Hochschild Homology:

classical, topological, & motivic

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Homotopy

Arithmetic $\swarrow$ $\searrow$ Geometry

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Outline

~~① Hochschild homology~~

② Topological & motivic variants

③ Computations over Spec $\mathbb{C}$

④ Two truths and a lie?

Hochschild homology of mod-p motivic cohomology over algebraically closed fields

Bjørn Ian Dundas, Michael A. Hill, Kyle Ormsby, Paul Arne Østvær

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Abstract

We perform Hochschild homology calculations in the algebro-geometric setting of motives. The motivic Hochschild homology coefficient ring contains torsion classes which arise from the mod-p motivic Steenrod algebra and from generating functions on the natural numbers with finite non-empty support. Under the Betti realization, we recover Bökstedt's calculation of the topological Hochschild homology of finite prime fields.

1 Introduction

Let $\mathscr{R}$ be a motivic ring spectrum such as algebraic cobordism, homotopy algebraic K-theory, or motivic cohomology [31]. Working in the stable motivic homotopy category $\mathscr{SH}(F)$ of a field F, we define the motivic Hochschild homology $\mathbf{MHH}(\mathscr{R})$ of $\mathscr{R}$ as the derived tensor product

$$\mathscr{R} \wedge_{\mathscr{R} \wedge \mathscr{R}^{\mathrm{op}}} \mathscr{R}. \tag{1}$$

The concepts of Hochschild homology for associative algebras and topological Hochschild homology for structured ring spectra inspire our constructions. In the event $\mathscr{R}$ is commutative one may equivalently to (1) form the tensor product in the category of commutative motivic ring spectra with the simplicial circle

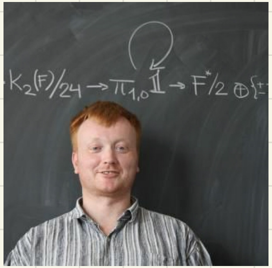
$$S^1 \otimes \mathscr{R}. \tag{2}$$

The primary purpose of this paper is to calculate the homotopy groups of motivic Hochschild homology of $\mathbf{M}\mathbb{F}_p$ over algebraically closed fields — the Suslin-Voevodsky mod-p motivic cohomology ring spectrum for p any prime number. When the base field F admits an embedding into the complex numbers $\mathbb{C}$, the Betti realization functor allows us to compare our $\mathbf{MHH}$ calculations with Bökstedt's pioneering work on topological Hochschild homology of the corresponding topological Eilenberg-Mac Lane spectrum $\mathbf{H}\mathbb{F}_p$. In fact, our calculation for $\mathbf{MHH}$ specializes to the one for $\mathbf{THH}$ in [8]. Additively, $\mathbf{THH}(\mathbb{F}_p)$ splits as a restricted product of Eilenberg-MacLane spectra in the stable homotopy category. This is not the case, however, for $\mathbf{MHH}(\mathbb{F}_p)$, $\mathbf{M}\mathbb{F}_p$, and $\mathscr{SH}(F)$. The source of this extra layer of complexity is the abundance of τ-torsion elements in the coefficients. Here τ is a canonical class in the mod-p motivic cohomology of F, which maps to the unit element in singular cohomology under Betti realization.

We express the coefficient ring $\mathbf{MHH}_\star(\mathbb{F}_p)$ in terms of algebra generators τ, μ_i, $\mathsf{x}_{S,f}$ arising from the mod-p motivic Steenrod algebra [17], [34], and generating endofunctions $f\colon \mathbb{N} \circlearrowleft$ with finite non-empty support containing some subset $S \subset \mathbb{N}$. The infinity of τ-torsion classes $\mathsf{x}_{S,f}$ is not witnessed in $\mathbf{THH}_\star(\mathbb{F}_p)$. For example, Kronecker delta functions give rise to such classes (in this case, S is either empty or a singleton set).

1

All work joint with Bjørn Dundas, Mike Hill, & Paul Arne Østvær



Topological Hochschild Homology

- For a ring spectrum R, $THH(R) := R \wedge_{R \wedge R^{op}} R$.
- If A is a classical commutative ring, $THH(A) := THH(\underbrace{HA}_{\text{Eilenberg-MacLane spectrum}})$
- When R is E_∞, we have

$$THH(R) = S^1 \underbrace{\otimes}_{\substack{\text{spectra are tensored} \\ \text{over simplicial sets}}} R$$

arising from

$$S^1 = \operatorname{colim}\begin{pmatrix} S^0 & \longrightarrow & * \\ \downarrow & & \\ I & & \end{pmatrix}.$$

Computations

Since $THH(A) = HA \wedge_{HA \wedge HA} HA$ and $\pi_* HA = A$ (in deg 0),

we have a Tor-spectral sequence

$$E^2_{h,t} = \mathrm{Tor}_{h,t}^{\pi_* HA \wedge HA}(A, A) \Longrightarrow THH_{h+t}(A)$$

and $d^r : E^r_{h,t} \longrightarrow E^r_{h-r,\, t+r-1}$.

For $A = \mathbb{F}_2$, $\pi_* H\mathbb{F}_2 \wedge H\mathbb{F}_2 = \mathcal{A}_* = \mathbb{F}_2[\xi_1, \xi_2, \xi_3, \ldots]$, $|\xi_i| = 2^i - 1$

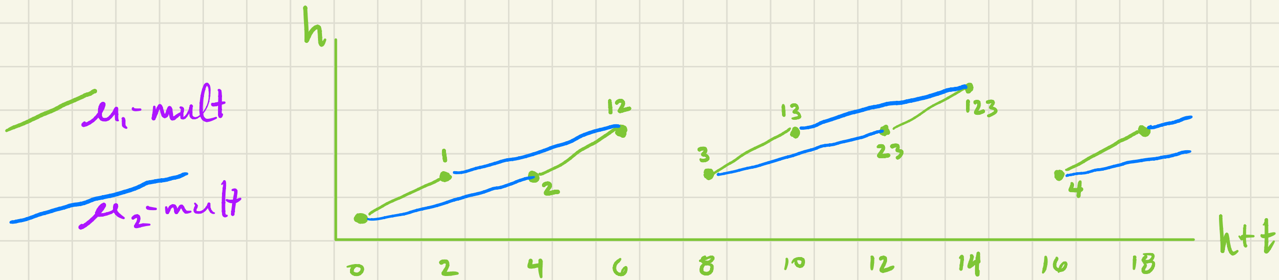
is the mod-2 dual Steenrod algebra.

Thus $E^2 \cong \Lambda_{\mathbb{F}_2}(\mu_1, \mu_2, \mu_3, \ldots)$ with $|\mu_i| = (1, 2^i - 1)$.

Pictorially ...

Computations (ct'd)

$E^2 \cong \Lambda_{\mathbb{F}_2}(\mu_1, \mu_2, \mu_3, \ldots)$ with $|\mu_i| = (1, 2^i - 1)$



Note ① One class in each even degree.

② Differentials go 1 left & down, so $E^2 = E^\infty$.

Fact By a power operations argument, $\mu_i^2 = \mu_{i+1} \in THH_*(\mathbb{F}_2)$.

Thm (Bökstedt) $THH_*(\mathbb{F}_p) = \mathbb{F}_p[\mu]$, $|\mu| = 2$ for all primes p.

⟨new⟩

Motivic Hochschild Homology

Now move from spectra to motivic spectra and replace HA with MA, the motivic Eilenberg-Mac Lane spectrum.

Morel-Voevodsky:

$$X \times \mathbb{A}^1 \longrightarrow Y$$

$$[Sm/k^{op}, [\Delta^{op}, Set]] \;+\; \text{Nisnevich \& } \mathbb{A}^1\text{-localize} \;=\; \text{motivic spaces } Spc_k$$

$$\left(Sm/k \xrightarrow{\;y\;} [Sm/k^{op}, [\Delta^{op}, Set]] \xleftarrow{\;const\;} [\Delta^{op}, Set] \right)$$

$$\Big\downarrow \;\text{invert } \mathbb{P}^1 \wedge -$$

motivic spectra Sp_k

Motivic Hochschild Homology (ct'd)

$$\begin{tikzcd} \mathbb{A}^1 \setminus 0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathbb{A}^1 \simeq * \arrow[d] \\ * \simeq \mathbb{A}^1 \arrow[r] & \mathbb{P}^1 \end{tikzcd}$$

$\implies \mathbb{P}^1 \simeq \underbrace{S^1}_{\text{simplicial circle}} \wedge \underbrace{(\mathbb{A}^1 \setminus 0)}_{\text{geometric circle}}$.

Upshot Bigraded spheres $S^{m,n} := (S^1)^{\wedge m-n} \wedge (\mathbb{A}^1 \setminus 0)^{\wedge n}$

total degree (m), weight (n)

Thus we have bigraded homotopy groups

$$\pi_{m,n} X := [S^{m,n}, X]$$

for $X \in Sp_k$.

Note Need homotopy sheaves to detect weak equivs.

Motivic Hochschild Homology (ct'd)

Important (co)homology theories are representable in Sp_k:

- MA = motivic cohomology with coefficients in A
- KGL = (homotopy) algebraic K-theory
- KQ = Hermitian K-theory
- MGL = algebraic cobordism

We define the motivic Hochschild homology of a commutative ring A to be the motivic spectrum

$$\boxed{MHH(A) := MA \wedge_{MA \wedge MA} MA = S^1 \otimes MA}$$

$$(\mathbb{A}^1 \setminus 0) \otimes MA$$

with coefficients $MHH_{*,*}(A) = \bigoplus_{m,n \in \mathbb{Z}} \pi_{m,n} MHH(A)$.

Computations over $\mathbb{C}$

Fix k algebraically closed, $A = \mathbb{F}_p$, $p \neq \mathrm{char}(k)$.

$\therefore \left\{ \begin{array}{l} THH_*(\mathbb{F}_p) \\ = \mathbb{F}_p[\mu] \end{array} \right.$

Theorem 1.1. *Over an algebraically closed field of exponential characteristic* $\mathsf{e}(F) \neq p$*, there is an algebra isomorphism*

$$\mathbf{MHH}_\star(\mathbb{F}_p) \cong \mathbb{F}_p[\tau, \mu_i, \mathsf{x}_{S,f}]_{i\in\mathbb{N},\, (S\subset \mathrm{supp}\, f,\, f\colon \mathbb{N}\circlearrowleft)}/\mathcal{I} \tag{3}$$

finite support

with the ideal of relations

$$\mathcal{I} = \begin{pmatrix} \mu_i^p - \tau^{p-1}\mu_{i+1}, \\ \tau^{p-1}\mathsf{x}_{S,f}, \\ \mathsf{x}_{S,f}\cdot \mathsf{x}_{T,g} - \sum\limits_{u\in \mathrm{supp}(f+g)-S\cup T} \epsilon_u \cdot \mathsf{x}_{S\cup T\cup\{u\}, f+g} \end{pmatrix}.$$

Here the support of f is a finite non-empty subset of the natural numbers and $S \subset \mathrm{supp} f \subset \mathbb{N}$ *does not contain the minimal element of* $\mathrm{supp} f$. *The coefficient* $\epsilon_u \in \mathbb{F}_p$ *is given explicitly in Definition 2.12. The algebra generators have bidegrees given by* $|\tau| = (0,-1)$, $|\mu_i| = (2p^i, p^i - 1)$, *and*

$$|\mathsf{x}_{S,f}| = (|S|+1)(-1, p-1) + p \sum_{j\in \mathrm{supp} f} f(j)(2p^j, p^j - 1).$$

Computations over $\mathbb{C}$ (ct'd)

Fix k algebraically closed, $A = \mathbb{F}_p$, $p \neq \mathrm{char}(k)$.

Theorem 1.1. *Over an algebraically closed field of exponential characteristic* $\mathsf{e}(F) \neq p$, *there is an algebra isomorphism*

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with the ideal of relations

$$\mathcal{I} = \begin{pmatrix} \mu_i^p - \tau^{p-1}\mu_{i+1}, \\ \tau^{p-1}\mathsf{x}_{S,f}, \\ \mathsf{x}_{S,f}\cdot \mathsf{x}_{T,g} - \sum\limits_{u\in \mathrm{supp}(f+g)-S\cup T} \epsilon_u \cdot \mathsf{x}_{S\cup T\cup\{u\}, f+g} \end{pmatrix}.$$

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$$|\mathsf{x}_{S,f}| = (|S|+1)(-1, p-1) + p \sum_{j\in \mathrm{supp} f} f(j)(2p^j, p^j - 1).$$

Definition 2.12. For functions $f, g\colon \mathbb{N} \circlearrowleft$ with finite support and non-empty finite sets $S, T \subseteq \mathbb{N}$ define $K_{S,T,f,g} \in \mathbb{F}_p$ by

$$K_{S,T,f,g} = \left(\prod_{s\in S} \binom{fs - 1 + gs}{fs - 1}\right)\left(\prod_{t\in T}\binom{ft + gt - 1}{ft}\right)\left(\prod_{c\notin S\cup T}\binom{fc + gc}{fc}\right)$$

if $(S, f), (T, g) \in J$ and $S \cap T = \emptyset$, and set $K_{S,T,f,g} = 0$ otherwise. Moreover, we define

$$\epsilon_{u,S,T,f,g} = K_{S\cup\{u\}, T\cup\{t_{f+g}\}, f+g} + K_{S\cup\{t_{f+g}\}, T\cup\{u\}, f+g}.$$

Yikes!

Goals ① Shape of the computation when $p = 2$.

② Consequences in motivic homotopy.

Computation Strategy

Input $\pi_{**} M\mathbb{F}_2 = \mathbb{F}_2[\tau]$, $|\tau| = (0,-1)$

$$\mathcal{A}_{**} = \pi_{**} M\mathbb{F}_2 \wedge M\mathbb{F}_2 \cong \mathbb{F}_2[\tau, \xi_1, \xi_2, \dots, \tau_0, \tau_1, \dots] / (\tau_i^2 - \tau\xi_{i+1} \mid i \geq 0)$$

$$|\xi_i| = (2^{i+1}-2, 2^i-1), \quad |\tau_i| = (2^{i+1}-1, 2^i-1)$$

Step 1 Calculate étale motivic Hochschild homology

$$MHH_{**}(\mathbb{F}_2)[\tau^{-1}] \cong \mathbb{F}_2[\tau^{\pm 1}, \mu_0, \mu_1, \dots] / (\mu_i^2 - \tau\mu_{i+1} \mid i \geq 0)$$

$$\cong THH_*(\mathbb{F}_2)[\tau^{\pm 1}].$$

Step 2 Calculate mod-τ MHH

$$MHH_*(\mathbb{F}_2)/\tau \cong \Gamma_{\mathbb{F}_2}(\bar{\mu}_0, \bar{\mu}_1, \dots) \otimes \Lambda_{\mathbb{F}_2}(\bar{\lambda}_1, \bar{\lambda}_2, \dots).$$

divided powers algebra

Computation Strategy

Step 3 τ-torsion in $MHH_{**}(\mathbb{F}_2)$ injects into $MHH_{**}(\mathbb{F}_2)/\tau$ with image that of the τ-Bockstein.

Step 4 Give a presentation of τ-torsion in $MHH_{**}(\mathbb{F}_2)$ in terms of generators $x_{S,f}$ where $f: \mathbb{N} \to \mathbb{N}$ has finite support and $S \subseteq \operatorname{supp} f$.

Step 5 Combine étale and τ-torsion computations via a pullback square.

Step 1 **Étale MHH**

$$\pi_{**} M\mathbb{F}_2 \wedge M\mathbb{F}_2[\tau^{-1}] \cong \mathbb{F}_2[\tau^{\pm 1}, \xi_1, \xi_2, \dots, \tau_0, \tau_1, \dots] / (\tau_i^2 - \tau\xi_{i+1} \mid i \geq 0)$$

$$\cong \mathbb{F}_2[\tau^{\pm 1}, \tau_0, \tau_1, \dots], \quad |\tau_i| = (2^{i+1}-1, 2^i - 1)$$

so the Tor-spectral sequence takes the form

$$E^2 = \mathrm{Tor}_{*,*,*}^{\mathbb{F}_2[\tau^{\pm 1}, \tau_0, \tau_1, \dots]}(\mathbb{F}_2[\tau^{\pm 1}], \mathbb{F}_2[\tau^{\pm 1}])$$

$$\cong \Lambda_{\mathbb{F}_2}(\mu_0, \mu_1, \dots)[\tau^{\pm 1}] \Longrightarrow MHH_{**}(\mathbb{F}_2)[\tau^{\pm 1}]$$

with $|\mu_i| = (1, 2^{i+1}-1, 2^i - 1)$.

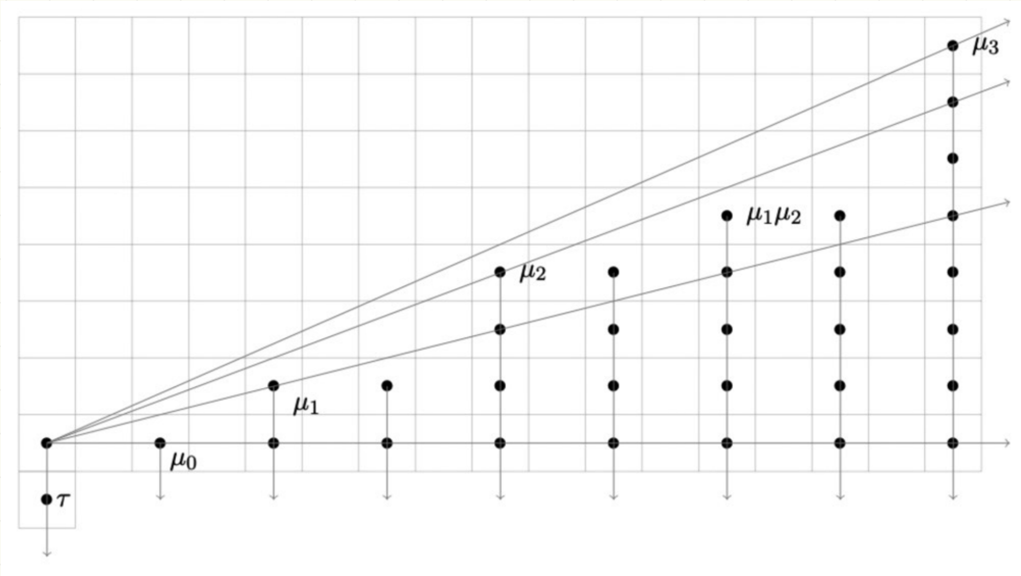
- Degree considerations $\Rightarrow E^2 = E^\infty$.
- Power operations $\Rightarrow \mu_i^2 = \tau \mu_{i+1}$.

Step 1 **Étale MHH** (ct'd)

Upshot $\quad MHH_{**}(\mathbb{F}_2)[\tau^{-1}] \cong \mathbb{F}_2[\tau^{\pm 1}, \mu_0, \mu_1, \ldots] / (\mu_i^2 - \tau \mu_{i+1} \mid i \geq 0)$

$$\cong \mathbb{F}_2[\tau^{\pm 1}, \mu_0]$$

$\mathrm{im}\left(MHH_{**}(\mathbb{F}_2) \to MHH_{**}(\mathbb{F}_2)[\tau^{-1}]\right)$



Step 2 # Mod-τ MHH

$\mathcal{A}_{**}/\tau \cong \mathbb{F}_2[\xi_1, \xi_2, \ldots] \otimes \Lambda_{\mathbb{F}_2}(\tau_0, \tau_1, \ldots)$ so the Tor-spectral sequence takes the form

$$E^2 = \mathrm{Tor}^{\mathcal{A}_{**}/\tau}_{*,*,*}(\mathbb{F}_2, \mathbb{F}_2)$$

divided powers algebra

$$\cong \Lambda_{\mathbb{F}_2}(\bar\lambda_1, \bar\lambda_2, \ldots) \otimes \Gamma_{\mathbb{F}_2}(\bar\mu_0, \bar\mu_1, \ldots) \Rightarrow \mathrm{MHH}_{**}(\mathbb{F}_2)/\tau.$$

Advanced degree yoga $\Rightarrow$ $E^2 = E^\infty$ with no hidden ext'ns.

Upshot

$$\mathrm{MHH}_{**}(\mathbb{F}_2)/\tau \cong \Lambda_{\mathbb{F}_2}(\bar\lambda_1, \bar\lambda_2, \ldots) \otimes \Gamma_{\mathbb{F}_2}(\bar\mu_0, \bar\mu_1, \ldots).$$

(See arXiv: 2204.0041 for Steps 3-5.)

④ Consequences

Since $THH(\mathbb{F}_p)$ is an $H\mathbb{F}_p$-module, we get the splitting

$$THH(\mathbb{F}_p) \simeq \bigvee_{i\geq 0} \Sigma^{2i} H\mathbb{F}_p .$$

The τ-torsion in $MHH_{**}(\mathbb{F}_p)$ implies that this fails wildly in $Sp_{\mathbb{C}}$:

$MHH(\mathbb{F}_p)$ is <u>not</u> a free $M\mathbb{F}_p$-module.

In Sp, the following are true:

1. $H\mathbb{F}_2$ is a Thom spectrum of an E_2-map with target $\Omega^2 S^3$
2. $THH(\mathrm{Thom}_1) = \mathrm{Thom}_2 \underset{\eta_{top}=0}{=} \mathrm{Thom}_1 \wedge B(\mathrm{base}_1)_+$.
3. $B\Omega^2 S^3 \simeq \bigvee(\text{spheres})$ stably.

Consequences (ct'd)

Behrens-Wilson: true C_2-equivariantly

Potential motivic analogues:

1. $M\mathbb{F}_2$ is a Thom spectrum over $\Omega^{2,1}S^{3,1}$.
2. $MHH(\mathrm{Thom}_1) \simeq \mathrm{Thom}_1 \wedge B(\mathrm{base}_1)_+$.
3. $B\Omega^{2,1}S^{3,1}$ is a wedge of spheres stably, i.e. $\Sigma^\infty \Omega^{1,1}\Sigma^{1,1}S^2$ satisfies "$\mathbb{G}_m$-James splitting".

Thm (Dundas, Hill, O, Østvær) At most two of these are true!

Consequences (ct'd)

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<u>Thm</u> (Dundas, Hill, O, Østvær) At most two of these are true!

<u>Poll</u> Rank from <u>most</u> to <u>least</u> likely <u>FALSE</u> at app.sli.do, code #MotivicHochschild.

Thank You!

Questions welcome.

Bonus content!

Power operations and $\mu_i^p = \tau^{p-1}\mu_{i+1}$:

- Mimic classical Dyer-Lashof operations via E_∞-structure.
- Show $Q^s\sigma = \sigma Q^{s}$: $\pi_{**} M\mathbb{F}_p \wedge M\mathbb{F}_p \longrightarrow MHH_{*+1+2s(p-1),p*}(\mathbb{F}_p)$ along classical lines. (Here $\mu_i = \sigma\tau_i$.)
- Rigidity + Betti realization$/\mathbb{C}$ imply the result, with τ^{p-1} compensating for weights.

Definition of $x_{S,f}$ for $f: \mathbb{N} \to \mathbb{N}$, $S \subseteq \mathrm{supp}(f)$, $|\mathrm{supp}(f)| < \infty$:

- $x_{S,f} = \tau$-Bockstein of $\chi_{S,f}$ where
- $\chi_{S,f} = \left(\prod_{m \in S} \bar{\lambda}_{m+1}\gamma_{pf(m)-p}\bar{\mu}_m\right)\left(\prod_{n \notin S}\gamma_{pf(n)}\bar{\mu}_n\right).$