Counting Matchstick Games on Modular Lattices

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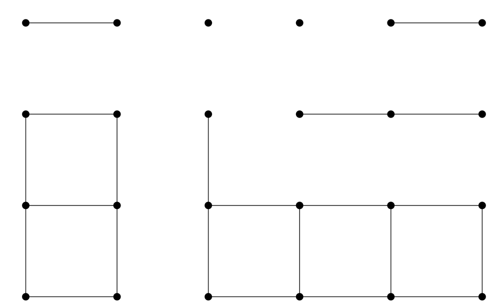
Matchstick Games

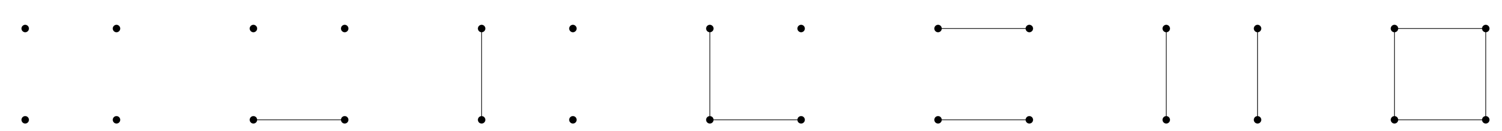
Consider the following grid, where lines can be drawn between adjacent vertices in the cardinal directions.



A matchstick game on $[5] \times [3]$.

Let $[m] := \{0, 1, ..., m\}$ and $[n] := \{0, 1, ..., n\}$. A matchstick game on an $[m] \times [n]$ grid is subject to the following rules:

1. If a vertical "matchstick" (line) is present, every matchstick directly to the left must be present.
2. If a horizontal matchstick is present, every matchstick directly below must be present.
3. No $[1] \times [1]$ square can have exactly 3 matchsticks.



All possible $[1] \times [1]$ games .

To generalize the rules to grids of arbitrary dimension, we just require every plane to be a matchstick game.

Matchstick Games in 2D

Let $s(m, n)$ denote the number of matchstick games on an $[m] \times [n]$ grid. By [HMOO22], we have

$$s(m,n) = \sum_{j=2}^{m+2} (-1)^{m-j} \left\{ {m+1 \atop j-1} \right\} \frac{j!}{2} j^n,$$

where $\left\{ {r \atop s} \right\}$ is the number of ways to partition a set of r elements into s nonempty subsets.

A max-closed relation R on a 2D grid is a relation where $x_1 \; R \; y_1$ and $x_2 \; R \; y_2$ implies $\max(x_1, x_2) \; R \; \max(y_1, y_2)$.

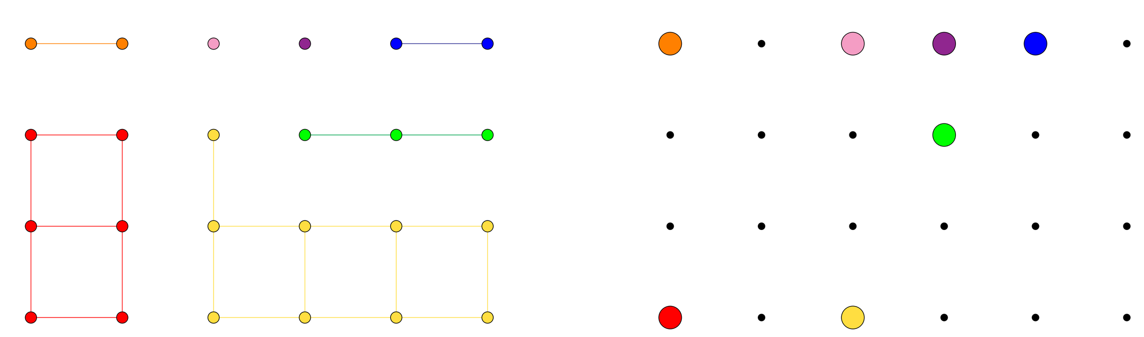
Let $B(m, n)$ denote the (m, n)-th poly-Bernoulli (pB) number. The pB numbers count max-closed relations, and, as written in [Knu24], a closed formula for the (m, n)-th pB number is given by

$$B(m,n) = \sum_{k=0}^{\min(m,n)} k! \left\{ {m+1 \atop k+1} \right\} k! \left\{ {n+1 \atop k+1} \right\}.$$

We have that

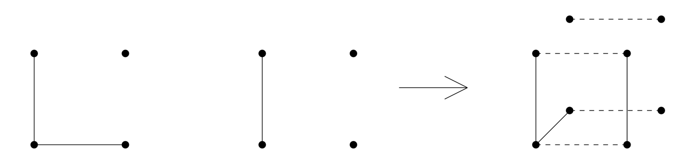
$$s(m,n) = \frac{1}{2} B(m+1, n+1).$$

While pB numbers do not have a nice higher dimensional generalization, max-closed relations, like matchstick games, generalize to higher dimensions.



Correspondence between a matchstick game and a max-closed relation

Facing matrix



Possible pole arrangements given two faces.

Consider the $[1] \times [1] \times [n]$ matchstick game. Order the 7 possible matchstick games on $[1] \times [1]$, then let A be the matrix where

$$A_{i,j} = |\{\text{Match-stick games with left face } i \text{ and right face } j\}|.$$

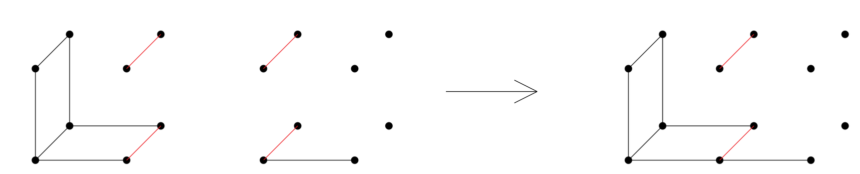
$$\begin{pmatrix} 6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 4 & 0 & 0 & 0 & 0 \\ 3 & 3 & 0 & 3 & 0 & 0 & 0 \\ 2 & 2 & 2 & 0 & 3 & 0 & 0 \\ 3 & 0 & 3 & 0 & 0 & 3 & 0 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{pmatrix}$$

Example: facing matrix for $[1] \times [1]$

We have that

$$\sum_{1 \le i,j \le 7} A_{i,j} = \text{ the number of match-stick games on } [1]^3.$$

Notice that matchstick games on the grid $[1] \times [1] \times [n]$ can be constructed by stacking $[1] \times [1] \times [1]$ cubes that share a face.



Constructing a matchstick game on $[1] \times [1] \times [2]$ from two cubes.

Since multiplying A by itself corresponds to this stacking, we get that $(A^n)_{i,j}$ is the number of matchstick games on $[1] \times [1] \times [n]$ with left face i and right face j. Generally we have that

$$\sum_{i,j} (A^n)_{i,j} = \text{ the number of match-stick games on } [1] \times [1] \times [n].$$

This matrix can be constructed for $[\ell] \times [m]$ for all $\ell, m \ge 0$.

Theorem (Diagonalizability)
The facing matrix A is diagonalizable.

Proof. Because the set of matchstick games forms a lattice, there exists an ordering s.t. A is lower triangular and the diagonal entries are monotone decreasing. These diagonal entries λ_i are exactly the eigenvalues. The blocks surrounding each distinct eigenvalue must be 0 off the diagonal because the ordered faces with the same eigenvalues cannot be subsets of each other. □

As a corollary, if A has eigenvalues $\lambda_1, \ldots, \lambda_k$, then $s(\ell, m, n) = \sum_{i=1}^{k} c_i \lambda_i^n$ for some rational numbers c_i depending only on ℓ, m. For example, the closed form for matchstick games on $[1] \times [1] \times [n]$ is

$$s(1,1,n) = \frac{35}{2} \cdot 6^n - 12 \cdot 4^n + 3^n + \frac{1}{2} \cdot 2^n.$$

Generating Functions and Recurrence

Theorem (Recurrence)
Let $F_{\ell,m}(x)$ be the ordinary generating function for $s(\ell, m, n)$ with ℓ, m fixed. Then by the diagonalization theorem we have

$$F_{\ell,m}(x) = \sum_n s(\ell,m,n) x^n = \sum_n \sum_{i=1}^{k} c_i \lambda_i^n x^n = \sum_{i=1}^{k} \frac{c_i}{1 - \lambda_i x} = \frac{P(x)}{\prod (1 - \lambda_i x)}.$$

Let $a_i = [x^i] \prod (1 - \lambda_i x)$, then for $n \ge k$ we have the recurrence

$$s(\ell,m,n) + a_1 s(\ell,m,n-1) + \cdots + a_k s(\ell,m,n-k) = 0.$$

Let G be the exponential generating function for $2s(\ell - 1, m - 1, n - 1)$ in three variables x, y, z. Based on initial conditions, we have

$$G(x,y,0) = e^{x+y} \qquad \left.\frac{\partial G}{\partial z}\right|_{z=0} = \frac{e^{x+y}}{e^x + e^y - e^{x+y}}.$$

Based on the closed formula given by the facing matrix, we get

$$\left.\frac{\partial G(x,y,z)}{\partial x^l \partial y^m}\right|_{x,y=0} = \sum_i b_i e^{\lambda_i z}.$$

Conjecture
The closed form of the exponential generating function G is a rational function on the variables e^x, e^y, e^z.

If the conjecture is true, then $G(x,y,z) = (e^x - 1)(e^y - 1)(e^z - 1) \frac{P}{Q}(e^x, e^y, e^z)$ for polynomials P, Q.

Connections

For a finite Abelian group G, matchstick games on $\mathrm{Sub}(G)$ are in bijection with saturated transfer systems of G. Saturated transfer systems are important in enumerative equivariant homotopy theory [BHO+24], which is the motivation for counting matchstick games.

Matchstick games on finite modular lattices P are in bijection with

1. submonoids of $(P, \vee)$,
2. interior operators on P,
3. comonads on P,
4. cofibrant model structures on P, and
5. coreflective factorization systems for P.

References

[BHO+24] Andrew J. Blumberg, Michael A. Hill, Kyle Ormsby, Angélica M. Osorno, and Constanze Roitzheim. Homotopical combinatorics. Notices of the American Mathematical Society, 2024.

[HMOO22] Usman Hafeez, Peter Marcus, Kyle Ormsby, and Angélica M. Osorno. Saturated and linear isometric transfer systems for cyclic groups of order $p^m q^n$. Topology and its Applications, 317:108162, 2022.

[Knu24] Donald Knuth. Parades and poly-bernoulli bijections. 2024.

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