CLOSURE OPERATORS AND COSATURATED TRANSFER SYSTEMS

SCOTT BALCHIN, ETHAN MACBROUGH, AND KYLE ORMSBY

ABSTRACT. We add cosaturated transfer systems to the list of objects cryptomorphically equivalent to a closure operator on a finite lattice. This leads to a new derivation of the number of closure systems on a rectangular lattice.

CONTENTS

1. INTRODUCTION

The emerging field of *homotopical combinatorics* studies combinatorial aspects of two related structures: (1) N_∞ operads and their associated homotopy category of transfer systems, and (2) model structures on complete lattices (typically finite posets admitting meets and joins). As discovered in [FOO+22], these are unified through the language of weak factorization systems on complete lattices. Indeed, a G-transfer system for G an Abelian group contains the same data as a weak factorization system on the subgroup lattice $\mathrm{Sub}(G)$, and a model structure consists of a compatible pair of weak factorization systems satisfying certain axioms.

In this work, we focus on cosaturated transfer systems, which correspond to model structures in which every morphism is a cofibration. These play a special role in G-N_∞ theory for G a cyclic group, where they are those transfer systems induced by Steiner operads. It turns out (see Dodecatheorem 1.20) that cosaturated transfer systems are naturally in bijection with *closure operators* on the underlying poset. Closure operators on Boolean posets originate in the work of Cantor, Dedekind, Moore, Riesz, and Schröder, and are closely related to (but more general than) closure in point-set topology. Closure operators are also well-studied under a number of crypotomorphic guises (Moore families, submonoids, monads, etc.,) so our work links cosaturated transfer systems with a rich web of ideas in order and category theory.

Now for a quick outline of this note: In the remainder of this introduction, we spell out some preliminaries and definitions in order theory and homotopical combinatorics, culminating in a precise statement of Dodecatheorem 1.20. In Section 2, we explain the correspondence between cosaturated transfer systems on P and saturated transfer systems[1] on P^{op}. This sets us up for Section 3 in which we prove Dodecatheorem 1.20. Finally, in Section 4 we use this new perspective to enumerate closure operators on a rectangular lattice. This amounts to a new proof of [HMOO22] which counted saturated transfer systems for a cyclic group of order $p^m q^n$, $p \neq q$ prime.

1.1. **Preliminaries on order theory.** A *lattice* is a partially ordered set (*poset*) $(P, \leq)$ admitting finite *meets* (infima, denoted $\wedge$) and *joins* (suprema, denoted $\vee$). If P is complete (*i.e.*, admits all meets and joins; this is automatic for P finite), we let $\hat{0}$ denote the minimal element of P, and $\hat{1}$ its maximal element. Let $P^{\mathrm{op}} := (P, \geq)$ denote the opposite poset of P with all relations reversed. Every poset P induces an associated category that we also denote P. The objects of P are the elements of the underlying poset, and

$$P(x,y) = \begin{cases} * & \text{if } x \leq y, \\ \varnothing & \text{otherwise.} \end{cases}$$

Composition is well-defined by transitivity of $\leq$, and the category associated with P^{op} is equal to the opposite of the category associated with P. Posets form a category in which the morphisms are *monotone* functions: $x \leq y \implies f(x) \leq f(y)$. A monotone map of posets is the same thing as a functor between associated categories.

Definition 1.1. A *closure operator* cl on a poset P is a function $\mathrm{cl}\colon P \to P$ that is

(1) *extensive*: $x \leq \mathrm{cl}(x)$,
(2) *monotone*: $x \leq y$ implies $\mathrm{cl}(x) \leq \mathrm{cl}(y)$, and
(3) *idempotent*: $\mathrm{cl}(\mathrm{cl}(x)) = \mathrm{cl}(x)$.

Example 1.2. If X is a topological space, then the Boolean poset $(2^X, \subseteq)$ of subsets of X admits topological closure $S \mapsto \overline{S}$ as a closure operator. Note, though, that topological closure additionally commutes with finite unions (joins in 2^X). If a closure operator presreves joins, then it is called a *Kuratowski closure operator*, while if it preserves meets, then it is a *nucleus*, a concept appearing in the theory of locales. Note that Kuratowski closure operators on 2^X are in bijection with topologies on X, while nuclei on a frame P are in bijection with sublocales on P.

Remark 1.3. Closure operators are notoriously challenging to enumerate. The most classical case is that of finite Boolean posets, recorded as sequence A102896 in the OEIS. For $n = 0, 1, \ldots, 7$, the number of closure operators on the poset of subsets of an n-element set is

1, 2, 7, 61, 2 480, 1 385 552, 75 973 751 474, 14 087 648 235 707 352 472,

respectively [Hig98, HN05, CIR10]. No additional values of this sequence are known, but its base-2 logarithm is known to grow at the rate of $\binom{n}{\lfloor n/2 \rfloor}$ and more precise asymptotics are given in [Kle76].

[1] Saturated transfer systems are a natural subclass of transfer systems related to linear isometries operads; see Remark **??**.

We may rephrase the closure operator conditions categorically as follows: monotonicity says that $\mathrm{cl}\colon P \to P$ is a functor, extensivity gives a natural transformation $\mathrm{id}_P \to \mathrm{cl}$, and idempotence gives a natural transformation $\mathrm{cl} \circ \mathrm{cl} \to \mathrm{cl}$. (For the last claim, note that extensivity automatically produces $\mathrm{cl} \to \mathrm{cl} \circ \mathrm{cl}$ so the content of idempotence is the map in the natural transformation in the other direction.) This is the data of a monad on the category P, and the coherence axioms hold trivially since P is a poset. As such, we have the following well-known result (see for example [LS86, Part 0]).

Proposition 1.4. Closure operators on a poset P are in bijection with monads on the category associated with P. □

Just as closed sets play a prominent role in topology, closed elements of a poset (relative to a closure operator) are important in this setting:

Definition 1.5. Fix a poset P and closure operator $\mathrm{cl}\colon P \to P$. An object $x \in P$ is *closed* relative to P if and only if $x = \mathrm{cl}(x)$.

It is in fact the case that the collection of closed objects in a complete lattice both determines and is determined by the closure operator. These collections cannot be arbitrary, but instead must satisfy the following:

Could get away with less, but is it worth it?

I would just keep it as it is, keep it short and sweet

Definition 1.6. Fix a complete lattice P. A *closure system* (or *Moore family*) on P is a subset $S \subseteq P$ containing $\hat{1}$ and closed under meets.

Remark 1.7. The algebraically inclined reader will immediately note that a closure system is the same thing as a submonoid of $(P, \wedge)$.

Given a closure operator cl on a complete lattice P, its collection of closed subsets forms a closure system $S_{\mathrm{cl}} = \{x \in P \mid x = \mathrm{cl}(x)\}$. Given a closure system $S \subseteq P$, we may define a closure operator cl_S on P by

$$\mathrm{cl}_S(x) = \bigwedge_{y \in [x, \hat{1}] \cap S} y,$$

i.e., $\mathrm{cl}_S(x)$ is the minimal element of S satisfying $x \leq cl_S(x)$. The following result is well-known and follows from the definitions (see for example [Pri02]):

Proposition 1.8. Fix a complete lattice P. The assignments $\mathrm{cl} \mapsto S_{\mathrm{cl}}$ and $S \mapsto \mathrm{cl}_S$ are mutually inverse bijections between closure operators and closure systems. □

Closure operators admit the dual notion of *interior operators* . These are monotone maps $\mathrm{int}\colon P \to P$ that are *contractive* ($\mathrm{int}(x) \leq x$) and idempotent. They are in natural bijection with *interior systems* (submonoids of $(P, \vee)$) and comonads on P.

1.2. **Preliminaries on transfer systems and model structures.** We now briefly recall transfer systems, weak factorization systems, and model structures on lattices. For additional details, see [FOO+22, §4].

Definition 1.9. A *transfer system* on a lattice $(P, \leq)$ is a relation R refining $\leq$ (so $x \, R \, y \implies x \leq y$) that is closed under restriction: $x \, R \, y$ and $z \leq y$ implies $(x \wedge z) \, R \, z$. Equivalently, a transfer system is a wide subcategory of P that is closed under pullbacks.

Transfer systems originated in the study of G-N_∞ operads, G a finite group [BH15]. The above definition is equivalent to the original one in the case G is Abelian (or Dedekind: all subgroups are normal) and $P = \mathrm{Sub}(G)$ is the subgroup lattice of G.

Though not given an explicit name, cosaturated transfer systems arose in J. Rubin's groundbreaking analysis of transfer systems induced by Steiner operads [Rub21].

Definition 1.10. A transfer system R on P is *cosaturated* when it is generated[2] by a set of relations $\{x \le \hat{1} \mid x \in S\}$ for some subset $S \subseteq P$.

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Given a finite group G, recall from [BH15]that each G-universe U has an associated Steiner operad $\mathscr{K}(U)$ which is G-N_∞. Write $R_{\mathscr{K}(U)}$ for the associated G-transfer system. In [Rub21], Rubin studies the image of the assignment $U \mapsto R_{\mathscr{K}(U)}$, proving the following results:

Lemma 1.11 ([Rub21, Lemma 4.1])**.** For any finite group G and G-universe U, $R_{\mathscr{K}(U)}$ is cosaturated.

Theorem 1.12 ([Rub21, Corollary 4.12])**.** *For any finite cyclic group G, a G-transfer system R is induced by a Steiner operad if and only if R is cosaturated.*

Thus, when P is the subgroup lattice of a finite cyclic group G, enumerating cosaturated transfer systems on P is equivalent to enumerating the homotopy classes of Steiner operads on G.

In *loc. cit.*, Rubin also introduces the notion of saturated transfer systems in relation to linear isometries operads. Although not immediately obvious (or previously observed), Rubin's saturated transfer systems are dual to cosaturated transfer systems in a precise sense as we will see in Dodecatheorem 1.20.

Definition 1.13. A transfer system R on P is *saturated* when $x\,R\,y$ and $x \le z \le y$ implies $z\,R\,y$.

Given a G-universe U, let $\mathscr{L}(U)$ denote the linear isometries operad associated with U. This operad is G-N_∞ and has an associated transfer system $R_{\mathscr{L}(U)}$. Rubin also studies the image of $U \mapsto R_{\mathscr{L}(U)}$:

Proposition 1.14 ([Rub21, Proposition 5.2])**.** For any finite group G and G universe U, $R_{\mathscr{L}(U)}$ is saturated.

Remark 1.15. Unlike in Theorem 1.12, it is not true that every saturated transfer system arises in this way, and this can already be seen in the case of C_6 (see [Rub21, Example 5.16]). Nonetheless, in [Rub21, Theorem 5.17], Rubin proves that every saturated transfer system for cyclic groups of prime power order is realized by some linear isometries operad, and in [HMOO22, Theorem 1.2], Hafeez–Marcus–Ormsby–Osorno prove that for $p, q > 3$ distinct primes, every saturated transfer system for cyclic groups of order $p^m q^n$ is realized by some linear isometries operad. Forthcoming work of MacBrough extends these results to other classes of cyclic groups.

[2] Given a set of relations $A \subseteq \le$, the minimal transfer containing A exists, is denoted $\langle A \rangle$, and is called the transfer system *generated* by A.

In Section 2 we prove that there is a natural bijection between saturated transfer systems on a lattice P and cosaturated transfer systems on P^{op}. This bijection is most easily conceived in the language of weak factorizaiton systems.

Weak factorization systems have a different flavor from transfer systems but ultimately encode the same data when P is a finite lattice. To set up their definition, recall that if R is a class of morphisms in a category, then we write ${}^{\boxslash}R$ for the set of morphisms in the category with left lifting property with respect to R. That is, ${}^{\boxslash}R$ consists of those morphisms i such that for all $p \in R$ and all commuting squares

$$\begin{array}{ccc} a & \longrightarrow & x \\ {\scriptstyle i}\big\downarrow & \overset{\lambda}{\nearrow} & \big\downarrow{\scriptstyle p} \\ b & \longrightarrow & y \end{array}$$

there exists a lift λ making the diagram commute. Given a class of morphisms L we can make a similar definition of $L^{\boxslash}$, the morphisms with the right lifting property with respect to L.

Definition 1.16. A *weak factorization system* on a category P consists of a pair (L,R) of classes of morphisms in P such that

(1) every morphism f is P factors as $f = pi$ with $i \in L$ and $p \in R$, and
(2) $L = {}^{\boxslash}R$ and $R = L^{\boxslash}$.

The collection of weak factorization systems on P is denoted $\mathrm{WFS}(P)$.

Theorem 1.17 ([HMOO22, Theorem 4.13])**.** *Let P be a finite lattice. Then*

$$\begin{aligned} \mathrm{Tr}\,P &\longrightarrow \mathrm{WFS}(P) \\ R &\longmapsto ({}^{\boxslash}R, R) \end{aligned}$$

is a bijection.

Weak factorization systems have particular prominence in homotopy theory because they can be used to define model structures. Here we present an alternative definition that is equivalent to Quillen's [Qui67] as given by Joyal and Tierney [JT07].

Definition 1.18. A *model structure* on a bicomplete category P consists of a pair of weak factorization systems $(C,AF),(AC,F)$ such that $AF \subseteq F$ (equivalently $AC \subseteq C$)) and $W := AF \circ AC$ satisfies the two-out-of-three property: if f and g are composable morphisms and two of the three maps f, g, fg are in W, then so is the third. The classes of morphisms have the following names:

- F — *fibrations*,
- C — *cofibrations*,
- AF — *acyclic fibrations*,
- AC — *acyclic cofibrations*, and
- W — *weak equivalences*.

Remark 1.19. Given Theorem 1.17, we may further streamline the data of a model structure when P is a finite lattice. In this case, each model structure determines and is determined by a pair of transfer systems $AF \leq F$ such that $W = AF \circ^{\boxslash} F$ satisfies the two-out-of-three property. We will frequently take this perspective headed forward.

We now have enough terminology to state our main theorem:

Dodecatheorem 1.20. *Given an arbitrary finite lattice P, the following twelve structures are in bijective correspondence:*

(1) cosaturated transfer systems on P,
(2) model structures on P in which every morphism is a fibration,
(3) closure operators on P,
(4) closure systems / Moore families on P,
(5) submonoids of $(P,\wedge)$*,*
(6) monads on the category associated with P.
(7) saturated transfer systems on P^{op}*,*
(8) model structures on P^{op} *in which every morphism is a cofibration,*
(9) interior operators on P^{op}*,*
(10) interior systems on P^{op}*,*
(11) submonoids of $(P^{\mathrm{op}},\vee)$*,*
(12) comonads on the category associated with P^{op}*.*

When P is self-dual, all instances of P^{op} *may be replaced with P.*

The primary content of the Dodecatheorem lies in the bijections between (1)–(6). These structures are each dual to the corresponding structures (7)–(12) on P^{op}. Nonetheless, our primary case of interest from N_∞-theory — $P = \mathrm{Sub}(G)$ for G a finite Abelian group — exhibits self-duality, so it is extremely useful to know, *e.g.*, that cosaturated and saturated transfer systems are bijectively interchanged by the duality exhibited in [FOO+22].

Corollary 1.21. The number of cosaturated transfer systems on the Boolean lattice $[1]^n$ is

$$1, \quad 2, \quad 7, \quad 61, \quad 2\,480, \quad 1\,385\,552, \quad 75\,973\,751\,474, \quad 14\,087\,648\,235\,707\,352\,472,$$

for $n = 0,1,\ldots,7$. This sequence also counts the number of saturated transfer systems on $[1]^n$.

Proof. This follows directly from the Dodecatheorem and Remark 1.3. □

We can also relate the Dodecatheorem to Steiner and linear isometries operads:

Corollary 1.22. The number of homotopy classes of Steiner operads for a cyclic group of order n is the same as the number closure operators on the divisor lattice of n.

Proof. This follows directly from Theorem 1.12 and Dodecatheorem 1.20. □

Corollary 1.23. Suppose G is an Abelian group for which every saturated transfer system is realized by a linear isometries operad (*e.g.*, $G = C_{p^n}$ for any prime p or $G = C_{pq^n}$ for primes $p,q > 3$; see Remark 1.15). Then the number of homotopy classes of linear isometries operads for G is the same as the number of closure operators on $\mathrm{Sub}(G)$.

Proof. This follows from Proposition 1.14, Dodecatheorem 1.20, and the fact that $\mathrm{Sub}(G)$ is self-dual when G is Abelian. □

Maybe one also adds to the above corollary that there is a bijection between linear isometries and Steiner operads in this setting.

2. Saturated and cosaturated transfer systems

In this section, we describe the basic properties of saturated and cosaturated transfer systems on a finite lattice. To warm up, we prove two lemmata on general transfer systems that will be useful in our analysis.

Lemma 2.1. Given a finite lattice $(P, \leq)$, suppose that R' is an arbitrary relation on P refining $\leq$. Let $R = \langle R' \rangle$ and define $L := {}^{\boxslash}R$, $L' := {}^{\boxslash}R'$. Then

$$L = L'.$$

Proof. Clearly $L \subseteq L'$, and we now prove the opposite inclusion. We can construct $\langle R' \rangle$ by iteratively taking pullback closure and then transitive closure. Thus it suffices to show (1) if R_p is the pullback closure of R', then $L_p := {}^{\boxslash}R_p \supseteq L'$, and (2) if R_t is the transitive closure of R', then $L_t := {}^{\boxslash}R_t \supseteq R'$.

We first prove (1). Suppose $z\, R_p\, w$. Then by definition there exists $x\, R'\, y$ with $w \leq y$ such that $z = x \wedge w$. Then if we have a diagram

$$\begin{array}{ccccc} a & \longrightarrow & x \wedge w & \longrightarrow & x \\ {\scriptstyle L'}\big\downarrow & & \big\downarrow{\scriptstyle R_p} & & \big\downarrow{\scriptstyle R'} \\ b & \longrightarrow & w & \longrightarrow & y \end{array}$$

(where $x \to y$ means $x \leq y$ and labeled arrows are in the corresponding classes) then by definition of L' we must have $b \leq x$, and hence $b \leq x \wedge w$ as well, providing the desired lifting and showing $L' \subseteq L_p$.

We now prove (2). Suppose $x\, R'\, x'\, R'\, y$ and we have a square

$$\begin{array}{ccc} a & \longrightarrow & x \\ & & \big\downarrow{\scriptstyle R'} \\ {\scriptstyle L'}\big\downarrow & & x' \\ & & \big\downarrow{\scriptstyle R'} \\ b & \longrightarrow & y. \end{array}$$

Then lifting with respect to $x'\, R'\, y$ gives $b \leq x'$, which in turn allows us to lift against $x\, R'\, x'$, giving $b \leq x$. This implies that $L_t \subseteq L'$, as desired. $\square$

Lemma 2.2. Suppose that R is a transfer system on a finite lattice P and let $L = {}^{\boxslash}R$. Then $x\, L\, y$ if and only if $(x \leq y)$ satisfies the left-lifting property with respect to all relations of the form $z\, R\, y$.

Proof. Given any square

$$\begin{array}{ccc} \hat{0} & \longrightarrow & y \\ \big\downarrow & & \big\downarrow{\scriptstyle R} \\ x & \longrightarrow & z \end{array}$$

we can form the pullback $x \wedge y$ of $x \to z \leftarrow y$ to get another square

$$\begin{array}{ccc} \hat{0} & \longrightarrow & x \wedge y \\ \big\downarrow & & \big\downarrow R \\ x & \longrightarrow & x. \end{array}$$

We have a lift $x \to x \wedge y$ of this square if and only if we have a lift $x \to y$ of the first square. □

Recall from [FOO+22, Remark 4.7] that $R^{\perp} := ({}^{\boxslash}R)^{\mathrm{op}}$ is a transfer system on P^{op} and the assignment $R \mapsto R^{\perp}$ is an involution $\mathrm{Tr}\,P \to \mathrm{Tr}\,P^{\mathrm{op}}$. This leads naturally to the following definition:

Emphasize this more? Put in preliminaries subsection?

Definition 2.3. A *cotransfer system* on a complete lattice P is a transitive relation L refining $\leq$ such that $x\,L\,y$ and $x \leq z$ implies that $z\,L\,(z \vee y)$. The cotransfer system *generated* by a set of relations on P is the smallest cotransfer system containing those relations.

Remark 2.4. The cotransfer systems are exactly those sets of maps that arise as sets of left morphisms for a weak factorization system on P.

We can now state and prove our main theorem linking saturated and cosaturated transfer systems.

Theorem 2.5. *A transfer system R on a finite lattice P is saturated if and only if $R^{\perp}$ is cosaturated.*

Proof. Suppose first that R is cosaturated and let $L = {}^{\boxslash}R$. To show that $R^{\perp}$ is saturated, it suffices by duality to show that if $x\,L\,y$ then for all $z \in [x,y]$, we have $x\,L\,z$. By Lemma 2.1 and the definition of cosaturation, there is a set $S \subseteq P$ such that

$$L = {}^{\boxslash}\{w \to \hat{1} \mid w \in S\}.$$

Thus $x\,L\,y$ is equivalent to $[x,\hat{1}] \cap S = [y,\hat{1}] \cap S$. But if $x \leq z$, then clearly $[x,\hat{1}] \cap S \supseteq [z,\hat{1}] \cap S$, so if $x\,L\,y$ and $z \in [x,y]$, then we have

$$[x,\hat{1}] \cap S \supseteq [z,\hat{1}] \cap S \supseteq [y,\hat{1}] \cap S \supseteq [x,\hat{1}] \cap S$$

and hence $x\,L\,z$, as desired. This proves the backwards implication.

Now suppose R is saturated, let $L = {}^{\boxslash}R$, and let

$$S = \{x \in P \mid \hat{0}\,L\,x\}.$$

To show that $R^{\perp}$ is cosaturated, is suffices by duality to show that the cotransfer system L' generated by S is equal to L. We check this by proving that (L', R) is a weak factorization system.

By Lemma 2.2, note that $x \in S$ if and only if there is no $y < z$ with $y\,R\,x$. Suppose $x \leq y$ and let $z \in [x,y]$ be a minimal element for which $z\,R\,y$. Suppose $w\,R\,z$ for some $w \in P$. Since R is saturated, $w'\,R\,z$ for all $w' \in [w,z]$, so by transitivity $w'\,R\,y$. Thus by minimality of z, we have $[w,z] \cap [x,z] = \{z\}$, *i.e.*, $x \vee w = z$.

Now let $w \in P$ be a minimal element for which $w\,R\,z$. Then there is no $w' < w$ with $w'\,R\,w$ since transitivity of R this would violate minimality of w. Thus we have $w \in S$. But then $\hat{0}\,L'\,w$ and $\hat{0} \leq x$ implies $x\,L'\,(x \vee w) = z$, so $x\,L'\,z\,R\,y$ gives a factorization of $x \leq y$. Thus (L', R) forms a weak factorization system, whence $L' = L$, concluding our proof. □

This proof probably needs to be revisited and edited for clarity. Make it more diagrammatic?

Corollary 2.6. A transfer system on a finite lattice $(P,\leq)$ is cosaturated if and only if it is the set of fibrations in a model structure in which the set of cofibrations $F = \leq$.

Proof. The argument in [BOOR22, Remark 4.17] shows that saturated transfer systems are the same thing as acyclic fibrations in model structures in which all maps are fibrations. By Theorem 2.5, cosaturated transfer systems form the fibrations in model structures for which every map is a cofibration. □

3. Cryptomorphic structures: more than Moore families

In this section, we connect cosaturated transfer systems to the classical web of structures related to closure operators, ultimately proving Dodecatheorem 1.20. We begin by relating cosaturated transfer systems and closure systems.

Let 2^P denote the lattice of subsets of P, ordered by inclusion, and let $\mathrm{Tr}\,P$ denote the lattice of transfer systems on P. We define the maps

$$\begin{aligned} F\colon 2^P &\longrightarrow \mathrm{Tr}\,P \\ S &\longmapsto \langle x \leq \hat{1} \mid x \in S\rangle \end{aligned}$$

and

$$\begin{aligned} G\colon \mathrm{Tr}\,P &\longrightarrow 2^P \\ R &\longmapsto \{x \in P \mid x\,R\,\hat{1}\} \end{aligned}$$

which are evidently monotone.

Lemma 3.1. The pair (F,G) forms a monotone Galois connection which is a Galois correspondence between cosaturated transfer systems and closure systems (*i.e.*, Moore families).

Proof. If $S \subseteq P$ and R is a transfer system on P, then $F(S) \leq R$ if and only if $x\,R\,\hat{1}$ for all $x \in S$ if and only if $S \leq G(R)$. This proves that (F,G) is a Galois connection.

Since every Galois connection induces a Galois correspondence between images, it only remains to identify $F(2^P)$ and $G(\mathrm{Tr}\,P)$. Clearly if $S \subseteq P$, then $F(S)$ is cosaturated and all cosaturated transfer systems arise in this way. Suppose R is a transfer system on P. Then $G(R)$ contains $\hat{1}$ and is closed under $\wedge$ since R is closed under pullbacks, whence $G(R)$ is a closure system. Given a closure system S, we have $G(F(S)) = S$, so $G(\mathrm{Tr}\,P)$ is the collection of closure systems on P. □

We are now ready to prove the Dodecatheorem. Fix a finite lattice P. For the reader's convenience, we reproduce the twelve equivalent structures from the theorem statement here:

(1) cosaturated transfer systems on P,
(2) model structures on P in which every morphism is a fibration,
(3) closure operators on P,
(4) closure systems / Moore families on P,
(5) submonoids of $(P,\wedge)$,
(6) monads on the category associated with P,
(7) saturated transfer systems on P^{op},
(8) model structures on P^{op} in which every morphism is a cofibration,
(9) interior operators on P^{op},

(10) interior systems on P^{op},
(11) submonoids of $(P^{\text{op}}, \vee)$,
(12) comonads on the category associated with P^{op}.

Proof of Dodecatheorem 1.20. Bijections between (3), (4), (5), and (6) are well-known (see Section 1.1), and the bijections with (9), (10), (11), and (12) are dual. Lemma 3.1 gives an order-preserving bijection between (1) and (4), while Corollary 2.6 gives the bijection between (1) and (2). Finally, Theorem 2.5 provides the bijection between (1) and (7). □

4. Closure operators on rectangular lattices

In [HMOO22], U. Hafeez, P. Marcus, K. Ormsby, and A. Osorno compute the number of saturated transfer systems on the rectangular lattice $[m] \times [n]$. Since this is a self-dual lattice, Dodecatheorem 1.20 implies that their count also gives the number of closure operators on $[m] \times [n]$, along with the ten other structures enumerated therein. The proof in [HMOO22] is inductive, depending on a recurrence

In this section, we give a new, direct (non-inductive) count of closure operators on $[m] \times [n]$, phrased in the language of submonoids and subsemigroups. We hope that our methods might inspire enumerations of closure operators over other lattices.

Need to complete the argument from Ethan's note. I think the first term in the inclusion-exclusion is spelled out, but the higher terms in the inclusion-exclusion are not proven yet.

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MAX PLANCK INSTITUTE FOR MATHEMATICS
E-mail address: balchin@mpim-bonn.mpg.de

REED COLLEGE
E-mail address: emacbrough@reed.edu

REED COLLEGE
E-mail address: ormsbyk@reed.edu