N_∞ operads, transfer systems, and the combinatorics of bi-incomplete Tambara functors

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(joint work with Linus Bao, Christy Hazel, Tia Karkos, Alice Kessler, Austin Nicolas, Jeremie Park, Cait Schleff, Scotty Tilton)

Let G be a finite group. The theory of N_∞ operads was created by Blumberg–Hill [4] to parametrize homotopy coherent normed multiplicative structures on G-equivariant ring spectra. The homotopy category of G-N_∞ operads is equivalent to the lattice $\operatorname{Tr} G$ of G-transfer systems. The combinatorial nature of $\operatorname{Tr} G$ makes it amenable to study by elementary means. In this talk, I report on work by the 2023 Electronic Computational Homotopy Theory REU to determine the structure of $\operatorname{Tr} G$ when $G = C_p \times C_p$ is an elementary Abelian p-group of rank two. This leads to an application in equivariant algebra: a quick derivation of the number of compatible pairs of transfer systems underlying bi-incomplete Tambara functors on $C_p \times C_p$.

Eschewing the standard homotopical conceit of writing Σ_n, let $\mathfrak{S}_n$ denote the symmetric group on n letters.

Definition 1. A G-N_∞ *operad* $\mathscr{O}$ is an operad in G-spaces such that (1) $\mathscr{O}(0)$ is G-contractible, (2) the action of $\mathfrak{S}_n = e \times \mathfrak{S}_n$ on $\mathscr{O}(n)$ is free, (3) for all $\Gamma \leq G \times \mathfrak{S}_n$, $\mathscr{O}(n)^\Gamma$ is either contractible or empty, and (4) $\mathscr{F}_\mathscr{O} := \{\Gamma \leq G \times \mathfrak{S}_n \mid \mathscr{O}(n)^\Gamma \simeq *\}$ is a $G \times \mathfrak{S}_n$-family[1] containing all subgroups of the form $H \times e$.

Let $H \leq G$ and let T be a finite H-set. Let $\Gamma(T)$ denote the graph of a permutation representation $H \to \mathfrak{S}_{|T|}$ of T. We say that $\mathscr{O}$ *admits T-norms* when $\mathscr{O}(n)^{\Gamma(T)} \simeq *$.

Note that when an $\mathscr{O}$-algebra X admits H/K-norms, we get a "wrong way" map

$$X^K \to X^H.$$

These are what practicitioners typically think of as norms (or transfers in an additive setting).

We write N_∞-$\mathbf{Op}^G$ for the category of G-N_∞ operads and G-equivariant maps of G-operads. A weak equivalence of G-N_∞ operads is map $\varphi\colon \mathscr{O}_1 \to \mathscr{O}_2$ such that the induced map $\mathscr{O}_1(n)^\Gamma \to \mathscr{O}_2(n)^\Gamma$ is a weak equivalence for all $n \geq 0$ and $\Gamma \leq G \times \mathfrak{S}_n$. We write $\operatorname{Ho}(N_\infty\text{-}\mathbf{Op}^G)$ for the associated homotopy category. Since we wish to classify N_∞ operads up to homotopy, it is desirable to have a tractable model for $\operatorname{Ho}(N_\infty\text{-}\mathbf{Op}^G)$, and this is provided by transfer systems.

Definition 2. Let $(P, \leq)$ be a partially ordered set (poset). A *transfer system* R on P is a partial order on the set P refining $\leq$ (so $x \mathrel{R} y \implies x \leq y$) such that

$$x \mathrel{R} y,\ z \leq y,\ \text{and } w \text{ maximal among } w' \leq x \implies w \mathrel{R} z. \tag{1}$$

[1] A collection of subgroups forms a *family* when it is closed under conjugation and taking subgroups.

If $(\operatorname{Sub} G, \leq)$ denotes the subgroup lattice of a finite group G ordered by inclusion, then a *G-transfer system* is a transfer system R on $\operatorname{Sub} G$ that is further closed under conjugation: $K\ R\ H \implies {}^gK\ R\ {}^gH$ where ${}^gH := gHg^{-1}$.

Note that when P is a lattice (like $\operatorname{Sub} G$), condition (1) reduces to

$$x\ R\ y \text{ and } z \leq y \implies x \wedge z\ R\ z, \tag{2}$$

which we refer to as the *restriction condition.* Categorically speaking, a transfer system on a lattice P is a wide subcategory of P closed under pullbacks.

Let $\operatorname{Tr} P$ denote the collection of transfer systems on P, and let $\operatorname{Tr} G$ denote the set of G-transfer systems. There is a canonical refinement partial order $\leq$ on $\operatorname{Tr} P$ given by

$$R \leq R' \iff (x\ R\ y \implies x\ R'\ y),$$

and when P is a finite lattice, $\operatorname{Tr} P$ is a finite lattice; the same is true of $\operatorname{Tr} G$.

Work of many authors [4, 8, 6, 9, 10, 1] establishes that $\operatorname{Tr} G$ models the homotopy category of G-N_∞ operads. Given an N_∞ operad $\mathscr{O}$, write $R_{\mathscr{O}} \in \operatorname{Tr} G$ for the transfer system given by

$$K\ R_{\mathscr{O}}\ H \iff \mathscr{O} \text{ admits } H/K \text{ norms.}$$

Theorem 3. *The assignment $R \mapsto R_{\mathscr{O}}$ is a functor N_∞-$\mathbf{Op}^G \to \operatorname{Tr} G$ and descends to an equivalence of categories*

$$\operatorname{Ho}(N_\infty\text{-}\mathbf{Op}^G) \simeq \operatorname{Tr} G.$$

Transfer systems are elementary but subtle, and enumerations of $\operatorname{Tr} G$ have only slowly appeared. Prior to our work, the only infinite family of transfer system lattices completely understood was for $G = C_{p^n}$, the cyclic group of order p^n, p prime.[2] Indeed, Balchin–Barnes–Roitzheim [1] prove that $\operatorname{Tr} C_{p^n}$ is isomorphic to the Tamari lattice $\mathcal{A}_{n+1}$ of planar binary rooted trees with $n+2$ leaves partially ordered by tree rotation. It follows that

$$|\operatorname{Tr} C_{p^n}| = \operatorname{Cat}_{n+1} = \frac{1}{2n+3}\binom{2n+3}{n+1},$$

the $(n+1)$-th Catalan number.

In our work [3], we completely determine and enumerate the lattice of transfer systems for $C_p \times C_p$, p prime. In order to state the theorem, set $[n] := \{0 < 1 < \cdots < n\}$ and note that $[1]^k$ is isomorphic to the lattice of subsets of a k-element set partially ordered by inclusion.

Theorem 4 (Bao, Hazel, Karkos, Kessler, Nicolas, O., Park, Schleff, Tilton [3, Theorem 5.4])**.** *For p prime there are exactly*

$$2^{p+2} + p + 1$$

transfer systems on $C_p \times C_p$, and the lattice of transfer systems consists of three disjoint induced subposets: $B, T \cong [1]^{p+1}$ and M consisting of $p+1$ incomparable

[2] Balchin–MacBrough–Ormsby [2] have also determined an elaborate set of interleaving recursions which determine the cardinalities of $\operatorname{Tr} D_{p^n}$ and $\operatorname{Tr} C_{qp^n}$

points. The only covering relations in $\mathrm{Tr}(C_p \times C_p)$ *not internal to* B *or* T *are of the following forms:*

(i) each element of B *covered by* $\max B$ *is also covered by exactly one element of* M*,*
(ii) each element of T *covering* $\min T$ *also covers exactly one element of* M*,*
(iii) $\min T$ *covers* $\max B$*.*

Proof idea. The subgroup lattice of $C_p \times C_p$ consists of the trivial subgroup e, $p+1$ rank 1 subgroups (isomorphic to C_p), and the full group. As such, $\mathrm{Sub}(C_p \times C_p) \cong [2]^{*(p+1)}$, the $(p+1)$-fold *fusion* of $[2]$ with itself. Here the fusion of two lattices is their disjoint union with minimal elements identified and maximal elements identified. We provide a general recursion for $|\,\mathrm{Tr}(P * Q)|$ in [3, Theorem 4.11], and leverage this to enumerate $\mathrm{Tr}(C_p \times C_p) \cong \mathrm{Tr}([2]^{*(p+1)})$.

With the enumeration in hand, it is easy to construct all transfer systems and check the covering relations between them. The subposet B consists of transfer systems that only have non-identity relations between e and some subset of rank 1 subgroups. The subposet T consists of transfer systems that have all relations from e to other subgroups and some subset of relations between rank 1 subgroups and $C_p \times C_p$. The transfer systems in M have all but one of the relations from e to rank 1 subgroups, along with one relation from the excluded rank 1 subgroup to $C_p \times C_p$. $\square$

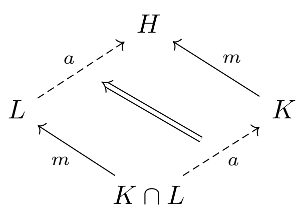
Such an explicit enumeration of transfer systems allows us to study other structures related to transfer systems on $C_p \times C_p$. By work of Chan [7], we know that special pairs of transfer systems enumerate compatible choices of transfers and norms for bi-incomplete Tambara functors in the sense of Blumberg–Hill [5].[3]

Definition 5. Let G be a finite group. A pair $(\overset{a}{\dashrightarrow}, \overset{m}{\longrightarrow})$ of G-transfer systems is called *compatible* when $\overset{m}{\longrightarrow} \leq \overset{a}{\dashrightarrow}$ and the following condition holds:

$$K, L \leq H \leq G,\ K \overset{m}{\longrightarrow} H, \text{ and } K \cap L \overset{a}{\dashrightarrow} K \implies L \overset{a}{\dashrightarrow} H. \tag{3}$$

We write $\mathrm{Comp}\, G$ for the collection of compatible pairs of G-transfer systems.

We may encode (3) diagramatically as



[3]Bi-incomplete Tambara functors arise in the context of equivariant ring spectra R defined over G-universes that might not be complete. In this scenario, $\underline{\pi}_* R$ is a bi-incomplete Tambara functor with additive transfers encoded by the G-universe and multiplicative norms encoded by an N_∞ operad over which R is an algebra.

where the double arrow indicates logical implication. Note that $K \cap L \xrightarrow{m} L$ is already guaranteed by (2).

Based on Theorem 4, we may enumerate the compatible pairs of transfer systems for $C_p \times C_p$ with relatively little pain.

Theorem 6 (O.). *For p prime, there are exactly*

$$2^p(2^{p+2} + p + 3) + 3^{p+1} + 2p + 2$$

compatible pairs of $(C_p \times C_p)$-transfer systems.

To give the reader a sense for these numbers, we record the first few values:

p	2	3	5	7	11	13
$\lvert \mathrm{Comp}(C_p \times C_p) \rvert$	117	393	5 093	73 393	17 337 353	273 349 525

Proof sketch. Set $n = p + 1$. For each $\xrightarrow{m} \in \mathrm{Tr}(C_p \times C_p)$ we determine which $\overset{a}{\dashrightarrow} \geq \xrightarrow{m}$ satisfy (3). First focus on the 2^n transfer systems in B. Since no relations in these transfer systems are restrictions of other relations, no conditions are imposed by (3) and we only need to count the size of the up-set of each $\xrightarrow{m}$ in B. If $\xrightarrow{m}$ has rank k, then there are 2^{n-k} elements of B at least as large as it, along with $n - k$ elements of M and all 2^n elements of T. Since there are $\binom{n}{k}$ elements of B of rank k, we find that there are exactly

$$\sum_{k=0}^{n} \binom{n}{k} (2^{n-k} + n - k + 2^n)$$

compatible pairs $(\overset{a}{\dashrightarrow}, \xrightarrow{m})$ with $\xrightarrow{m}$ in B. Standard combinatorial identities reduce this expression to

$$3^n + 2^{n-1} \cdot n + 2^{2n}.$$

Now let $\xrightarrow{m}$ be one of the n transfer systems in M. While there are $1 + 2^{n-1}$ transfer systems at least as large as $\xrightarrow{m}$, only $\xrightarrow{m}$ and the complete transfer system $\leq$ pair with $\xrightarrow{m}$ to satisfy (3). Thus there are $2n$ compatible pairs $(\overset{a}{\dashrightarrow}, \xrightarrow{m})$ with $\xrightarrow{m}$ in M.

Finally, if $\xrightarrow{m}$ is in T, then it can only pair with the complete transfer system to satisfy (3), so there are 2^n compatible pairs $(\overset{a}{\dashrightarrow}, \xrightarrow{m})$ with $\xrightarrow{m}$ in T. Adding things up, we see that there are exactly

$$3^n + 2^{n-1} \cdot n + 2^{2n} + 2n + 2^n = 2^{n-1}(2^{n+1} + n + 2) + 3^n + 2n$$

compatible pairs for $C_p \times C_p$. Substituting $n = p + 1$ gives the expression from the theorem statement. ☐

Acknowledgments. The 2023 Electronic Computational Homotopy Theory Research Experience for Undergraduates was supported by NSF grant DMS–2135884; we thank Dan Isaksen for his support and leadership within the eCHT. The author was partially supported by NSF grant DMS–2204365.

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