{b_1} = (a_{ij})$

On diagonal forms,

$$\langle a_1,\dots,a_n\rangle \perp \langle b_1,\dots,b_m\rangle = \langle a_1,\dots,a_n,b_1,\dots,b_m\rangle$$

$$\langle a,b\rangle \otimes \langle c,d,e\rangle = \langle ac, ad, ae, bc, bd, be\rangle$$

Let $S(k) := \{\text{regular } k\text{-sbf's}\}/\text{isometry}$.

Then $(S(k), \perp, \otimes)$ is a commutative ~~semiring~~ rig.

which is cancellative:

$$b\perp c \cong b\perp d \Longrightarrow c\cong d$$

Apply the Grothendieck construction: group completion

$$GW(k) := S(k)^{gp} = S(k) \times S(k) \Big/ (b,c) \sim (b',c')$$

for $b \perp c' \cong b' \perp c$

$[b,c]$ is "$b-c$"

to get the Grothendieck-Witt ring of k:

$$a \otimes [b,c] := [a \otimes b, a \otimes c]$$

Thm $GW(k)$ is generated by $\{\langle a \rangle \mid a \in k^\times\}$ subject to relations

(1) $\langle a \rangle = \langle ab^2 \rangle$

(2) $\langle a \rangle \langle b \rangle = \langle ab \rangle$

(3) $\langle a \rangle + \langle b \rangle = \langle a+b \rangle + \langle ab(a+b) \rangle$

$a+b, a, b \in k^\times$

Exc $\Rightarrow$ $\langle a, -a \rangle \cong \langle 1, -1 \rangle =: h$

E.g. (1) If $k^{\boxtimes} := \{a^2 \mid a \in k^\times\}$ is all of $k^\times$, then

$\langle a \rangle = \langle 1 \rangle$

so $GW(k) \cong \mathbb{Z}$.

$GW(\mathbb{C}) \cong \mathbb{Z}$

$\Rightarrow q \otimes h \cong (\operatorname{rank} q) h$

(2) $GW(\mathbb{R}) \cong \{(\overset{\text{dimn}}{n}, \overset{\text{signature } n_+ - n_-}{s}) \in \mathbb{Z}\times\mathbb{Z} \mid n+s \equiv 0 \pmod 2\}$

$\cong \mathbb{Z}[h]/(h^2-2h)$ Exc

poly ring over $\mathbb{Z}$ w/ variable h

$\mathbb{Z}[x]/(x^2-2x)$

h = hyperbolic plane

(3) For k finite, $GW(k) \cong \overset{\text{additively dimn}}{\mathbb{Z}} \times \overset{\text{discriminant} = \det(\text{Gram})}{k^\times/k^{\times 2}}$

$\cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

(4) $GW(\mathbb{Q})$ is not finitely generated!

<u>Defn</u> The Witt ring of k is $W(k) := GW(k)/(h)$

$= GW(k)/\mathbb{Z}h$

Its elements are in bijective correspondence with isometry classes of anisotropic sbf's.

<u>E.g.</u> (1) $W(\mathbb{C}) \cong \mathbb{Z}/2\mathbb{Z}$

(2) $W(\mathbb{R}) \cong \mathbb{Z}$

$$\langle 1,-1\rangle \longmapsto 0$$

$$\begin{tikzcd} GW(\mathbb{R}) \arrow[r] \arrow[r, bend right, "\text{sgn}"'] & W(\mathbb{R}) \cong \mathbb{Z} \end{tikzcd}$$

(3) $W(\mathbb{F}_q) \cong \begin{cases} \mathbb{Z}/4\mathbb{Z} & q \equiv 3 \pmod 4 \\ \mathbb{F}_2[k^\times/k^{\times 2}] & q \equiv 1 \pmod 4 \end{cases}$

ring

(4) $W(\mathbb{Q}) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \bigoplus_{p>2} W(\mathbb{F}_p)$

products: quadratic reciprocity!

<u>Fact</u> There is a pullback square of rings

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] \arrow[d] & \mathbb{Z}h \arrow[r, "\cong"] \arrow[d] & 2\mathbb{Z} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & GI(k) \arrow[r] \arrow[d, "\cong"'] & GW(k) \arrow[r, "\mathrm{rank}"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I(k) \arrow[r] \arrow[d] & W(k) \arrow[r, "\mathrm{rank}_2"'] \arrow[d] & \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

<u>Extensions & transfers</u>

Field extension L/k. Get

$$\mathrm{ext}_{L/k} : GW(k) \longrightarrow GW(L) \quad \text{ring map}$$

$$q \longmapsto q$$

for free. $$V \longmapsto V \otimes_k L$$

If L/k finite, have $\mathrm{tr}_{L/k} : L \to k$ the trace map. Then $\mathrm{tr}_{L/k} : GW(L) \to GW(k)$

k-linear

additive hom

$$\left(\begin{array}{c} V\times V \\ b\downarrow \\ L \end{array}\right) \longmapsto \left(\begin{tikzcd} V\times V \arrow[r, "b"] \arrow[dr, "\mathrm{tr}_{L/k}(b)"'] & L \arrow[d, "\mathrm{tr}"] \\ & k \end{tikzcd}\right)$$

is an additive homomorphism.

$\mathrm{tr}_{L/k}(a) = \mathrm{trace}_k(m_a : L \to L)$

$= \sum_{\sigma \in \mathrm{Aut}(L/k)} \sigma(a)$

E.g. $\mathrm{tr}_{\mathbb{C}/\mathbb{R}} : \mathbb{C} \to \mathbb{R}$

$z \longmapsto z + \bar{z} = 2\,\mathrm{Re}(z)$

$GW(\mathbb{C}) \longrightarrow GW(\mathbb{R})$

$$\begin{array}{c} (z,w) \\ \downarrow \\ azw \end{array} \quad \left(\begin{array}{c} \mathbb{C}\times\mathbb{C} \\ \langle a\rangle \downarrow \\ \mathbb{C} \end{array}\right) \longmapsto \left(\begin{tikzcd} \mathbb{C}\times\mathbb{C} \arrow[r, "\langle a\rangle"] \arrow[dr] & \mathbb{C} \arrow[d, "\mathrm{tr}_{\mathbb{C}/\mathbb{R}}"] \\ & \mathbb{R} \end{tikzcd}\right)$$

$(\mathrm{Re}(z), \mathrm{Im}(z), \mathrm{Re}(w), \mathrm{Im}(w)) \mapsto$

" (r,s,t,u)

$a(rt-su, ru+st)$

$a=1$

$2(rt-su)$

Think of V as a k-vs (since $k \subseteq L$)

$\dim(\mathrm{tr}_{L/k}(b))$

rank

$= [L:k] \cdot \dim(b)$

rank

Exc

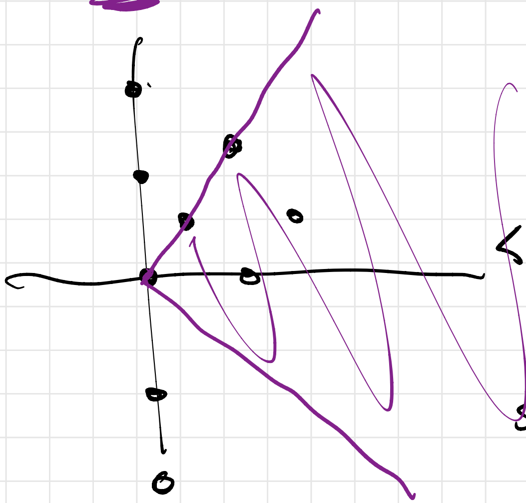
What is $\mathrm{tr}_{\mathbb{C}/\mathbb{R}} \langle 1 \rangle_{\mathbb{C}} \overset{?}{=}$ as diagonalized quad form?

$$GW(\mathbb{R}) \xleftarrow{\ \cong\ } \mathbb{Z}[x]\big/(x^2-2x)$$

$$\langle 1,-1\rangle = h \mapsfrom x$$

$$\langle 1\rangle \mapsfrom 1$$

$$\underline{S(\mathbb{R})} \hookrightarrow GW(\mathbb{R})$$



$\langle 2,4\rangle$

rank 2

signature $2-0$

$\underline{\langle 1,1,1\rangle - \langle -1\rangle}$

rank $3-1=2$

sgn $3-(-1)=4$

All k: $\quad q\otimes h = \mathrm{rank}(q)\cdot h$

$$h\otimes h = 2h = h+h$$

$$h^2 = 2h$$

Milnor conjecture

$$I(k)^n / I(k)^{n+1} \cong K_n^M(k)/(2)$$

Milnor K-theory

4 Aug 2021

Motivic degree

k field of char $\neq 2$

Recall $\mu_p(f) = \deg_p(\nabla f) \in \mathbb{Z}$ for $f \in \mathbb{C}[x_1,\dots,x_n]$

We want $\mu_p^{\mathbb{A}^1}(f) = \deg_p^{\mathbb{A}^1}(\nabla f) \in GW(k)$

for $f \in k[x_1,\dots,x_n]$.

$$\boxed{S^{m+n\alpha} = (S^1)^{\wedge m} \wedge (\mathbb{A}^1 \setminus 0)^{\wedge n} = S^{m+n,n}}$$

Motivic homotopy

(everything pointed)

$$\begin{tikzcd} \underset{\text{smooth } k\text{-schemes}}{Sm_k} \arrow[dr, hook] & \\ & \underset{\text{(motivic) } k\text{-spaces}}{Spc_k} \\ \underset{\text{simplicial sets}}{Set^{\Delta^{op}}} \arrow[ur, hook] & \end{tikzcd}$$

(motivic) k-spaces = simplicial presheaves on Sm_k: $[Sm_k, Set^{\Delta^{op}}]$

Simplicial spheres

$$\boxed{\begin{aligned} \wedge &= \text{smash} \\ \vee &= \text{wedge} = \text{1-pt union} \end{aligned}}$$

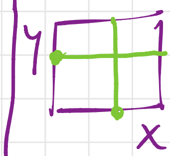
$$S^{n,0} := \underbrace{S^1 \wedge \cdots \wedge S^1}_{n \text{ times}}$$

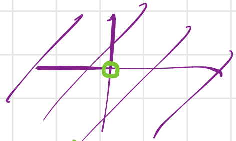
$$\boxed{X \wedge Y := X \times Y \big/ X \times * \cup * \times Y}$$

Geometric spheres

$$S^{1,1} = \mathbb{A}^1 \setminus 0$$

$$S^{n,n} := \underbrace{(\mathbb{A}^1 \setminus 0) \wedge \cdots \wedge (\mathbb{A}^1 \setminus 0)}_{n \text{ times}}$$



$\mathbb{A}^1/\mathbb{C}$  $/\mathbb{R}$ 

$S^{m,n}(\mathbb{C}) = S^{m-n} \wedge (\mathbb{C}\setminus 0)^{\wedge n} \simeq S^{m-n} \wedge S^n \cong S^m$

$S^{m,n}(\mathbb{R}) = S^{m-n} \wedge (S^0)^{\wedge n} = S^{m-n}$

Mixed spheres $(m-n \geq 0)$

$$\underline{S^{m,n}} := S^{m-n,0} \wedge S^{n,n}$$

$$S^{1,0} \wedge S^{1,1} = S^1 \wedge (\mathbb{A}^1 \setminus 0)$$

so m = total # "circles" (simplicial & geometric)

n = # geometric circles.

$\mathbb{A}^1$-homotopy puts a model structure on Spc_k witnessing

- weak equivalences on Nisnevich stalks
- contractibility of $\mathbb{A}^1$

Facts (1) $\mathbb{P}^1 \simeq S^{2,1}$

(2) $\underbrace{\mathbb{A}^n/\mathbb{A}^n \setminus p}_{\text{really the (homotopy) cofiber of } \mathbb{A}^n\setminus p \hookrightarrow \mathbb{A}^n} \simeq S^{2n,n} \simeq (\mathbb{P}^1)^{\wedge n}$

$\mathbb{R}^2 \setminus 0 \simeq \mathbb{R}^2 \setminus D^2$

$\mathbb{R}^2 / \mathbb{R}^2 \setminus D^2 \cong D^2/\partial D^2 = S^2$

THM (Morel) $\exists \deg^{\mathbb{A}^1} : [S^{2n,n}, S^{2n,n}]_{\mathbb{A}^1} \longrightarrow GW(k)$ isomorphism for $n \geq 2$.

$\begin{cases} \mathbb{Z} & k = \mathbb{C} \\ \mathbb{Z}[h]/ & k = \mathbb{R} \end{cases}$

For $\bar{f}: S^{2n,n} \longrightarrow S^{2n,n}$ induced by $f: \mathbb{A}^n \to \mathbb{A}^n$ (h^2-2h)

and q a regular point of f,

$$\begin{tikzcd} p \arrow[r, mapsto] \arrow[d] & q \arrow[d] \\ \mathbb{A}^n / \mathbb{A}^n - p \arrow[r, "\bar{f}"] & \mathbb{A}^n / \mathbb{A}^n - q \end{tikzcd}$$

$\operatorname{Spec} \mathbb{R}[x]$
$= \{(f) \mid f \text{ is irred}\}$
$x^2+bx+c,\ x-c$ or 0

$b^2-4c<0$

$$\deg^{\mathbb{A}^1}(\bar{f}) := \sum_{p \in f^{-1}q} \operatorname{tr}_{k(p)/k} \langle \underbrace{\det Jf(p)}_{\text{Jacobian of } f \text{ eval'd @ } p.} \rangle$$

$\underbrace{\hspace{4cm}}_{\text{local } \mathbb{A}^1\text{-degree of } f \text{ at } p,\ \deg_p^{\mathbb{A}^1}(f)}$

$k(p)$ = residue field of p

$p \in (\operatorname{Spec} k[x_1, \dots, x_n]) = \mathbb{A}^n_k$

$k[x_1, \dots, x_n]_p / m_p =: k(p)$

$m_p = (p) \subseteq k[x_1, \dots, x_n]_p$

<u>Fact</u> For $k \subseteq \mathbb{R}$,

$$\begin{tikzcd} {[S^n, S^n]} \arrow[d, "\deg"'] & {[S^{2n,n}, S^{2n,n}]_{\mathbb{A}^1}} \arrow[l, "\mathbb{R}\text{-pts}"'] \arrow[r, "\mathbb{C}\text{-pts}"] \arrow[d, "\deg^{\mathbb{A}^1}"] & {[S^{2n}, S^{2n}]} \arrow[d, "\deg"] \\ \mathbb{Z} & GW(k) \arrow[l, "\operatorname{sgn}"] \arrow[r, "\operatorname{rank}"'] & \mathbb{Z} \end{tikzcd}$$

$/\mathbb{R}$ $\quad \mathbb{Z}[h]/(h^2-2h)$

<u>Cor</u> The degree of the $\mathbb{C}$-points of any algebraic map is nonnegative.

$\underline{Q}$ Why? $\det\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = a^2+b^2 > 0$

$\underline{\text{E.g.}}$ (1) Suppose $f: \mathbb{A}^n \to \mathbb{A}^n$ has an isolated simple zero at p with $\det Jf(p) \neq 0$. Then

$$\deg^{\mathbb{A}^1}(\bar{f}) = \deg_p^{\mathbb{A}^1}(f) = \langle \det Jf(p) \rangle$$

f induced by $A \in GL_n(k) \Rightarrow \deg^{\mathbb{A}^1}(\bar{f}) = \langle \det A \rangle$

(2) $f: \mathbb{A}^1 \to \mathbb{A}^1$, $f(x) = x^2$. $Jf = (2x)$

Use $q = 1$. Then $f^{-1}1 = \{\pm 1\}$ and

$$\deg^{\mathbb{A}^1}(\bar{f}) = \deg_1^{\mathbb{A}^1}(f) + \deg_{-1}^{\mathbb{A}^1}(f)$$

$$= \langle 2 \rangle + \langle -2 \rangle = h.$$

Use $q = -1$. Sol'ns of $x^2 + 1 = 0$. Unique elt of $f^{-1}\{-1\} = \{(x^2+1)\}$.

$$\deg^{\mathbb{A}^1}(\bar{f}) = tr_{\mathbb{C}/\mathbb{R}} \langle 1 \rangle = h$$

$\to$ What is $\deg_p^{\mathbb{A}^1}(f)$ when $Jf(p)$ is singular?

Eisenbud - Levine / Khimshiashvili forms

(after J. Kass, K. Wickelgren, et al)

$f: \mathbb{A}^n \longrightarrow \mathbb{A}^n$ s.t. $f(p) = q$ for p, q k-pts

Local algebra of f at p

$$Q_p(f) := k[x_1, \ldots, x_n]_p \Big/ (f_1 - q_1, f_2 - q_2, \ldots, f_n - q_n)$$

E.g. $f(x) = x^2 \quad \mathbb{A}^1 \xrightarrow{f} \mathbb{A}^1$

$$Q_0(f) = k[x]_{(x)} \Big/ (x^2) \cong k[x] \Big/ (x^2)$$

i.e. (x)

Defn Assume char $k \nmid \dim_k Q_p(f)$.

The EL/K form of f at p is

$$\omega = \omega_p(f) : Q_p(f) \times Q_p(f) \longrightarrow k$$
$$(a, b) \longmapsto \eta(ab)$$

where $\eta : Q_p(f) \to k$ is any k-linear map s.t. $\eta(\det Jf) = \dim_k Q_p(f)$.

Note $\exists$ workaround for $\underset{\text{char } k}{\wedge} \mid \dim_k Q_p(f)$

... distinguished socle ...

E.g. $f : \mathbb{A}^1 \longrightarrow \mathbb{A}^1$
$x \longmapsto x^2$

$f(0) = 0$

$k[x]_{(x)} / (x^2 - 0)$

$Q_0(f) = k[x]/(x^2)$ $\quad \det Jf = 2x$

basis $1, x$

$\eta(1) = 0$

$\eta(x) = 1 \quad (\Rightarrow \eta(2x) = 2 \checkmark)$

Gram matrix of ω:

$$\begin{array}{c c} & \begin{array}{cc} 1 & x \end{array} \\ \begin{array}{c} 1 \\ x \end{array} & \begin{pmatrix} \eta(1\cdot 1) & \eta(1\cdot x) \\ \eta(x \cdot 1) & \eta(x^2) \end{pmatrix} \end{array} = \begin{pmatrix} \eta(1) & \eta(x) \\ \eta(x) & \eta(0) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cong h$$

E.g. $f: \mathbb{A}^1 \longrightarrow \mathbb{A}^1 \quad f(x) = ax^n, \ a \in k^\times$

$Q_0(f) \cong k[x]/(x^n)$ w/ dim'n n,

basis $1, x, x^2, \dots, x^{n-1}$

$\det Jf = nax^{n-1}$

$$\eta: \quad 1 \mapsto 0, \quad x \mapsto 0, \quad x^2 \mapsto 0, \quad \dots, \quad x^{n-1} \mapsto \tfrac{1}{a}$$

$$\Rightarrow \eta(nax^{n-1}) = na \cdot \frac{1}{a} = n$$

Gram of ω:

$$\begin{array}{c|c}
 & \begin{matrix} 1 & x & x^2 & \cdots & x^{n-2} & x^{n-1} \end{matrix} \\
\hline
\begin{matrix} 1 \\ x \\ x^2 \\ \vdots \\ x^{n-2} \\ x^{n-1} \end{matrix} &
\begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & a^{-1} \\
0 & 0 & 0 & \cdots & a^{-1} & 0 \\
0 & 0 & 0 & \iddots & & \\
\vdots & \vdots & \iddots & & & \\
0 & a^{-1} & 0 & & 0 & \\
a^{-1} & 0 & 0 & & &
\end{pmatrix}
\end{array}$$

If n is even, get $\frac{n}{2}h$

n odd, get $\frac{n-1}{2}h + \langle a \rangle$

General points

Suppose $k(p)$ is a finite separable extn of k.

What is $\deg_p^{\mathbb{A}^1}(f)$?

Thm (Kass-Wickelgren-Pauli)

$$\deg_p^{\mathbb{A}^1}(f) = \mathrm{tr}_{k(p)/k} \underbrace{\deg_p^{\mathbb{A}^1}(f \otimes k(p))}_{\in GW(k(p))}$$

Thm If p is regular, then

$$\omega_p(f) = \text{Morel } \deg_p^{\mathbb{A}^1}(f).$$

Also $$\deg^{\mathbb{A}^1}(\bar{f}) = \sum_{p \in f^{-1}q} \mathrm{tr}_{k(p)/k}\, \omega_p(f)$$

5 Aug 2021

Milnor forms

$\boxed{\text{Milnor \#} \ \mu_p(f) = \deg_p(\nabla f)}$

Defn Suppose $f : \mathbb{A}^n \to \mathbb{A}^1$ is algebraic with isolated singularity at p. The Milnor form (or $\mathbb{A}^1$-Milnor number) of f (or $V(f)$) at p is

$$\mu_p^{\mathbb{A}^1}(f) := \deg_p^{\mathbb{A}^1}(\nabla f).$$

$$= \omega_p^{EL/K}(\nabla f) \in GW(k).$$

Unpack: Since $\nabla f(p) = 0$,

$$Q_p(\nabla f) = k[x_1, \ldots, x_n]_p / (\partial f/\partial x_1, \ldots, \partial f/\partial x_n).$$

We have $\quad J(\nabla f) = Hf = \left(\partial^2 f/\partial x_i \partial x_j\right)_{i,j}$.

So $\quad \omega_p(\nabla f) : Q_p(\nabla f) \times Q_p(\nabla f) \longrightarrow k$

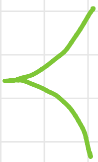
$$(a,b) \longmapsto \eta(ab)$$

for $\eta : Q_p(\nabla f) \to k$ k-linear s.t.

$$\eta(\det Hf) = \dim_k Q_p(\nabla f) = \mu_p(f).$$

at least in char 0

E.g. $f(x,y) = x^3 - y^2$, char $k \nmid \mathbb{Q}$.



$$\nabla f = (3x^2, -2y)$$

$$Hf = \begin{pmatrix} 6x & 0 \\ 0 & -2 \end{pmatrix} \qquad \det Hf = -12x$$

$$Q_0(\nabla f) \cong k[x,y]/(x^2, y) \cong k[x]/(x^2)$$

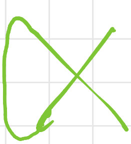
basis $1, x$

Define $\eta(1) = 0$, $\eta(x) = -\frac{1}{6} \Rightarrow \eta(-12x) = 2$.

Then ω has Gram matrix

$$\begin{array}{cc} & \begin{array}{cc} 1 & x \end{array} \\ \begin{array}{c} 1 \\ x \end{array} & \begin{pmatrix} 0 & -1/6 \\ -1/6 & 0 \end{pmatrix} \end{array} \cong h = \mu_0^{\mathbb{A}^1}(x^3 - y^2).$$

E.g. f is a node when $\det Hf(p) \neq 0$.



In this case, p is a regular point of ∇f and we may compute

$$\mu_p^{\mathbb{A}^1}(f) = \mathrm{tr}_{k(p)/k} \langle \det Hf(p) \rangle.$$

For $p = 0$ and $n = 2$, nodes look like

$xy + \text{h.o.t.}$
$\cong x^2 - y^2 + \text{h.o.t.}$

$$f(x,y) = ax^2 + by^2 + \text{h.o.t.}$$

for some $a, b \in k^\times$. In this case,

$$Hf(0) = \begin{pmatrix} 2a & 0 \\ 0 & 2b \end{pmatrix}$$

$$\det Hf(0) = 4ab$$

$$\Rightarrow \mu_0^{\mathbb{A}^1}(f) = \langle 4ab \rangle = \langle ab \rangle.$$

In arbitrary dimn,

$$Hf(0) = \begin{pmatrix} 2a_1 & & & \\ & 2a_2 & & \\ & & \ddots & \\ & & & 2a_n \end{pmatrix}$$

$$f(x_1, \dots, x_n) = a_1 x_1^2 + \cdots + a_n x_n^2 + \text{h.o.t.}$$

$$\Rightarrow \mu_0^{\mathbb{A}^1}(f) = \langle 2^n a_1 \cdots a_n \rangle = \begin{cases} \langle a_1 \cdots a_n \rangle & n \text{ even} \\ \langle 2a_1 \cdots a_n \rangle & n \text{ odd.} \end{cases}$$

E.g.: (Kass-Wickelgren, Pauli)

name	equation f	$\mu_0^{\mathbb{A}^1}(f)$
A_n, n odd	$x^2 + y^{n+1}$	$\frac{n-1}{2}h + \langle 2(n+1) \rangle$
A_n, n even	$x^2 + y^{n+1}$	$\frac{n}{2}h$
D_n, n even	$y(x^2 + y^{n-2})$	$\frac{n-2}{2}h + \langle -2, 2(n-1) \rangle$
D_n, n odd	$y(x^2 + y^{n-2})$	$\frac{n-1}{2}h + \langle -2 \rangle$
E_6	$x^3 + y^4$	$3h$
E_7	$x(x^2 + y^3)$	$3h + \langle -3 \rangle$
E_8	$x^3 + y^5$	$4h$
E_{12}	$x^7 + y^3 + z^2$	$6h$
Z_{11}	$x^5 + xy^3 + z^2$	$5h + \langle -6 \rangle$
Q_{10}	$x^4 + y^3 + xz^2$	$5h$
E_{13}	$x^5y + y^3 + z^2$	$6h + \langle -10 \rangle$
Z_{12}	$x^4y + y^3 + z^2$	$5h + \langle -22 \rangle + \langle -66 \rangle$

Perturbation

Fix $f \in k[x_1, \dots, x_n]$.

$\exists$ Zariski open $U \subseteq \mathbb{A}^n$ s.t. for fixed $a_1, \dots, a_n \in U(k)$ and all $t \in k$

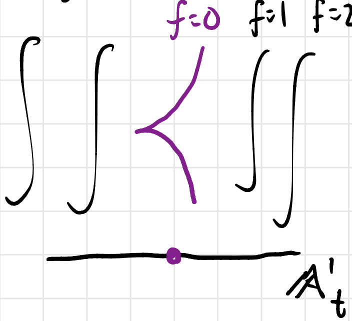
For generic $a_1, \dots, a_n \in k$ and all $t \in k$,

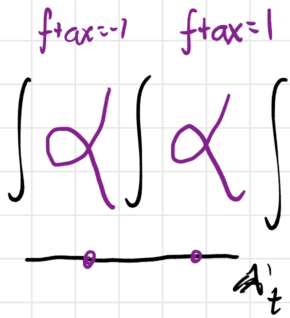
the hypersurface

$$f(x_1, \dots, x_n) + a_1 x_1 + \cdots + a_n x_n = t$$

smooth or nodal

only has nodal singularities





Thm (Kass-Wickelgren-Pauli)

Fix $a_1, \dots, a_n \in k$ s.t. $\boxed{f(x) + ax = t}$ ~~only has nodal singularities~~ for all $t \in k$. Then

$f(x_1,\dots,x_n) + a_1x_1 + \cdots + a_nx_n = t$ is smooth or nodal

$$\sum_{p \in \mathrm{Sing}(f)} \mu_p^{\mathbb{A}^1}(f) = \sum_{\substack{q \text{ node of} \\ \text{some } f(x)+ax=t}} \mu_q^{\mathbb{A}^1}(f(x) + ax - t)$$

Upshot Milnor form of f constrains the types of nodes f can bifurcate into.

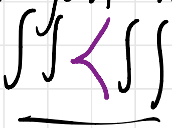
E.g. $f = x^3 - y^2 \Rightarrow \mu_0^{\mathbb{A}^1}(f] = h$ as before.

Consider $f_a = x^3 - y^2 + ax$ and the 1-parameter family of curves $f_a = t$.

$f_a = t$ has a singularity iff $x^3 + ax - t$ has a double root iff discriminant

$-4a^3 - 27t^2 = 0$.

For $a = 0$ (i.e. $f = f_0$) have just one singularity at $t = 0$, the cusp.



$f_0 = f$

$\mathbb{A}^1_t$

For $a \in k^\times$ fixed, get two singularities at

$$t^2 = \frac{-4}{27} a^3 .$$

If $\frac{-4}{27} a^3 \in k^{\boxtimes}$, have 2 rational nodes;

o/w a node with residue field $k\left(\sqrt{\frac{-4}{27} a^3}\right)$

In this case, we can explicitly determine the nodes and their Milnor forms (exc!). But even w/o doing so, we know they will add up to $h = \mu_0^{\mathbb{A}^1}(f)$.

For more gen'l singularities (and perhaps specific fields) this places interesting constraints on nodes in bifurcations.

Research problem (1) Explore these constraints systematically.

Other research problems

(2) Answer the following questions:

(a) Which $GW(k)$ classes are realized as local motivic degrees of algebraic maps $\mathbb{A}^n \to \mathbb{A}^n$?

(Up to rank 7 done by Quick-Strand-Wilson)

(b) Which $GW(k)$ classes are realized as Milnor forms?

(3) "Interpret" discriminant and other invariants of $\deg_p^{\mathbb{A}^1}(g)$, $\mu_p^{\mathbb{A}^1}(f)$.

rank = classical Milnor #

sgn = "real Milnor #"

(4) Use Newton polygons or Puiseux series (+ ?) of curves to determine $\mu_p^{\mathbb{A}^1}(f)$.

(5) Prove the following conjecture:

$$\mu_p^{\mathbb{A}^1}(fg) = \mu_p^{\mathbb{A}^1}(f) + \mu_p^{\mathbb{A}^1}(g) + 2\,\deg_p^{\mathbb{A}^1}(f,g) - 1.$$

for $(f,g): \mathbb{A}^2 \to \mathbb{A}^2$

Replacement for local int'n multiplicity $[V(f).V(g)]_p$ in classical version

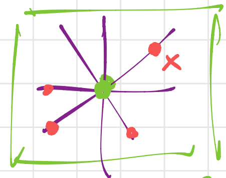
$\dim_{\mathbb{C}} \mathbb{C}[x,y]_p/(f,g)$

(6) Connections with tropical geometery?

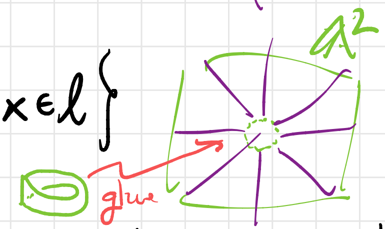
6 Aug 2021

Blowups and resolutions

Idea Replace a pt of $\mathbb{A}^2$ with all the lines passing through it without disturbing the rest of $\mathbb{A}^2$.



Implementation



$$Bl_0\mathbb{A}^2 := \{(x,\ell) \in \mathbb{A}^2 \times \mathbb{P}^1 \mid x \in \ell\}$$

$$\begin{array}{ccc} Bl_0\mathbb{A}^2 & & (x,\ell) \\ \pi \downarrow & & \downarrow \\ \mathbb{A}^2 & & x \end{array}$$

iso over $\mathbb{A}^2 \setminus 0$ w/ $\pi^{-1}0 = 0 \times \mathbb{P}^1$ $=: E$, the exceptional divisor

Coordinate charts

$$U_0 = \{((x,y),[z:w]) \mid z \neq 0,\ (x,y) \in [z:w]\}$$

$$\begin{array}{ccc} U_0 & & ((x,y),[z:w]) \\ \cong \downarrow & & \downarrow \\ \mathbb{A}^2 & & (\underset{u_0}{x}, \underset{v_0}{w/z}) \end{array}$$

$$U_1 = \{((x,y),[z:w]) \mid w \neq 0,\ (x,y) \in [z:w]\}$$

$$\begin{array}{ccc} U_1 & & ((x,y),[z:w]) \\ \cong\downarrow & & \downarrow \\ \mathbb{A}^2 & & (z/w,\ y) \\ & & \overset{\shortparallel}{u_1} \quad \overset{\shortparallel}{v_1} \end{array}$$

$$Bl_0\mathbb{A}^2 = U_0 \cup U_1$$

Then $\pi|_{U_0}(u_0, v_0) = (u_0, u_0 v_0)$

$\pi|_{U_1}(u_1, v_1) = (u_1 v_1, v_1)$

and $E \longleftrightarrow u_0 = 0$ in U_0

$v_1 = 0$ in U_1

Suppose $V = V(f) \subseteq \mathbb{A}^2$ is a curve. The strict transform $\tilde{V}$ of V is the closure of $(\pi^{-1}V) \setminus E$ in $Bl_0\mathbb{A}^2$.

We have $\pi^{-1}V \cap U_0 = V(f(u_0, u_0 v_0))$

$\pi^{-1}V \cap U_1 = V(f(u_1 v_1, v_1))$

If $0 \in V$, then

$$f(u_0, u_0 v_0) = u_0^m f_0^{(1)}(u_0, v_0), \quad u_0 \nmid f_0^{(1)}$$

$$f(u_1 v_1, v_1) = v_1^n f_1^{(1)}(u_1, v_1). \quad v_1 \nmid f_1^{(1)}$$

The eq'ns for $\tilde{V}$ are $f_0^{(1)}$ in U_0, $f_1^{(1)}$ in U_1.

E.g. Cusp $V = V(x^3 - y^2)$. Then

$\pi^{-1}V \cap U_0$ has eqn $u_0^3 - (u_0 v_0)^2 = u_0^2(u_0 - v_0^2) = 0$

$\Rightarrow f_0^{(1)}(u_0, v_0) = u_0 - v_0^2$ — a smooth parabola

$\pi^{-1}V \cap U_1$ has eqn $(u_1 v_1)^3 - v_1^2 = v_1^2(u_1^3 v_1 - 1) = 0$

$\Rightarrow f_1^{(1)}(u_0, v_0) = u_1^3 v_1 - 1$ w/ smooth zero locus not intersecting E.

Thus $\tilde{V}$ is smooth and $\tilde{V} \longrightarrow V$ is a resolution of singularities.

Thm The singularities of any plane algebraic curve may be resolved by a finite sequence of blowups, even so that proper preimage of singular points meets exceptional divisors transversely.

Q Can we compute $\mu_p^{\mathbb{A}^1}(f)$ in terms of its resolution by blowups?

Thm $V \subseteq \mathbb{A}^2_{\mathbb{C}}$ plane curve w/ isolated sing p of multiplicity d, V has r distinct tangent lines at p. Then

$$\mu_p(V) = d(d-1) + \sum_{x \in \mathrm{Sing}(\tilde{V}) \cap E} \mu_x(\tilde{V}) + 1 - r$$

Q Does this admit a quadratic refinement?

Usman studied this by computing

$$\Delta_p(f) := \mu_p^{\mathbb{A}^1}(f) - \sum_{x \in \mathrm{Sing}(\tilde{V}) \cap E} \mu_x^{\mathbb{A}^1}(\tilde{f})$$

for a number of examples.

E.g. (1) $f = x^n + y^m$ then $\quad r=1$

$$\Delta_0(f) = \begin{cases} \frac{n(n-1)}{2} h & n \text{ odd, or } n \text{ even and } m \text{ odd} \end{cases}$$

$$\left(\frac{n(n-1)}{2}h + \langle mn\rangle - \langle n(m-n)\rangle\right) \quad \text{o/w}$$

(2) $$\Delta_0(D_n) = \begin{cases} 2h + \langle -2, 2(n-1)\rangle - \langle 2(n-4)\rangle & n \text{ even} \\ 2h + \langle -2\rangle & n \text{ odd} \end{cases}$$

$\therefore$ $\frac{d(d-1)}{2}h$ replaces $d(d-1)$, but what replaces $1-r$?

Field of defn of tangent lines could play a role

One approach to the classical fmla is via a Plücker formula for polar curves.

D_n : $y(x^2+y^{n-2})$ 2 branches $\Rightarrow$ link(D_n) has 2 component