Milnor forms of algebraic singularities

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Preliminary version.

These notes were produced for the 2021 PCMI Undergraduate Faculty Program and form a breezy introduction to algebraic curve and hypersurface singularities and their Milnor forms, *i.e.*, $\mathbb{A}^1$-Milnor numbers. They include some recollections on classical Milnor numbers, a quick development of the algebraic theory of quadratic forms, and the construction of (local) motivic degree and its relation with the Eisenbud–Levine/Khimshiashvili form. Everything is motivated by the second derivative test from multivariable calculus, and we conclude with some open research problems regarding resolution of singularities over non-algebraically closed fields.

1 *A Hessian telescope*

Every multivariable calculus student learns the second (partial) derivative test for classifying critical points of a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ with continuous second order derivatives:

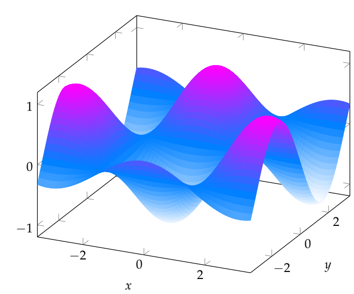

Figure 1: The second derivative test allows us to classify the critical points of a function like $f(x,y) = \cos(x)\sin(y)$. Exercise: Visually identify some local extrema and saddle points in this picture, then check that the second derivative test verifies your inspection.

Theorem 1.1 (Second derivative test)**.** *Suppose $f\colon \mathbb{R}^2 \to \mathbb{R}$ is a C^2 function and that $\nabla f(a,b) = (\frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b)) = (0,0)$. Let*

$$Hf(a,b) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(a,b) & \frac{\partial^2 f}{\partial x \partial y}(a,b) \\ \frac{\partial^2 f}{\partial y \partial x}(a,b) & \frac{\partial^2 f}{\partial y^2}(a,b) \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \beta & \delta \end{pmatrix}$$

denote the Hessian matrix of f at (a,b). Then

(1) if $\alpha > 0$ and $\alpha\delta - \beta^2 > 0$, then $f(a,b)$ is a local minimum;

(2) if $\alpha < 0$ and $\alpha\delta - \beta^2 > 0$, then $f(a,b)$ is a local maximum;

(3) if $\alpha\delta - \beta^2 < 0$, then $f(a,b)$ is a saddle point.

Being a symmetric matrix, $Hf(a,b)$ has an assocaited quadratic form

$$(x,y) \in \mathbb{R}^2 \longmapsto Qf_{(a,b)}(x,y) := (x\ y)Hf(a,b)\begin{pmatrix} x \\ y \end{pmatrix} = \alpha x^2 + 2\beta xy + \delta y^2.$$

Quadratic Taylor approximation tells us there is some $c \in [0,1]$ such that for all small enough $h,k \in \mathbb{R}$,

$$f(a+h, b+k) = f(a,b) + \frac{\partial f}{\partial x}(a,b)h + \frac{\partial f}{\partial y}(a,b)k + \frac{1}{2}Qf_{(a+ch,b+ck)}(h,k).$$

When $Qf_{(a,b)}$ is nondegenerate — meaning that $\alpha\delta - \beta^2 \neq 0$ — it is a consequence of continuity of second order derivatives that the "shape" of $Qf_{(a,b)}$ is the same for small perturbations of a and b. Thus, when it is nondegenerate, $Qf_{(a,b)}$ controls the local behavior of f near critical points (where $\nabla f(a,b) = 0$). Indeed, the conditions of the second derivative test precisely describe when $Qf_{(a,b)}$ is (1) *positive definite*, (2) *negative definite*, or (3) *indefinite*.

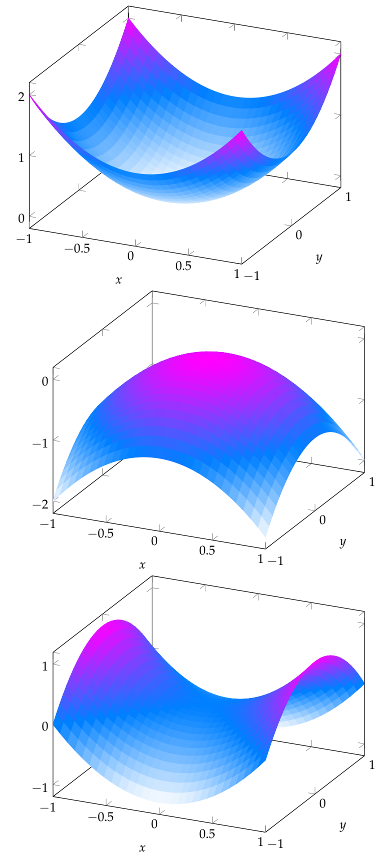

Figure 2: Graphs of positive definite, negative definite, and indefinite regular quadratic forms of rank 2.

Presently, I would like to use the second derivative test as a telescope through which we will spy some fragments of further mathematics:

(1) We only sacrifice the convenient numerology of Theorem 1.1 in passing to functions of more variables. The Hessian matrix of $f\colon \mathbb{R}^n \to \mathbb{R}$ still has an associated quadratic form Qf, and the definiteness of Qf (when it is nondegenerate) still controls the extremal properties of f near critical points. The isometry class of Qf is encoded in its *Sylvester type*: there is an invertible matrix S such that $SHS^\top$ is diagonal with only 0's, 1's, and -1's on the diagonal. The triple (n_0, n_+, n_-) counting the number of such entries in $SHS^\top$ is the Sylvester type of Qf and is invariant under choice of S. This corresponds to the n-ary quadratic form

$$\sum_{i=1}^{n_+} x_i^2 - \sum_{j=n_++1}^{n_++n_-} x_j^2.$$

and is nondegenerate if and only if $n_0 = 0$. Sylvester type $(0, n, 0)$ gives a local minimum, while $(0, 0, n)$ gives a local maximum. When the Sylvester type is $(0, p, q)$ for $q = n - p$, we call the critical point nondegenerate with *index* (p, q) or *signature* $p - q$.

(2) In the $n = 2$ case, we can study the shape of f via its *contour lines*, also known as *level curves*. These are solutions to $f(x, y) = d$ for some fixed $d \in \mathbb{R}$. For general n, these solution sets are *level hypersurfaces*. If f is a polynomial function, then we are working with algebraic hypersurfaces. The critical points of f correspond to *singularities* in the associated hypersurfaces.

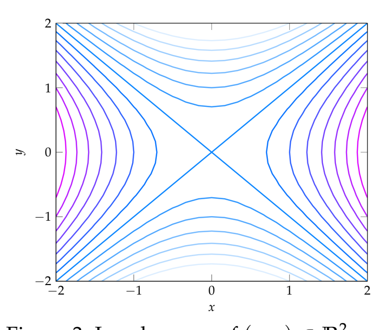

Figure 3: Level curves of $(x, y) \in \mathbb{R}^2 \mapsto x^2 - y^2$.

(3) What can we do with holomorphic functions $\mathbb{C}^n \to \mathbb{C}$? Or polynomial functions $k^n \to k$ for a general field k? Our primary focus will be on plane algebraic curves, which are the level curves of polynomials $f(x, y) \in k[x, y]$, but we will also consider algebraic hypersurfaces. Questions about extreme values quickly lose meaning in this generality, but there is still much that we can say about the "shape" of singularities.

(4) The classification of real quadratic forms by Sylvester type just scratches the surface of the algebraic theory of quadratic forms.

As we move to a general base field k, we will use the *Grothendieck–Witt ring* $\mathrm{GW}(k)$ to track isometry types of nondegenerate quadratic forms. Over $k = \mathbb{C}$, isometry type is completely determined by dimension and $\mathrm{GW}(\mathbb{C}) \cong \mathbb{Z}$. To the eyes of *quadratic enumerative geometry*, this isomorphism is responsible for classical enumerations producing numbers. Over $k = \mathbb{R}$, both dimension and signature are necessary in order to classify forms, but not every dimension-signature pair is achievable; indeed, $\mathrm{GW}(\mathbb{R}) \cong \mathbb{Z}[h]/(h^2 - 2h)$ where h corresponds to the hyperbolic plane $x^2 - y^2$. For more general fields, the structure of $\mathrm{GW}(k)$ can be fairly exotic. For instance, $\mathrm{GW}(\mathbb{Q})$ is countably-infinitely generated as an Abelian group and "knows about" Hilbert and quadratic reciprocity.

The invariant we aim to study is the *Milnor form* of a plane curve (or hypersurface) singularity. We begin with the story of classical Milnor numbers; these are defined for hypersurfaces over $\mathbb{C}$ and agree with the rank of the associated Milnor form.

2 Classical Milnor fibers and numbers

Suppose that $f \in \mathbb{C}[x_1, \ldots, x_n]$ is a complex polynomial in n variables. The *hypersurface* cut out by f is

$$V(f) := \{x \mid f(x) = 0\} \subseteq \mathbb{C}^n.$$

A *singular point* p of $V(f)$ is a simultaneous solution of the equations

Recall that $\nabla f = (\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$ is the *gradient* of f.

$$f(p) = 0, \qquad \nabla f(p) = 0.$$

We let $\mathrm{Sing}(f) \subseteq V(f)$ denote the singular points of $V(f)$, and we call elements of $V(f) \smallsetminus \mathrm{Sing}(f)$ *regular points* of f.

Our initial goal is to get a feeling for the local topology of $V(f)$. We begin with a regular point p of $V(f)$ and assume for simplicity that $\frac{\partial f}{\partial x_n}(p) \neq 0$. Then, by the implicit function theorem, there is a neighborhood of p in $V(f)$ that is the graph of an analytic function of the first $n-1$ variables. Thus, near a regular point, $V(f)$ has the topology of $\mathbb{C}^{n-1}$, with real dimension $2n-2$. This is why the algebraic geometer's plane curves (which have real dimension 2 in $\mathbb{C}^2 \cong \mathbb{R}^4$) are sometimes thought of as surfaces.

Now consider $p \in \mathrm{Sing}(f)$. Since $\nabla f(p) = 0$, we no longer have recourse to the implicit function theorem. To simplify matters, let us assume that p is an *isolated* singularity, *i.e.*, that there exists a neighborhood of p in $V(f)$ in which p is the only singularity. The idea used to such great effect by Milnor[1] is to study the neighborhood of p in "slices" according to distance (in $\mathbb{C}^n$) from p. To be more precise, for $\varepsilon > 0$, let

$$S_\varepsilon^{2n-1}(p) = \{x \in \mathbb{C}^n \mid \|x - p\| = \varepsilon\}$$

[1] Milnor, J. (1968). *Singular points of complex hypersurfaces*. Annals of Mathematics Studies, No. 61. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo

denote the radius ε sphere centered at p. Since p is an isolated singularity of f, for small enough ε, $S^{2n-1}_\varepsilon(p)$ only intersects $V(f)$ at regular points, so

$$K_{p,\varepsilon}(f) := S^{2n-1}_\varepsilon(p) \cap (V(f) \smallsetminus \mathrm{Sing}(f))$$

is the intersection of a real codimension 1 manifold and real codimension 2 manifold. As such, $K_{p,\varepsilon}(f)$ is a real codimension 3 submanifold of $\mathbb{C}^n$ and real codimension 2 submanifold of the sphere $S^{2n-1}_\varepsilon(p)$. In the plane curve case ($n = 2$), we get a link in S^3! The number of components in the link corresponds to the number of branches — irreducible local components — of $V(f)$ at p.

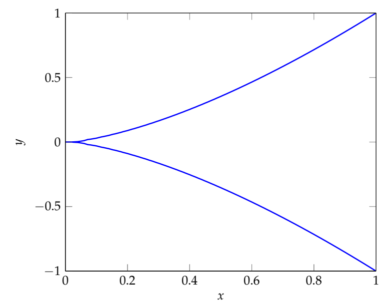

Figure 4: The real points of $V(x^3 - y^2)$.

Remark 2.1. In what follows, I will make statements about hypersurfaces in $\mathbb{C}^n$ whenever their complexity is bounded above by the specialization to plane algebraic curves (the $n = 2$ case). The reader is encouraged to mentally engage with the $n = 2$ specialization throughout, and most of my examples will be curves.

Example 2.2. Consider the polynomial $f(x,y) = x^3 - y^2$. The real points of $V(f)$ form a cusp.

One may check[2] that $K_{0,\varepsilon}(f)$ lies on a torus of the form

$$\{(x,y) \mid \|x\| = \xi,\ \|y\| = \eta\}$$

for some positive constants ξ, η. In fact, $K_{0,\varepsilon}(f)$ is a $(2,3)$-torus knot, *aka* a trefoil knot.

[2] Exercise!

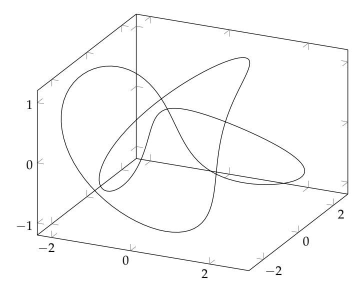

Figure 5: A projection of $K_{0,\varepsilon}(x^3 - y^2)$ into $\mathbb{R}^3$.

What Milnor proves is that the topology of $K_{p,\varepsilon}(f)$ is independent of ε for small enough $\varepsilon > 0$. In fact, Milnor does even more: if $D^{2n}_\varepsilon(p)$ is the radius ε closed disk centered at p, then for small ε,

$$D^{2n}_\varepsilon(p) \cap V(f) \cong C(K_{p,\varepsilon}(f))$$

where the right-hand side is the *cone* on $K_{p,\varepsilon}(f)$. In this sense, $K_p(f) = K_{p,\varepsilon}(f)$ (for $\varepsilon > 0$ sufficiently small) determines the local topology of $V(f)$ near p.

Crucially, the polynomial f endows the pair $K_{p,\varepsilon}(f) \subseteq S^{2n-1}_\varepsilon$ with some additional structure. Consider the map

$$\begin{aligned} M_f \colon S^{2n-1}_\varepsilon \smallsetminus K_{p,\varepsilon}(f) &\longrightarrow S^1 \\ x &\longmapsto \frac{f(x)}{\|f(x)\|}. \end{aligned}$$

Here x is viewed as an element of $\mathbb{C}^n$ which is distance ε from p such that $f(x) \neq 0$, and S^1 is the space of unit length complex numbers. Milnor proves that M_f is a fiber bundle with each fiber $F_\theta := M_f^{-1}(e^{i\theta})$ a smooth parallelizable[3] $(2n-2)$-dimensional manifold.

Thinking about the $n = 2$ case is particularly instructive. Then the fibers F_θ are all homeomorphic surfaces bounded by $K_{p,\varepsilon}(f)$. In

[3] Parallelizable m-manifolds admit m smooth vector fields $V_1, \ldots, V_m$ such that $\{V_1(x), \ldots, V_m(x)\}$ is a basis of the tangent space at each point x in the manifold. Equivalently, the tangent bundle is trivializable. Every parallelizable manifold is orientable.

this fashion, we can think of M_f as instructions for viewing the F_θ as pages of a book all glued together along the common spine $K_{p,\varepsilon}(f)$; the entirety of this book is all of S^3.

We write

$$\overline{F}_\theta = F_\theta \cup K_{p,\varepsilon}(f)$$

for the closure of F_θ in S_ε^{2n-1}. This makes $\overline{F}_\theta$ a $(2n-2)$-dimensional manifold with boundary

$$\partial \overline{F}_\theta = K_{p,\varepsilon}(f).$$

Milnor proves the following remarkable fact about $\overline{F}_\theta$.

Theorem 2.3. *The space $\overline{F}_\theta$ is homotopy equivalent to a bouquet of $(n-1)$-dimensional spheres,*

$$\overline{F}_\theta \simeq \underbrace{S^{n-1} \vee \cdots \vee S^{n-1}}_{\mu \text{ copies}}.$$

In particular, the middle homology $H_{n-1}(\overline{F}_\theta; \mathbb{Z})$ is free of rank μ.

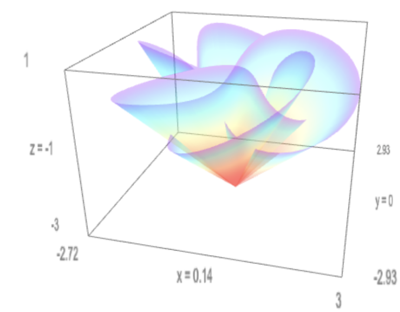

Figure 6: A projection of $D_\varepsilon^{2n}(0) \cap V(x^3+y^5)$. In this case, $K_{0,\varepsilon}(x^3+y^5)$ is a $(3,5)$-torus knot. Each color corresponds to $K_{0,\eta}(x^3+y^5)$ for some $0 \leq \eta \leq \varepsilon$, and the visually apparent self-intersections are artificial. Access a manipulatable model in CoCalc (click the green "Open with one click!" button).

The number $\mu = \mu_p(f)$ is called the *Milnor number* of f at p. Perhaps surprisingly, it measures the degeneracy of our singularity. Recall that an isolated critical point p of f is nondegenerate when the Hessian matrix

$$Hf(p) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p) \right)_{1 \leq i,j \leq n}$$

is nonsingular. We claim that this is equivalent to the multiplicity of ∇f at p being 1. Indeed, note that $p \in \mathrm{Sing}(f)$ implies that the components of ∇f (that is, the partial derivatives of f) have intersection multiplicity at least one. The multiplicity is exactly one if and only if

$$V(\nabla f) = V(f_1) \cap \cdots \cap V(f_n)$$

is smooth at p. This is equivalent to the Jacobian of ∇f being nonsingular at p. But the Jacobian of ∇f is exactly Hf. Thus the degeneracy of p is measured by the multiplicity with which $\partial f/\partial x_1, \ldots, \partial f/\partial x_n$ intersect. In order to state this precisely, let $\mathbb{C}[x_1, \ldots, x_n]_p$ denote the localization of $\mathbb{C}[x_1, \ldots, x_n]$ at the ideal $(x_1 - p_1, \ldots, x_n - p_n)$. This is the subring of rational functions in $x_1, \ldots, x_n$ with denominator not vanishing at p.

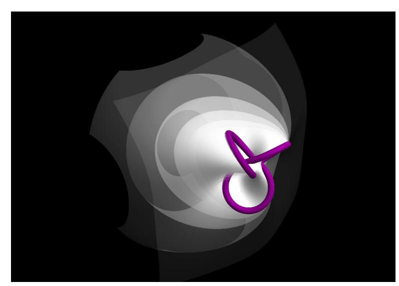

Figure 7: Several fibers of $M_{x^3-y^2}$. See this blog post for more visualizations by Henry Blanchette.

Theorem 2.4. *The Milnor number $\mu_p(f)$ is equal to the intersection multiplicity*

$$\dim_{\mathbb{C}} \mathbb{C}[x_1, \ldots, x_n]_p / (\nabla f)$$

of the partial derivatives $\partial f/\partial x_1, \ldots, \partial f/\partial x_n$ of f at p.

Milnor's proof uses the identity

Here χ denotes Euler characteristic.

$$\chi(\overline{F}_\theta) = 1 + (-1)^{n-1}\mu_p(f),$$

the Lefschetz trace formula, and a "topologization" of intersection multiplicity (in terms of degree — see below).

Example 2.5. Let's compute the Milnor number of the cusp $V(x^3 - y^2)$ in two ways, beginning with the middle Betti number of $\overline{F}_\theta$. Since this is a Seifert surface for a trefoil knot, we find our local knot theorist and learn that the genus of this knot is 1. This corresponds to Euler characteristic

$$\chi(\overline{F}_\theta) = 2 - 2 \cdot 1 - 1 = -1$$

and the above relationship between Euler characteristic and Milnor number gives

$$-1 = 1 - \mu_0(x^3 - y^2) \implies \mu_0(x^3 - y^2) = 2.$$

The partial derivatives of $x^3 - y^2$ are $3x^2$ and $-2y$. Thus

$$\mathbb{C}[x,y]_{(x,y)}/(3x^2, -2y) \cong \mathbb{C}[x,y]/(x^2, y) \cong \mathbb{C}[x]/(x^2)$$

has dimension 2 as a $\mathbb{C}$-vector space. This agrees with the above computation.

We conclude this brief overview of classical Milnor numbers by discussing their connection with topological degree. Suppose $g\colon M \to N$ is a smooth map between connected orientable manifolds of the same dimension. Endow M with charts that have consistent orientations (so the transfer maps are orientation-preserving) and do the same for N. Given a point $p \in M$ at which g is regular, define the *local degree* of g at p to be

Here $(dg)_p\colon T_pM \to T_pN$ is the derivative of g at p.

$$\deg_p(g) := \operatorname{sign}(\det(dg)_p) \in \{\pm 1\}.$$

If $q \in N$ is a regular value of g, then the (global) *degree* of g is

$$\deg(g) := \sum_{p \in g^{-1}\{q\}} \deg_p(g) \in \mathbb{Z}.$$

It is a theorem that $\deg(g)$ is independent of the choice of q.

In our case of interest, we want to view ∇f as a map $\mathbb{C}^n \to \mathbb{C}^n$ (where again $f \in \mathbb{C}[x_1, \ldots, x_n]$). Beware, though, that ∇f is not smooth at degenerate critical points of f. Milnor's workaround is to instead work with the normalized function $\nabla f / \|\nabla f\|$ on a small sphere centered at some $p \in \operatorname{Sing}(f)$. This is a smooth function $S^{2n-1}_\varepsilon \to S^{2n-1}$ (where the latter sphere has radius 1). The topological degree of $\nabla f / \|\nabla f\|$ as a function $S^{2n-1}_\varepsilon \to S^{2n-1}$ is also called the *local degree* of ∇f at p, and is denoted $\deg_p(\nabla f)$; well-definition of $\deg_p$ is an exercise, and the proof of the following theorem may be found in Milnor.

Theorem 2.6. *For $f \in \mathbb{C}[x_1, \ldots, x_n]$ and $p \in \mathrm{Sing}(f)$,*

$$\mu_p(f) = \deg_p(\nabla f).$$

This is our first indication that Milnor numbers are related to degree. The insight of Kass–Wickelgren[4] (building off work of Eisenbud–Levine,[5] Khimshiashvili,[6] and Scheja–Storch[7]), is that the local motivic degree of ∇f (valued in the Grothendieck–Witt ring of quadratic forms) plays this role over fields other than $\mathbb{C}$.

First envisioned by Barge, Lannes, and Morel,[8] motivic degree is a refined version of topological degree that retains information about the Jacobian determinant $\det(dg)_p$ beyond its sign. For a smooth map between equidimensional varities over a field k, this is packaged as the image of a quadratic form over k in $\mathrm{GW}(k)$, the Grothendieck–Witt ring of k. Before constructing this degree function, we will introduce the necessary prerequisites on quadratic forms and $\mathrm{GW}(k)$.

[4] Kass, J. L. and Wickelgren, K. (2019). The class of Eisenbud-Khimshiashvili-Levine is the local $\mathbf{A}^1$-Brouwer degree. *Duke Math. J.*, 168(3):429–469

[5] Eisenbud, D. and Levine, H. I. (1977). An algebraic formula for the degree of a C^∞ map germ. *Ann. of Math. (2)*, 106(1):19–44

[6] Himšiašvili, G. N. (1977). The local degree of a smooth mapping. *Sakharth. SSR Mecn. Akad. Moambe*, 85(2):309–312

[7] Scheja, G. and Storch, U. (1975). Über Spurfunktionen bei vollständigen Durchschnitten. *J. Reine Angew. Math.*, 278(279):174–190

[8] Morel, F. (2006). $\mathbb{A}^1$-algebraic topology. In *International Congress of Mathematicians. Vol. II*, pages 1035–1059. Eur. Math. Soc., Zürich

3 *The Grothendieck–Witt ring*

We will quickly introduce quadratic (and symmetric bilinear) forms and the Grothendieck–Witt ring of a field, which captures the isometry classes and arithmetic of these structures. The presentation here is heavily influenced by Lam's exceptional book.[9]

[9] Lam, T. Y. (2005). *Introduction to quadratic forms over fields*, volume 67 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI

3.1 *Basic notions*

A *quadratic form* q over a field k (of characteristic not 2) is a function $q\colon V \to k$ where

» V is a k-vector space,

» q is homogeneous of degree 2:

$$q(\lambda v) = \lambda^2 q(v),$$

and

» the *polarization*

$$\begin{aligned} b_q\colon V \times V &\longrightarrow k \\ (v, w) &\longmapsto \frac{1}{2}(q(v+w) - q(v) - q(w)) \end{aligned}$$

is symmetric bilinear.

Remark 3.1. We will only concern ourselves with quadratic forms on finite-dimensional vector spaces. As such, we will assume this "finite rank" condition without comment henceforth.

Remark 3.2. Once V is given coordinates, we may view q as a degree 2 homogeneous polynomial.[10]

[10] Exercise!

The polarization portion of the definition may not feel particularly natural, but it is necessary to build a bridge with symmetric bilinear forms. In fact, away from characteristic 2, quadratic forms and symmetric bilinear forms are equivalent by taking (squared) norms:

$$q(v) = b(v, v).$$

In other words, polarization and $v \mapsto b(v, v)$ are inverse operations. Since the definition is more convenient, we will work with symmetric bilinear forms from here on. In order to save ink/pixels, we will sometimes refer to symmetric bilinear forms as sbf's.

Let $V^* := \mathrm{Hom}_k(V, k)$. A symmetric bilinear form b on V supplies, for each $w \in V$, a functional $b(-, w) \in V^*$ given by $v \mapsto b(v, w)$. The form b is *nondegenerate* (or *regular*) when $V \to V^*$, $w \mapsto b(-, w)$ is an isomorphism.

The relevant notion of equivalence of symmetric bilinear forms is isometry. We say that k-sbf's b_1 on V_1 and b_2 on V_2 are *isometric* when there is a k-linear isomorphism $\phi\colon V_1 \to V_2$ such that

$$b_2(\phi u, \phi v) = b_1(u, v)$$

for all $u, v \in V_1$.

When V has ordered basis $\alpha = (e_1, \ldots, e_n)$, we may represent a sbf on V by its *Gram matrix*

$$G_b := (b(e_i, e_j))_{i,j}.$$

Indeed, we get the identity

$$b(v, w) = [v]_\alpha^\top G_b [w]_\alpha.$$

Here we are writing $[w]_\alpha$ for the α-coordinates of w written as a column vector.

Since b is symmetric, G_b is a symmetric matrix. A standard (and important) exercise shows that b is nondegenerate if and only if $\det G_b \neq 0$. Tracing through definitions,[11] we get that b and b' sbf's on V are isometric if and only if their Gram matrices are *congruent*: there exists $A \in \mathrm{GL}_n(k)$ such that

[11] Exercise!

$$G_{b'} = A^\top G_b A.$$

The matrix A is a change of basis corresponding to a linear isomorphism $\phi\colon V \to V$ which witnesses that b and b' are isometric.

It is also easy to recover the Gram matrix of a sbf from its associated quadratic form. If the quadratic form is written as a homogeneous degree 2 polynomial

$$q(x_1, \ldots, x_n) = \sum_{1 \leq i \leq j \leq n} a_{ij} x_i x_j$$

then the corresponding Gram matrix has (i, j) entry

$$\begin{cases} a_{ii} & \text{if } i = j, \\ \frac{1}{2}a_{ij} & \text{if } i < j, \\ \frac{1}{2}a_{ji} & \text{if } i > j. \end{cases}$$

At this point, some examples are due. The below table lists a k-vector space,[12] a quadratic form (presented as a homogeneous degree 2 polynomial), and its Gram matrix. I recommend developing fluency in translating between each of these perspectives.

[12] If the vector space is k^n, then we will assume it is endowed with the standard basis and that x_i is the dual vector of e_i. If $1 \leq n \leq 3$, we will write $x = x_1$, $y = x_2$, $z = x_3$.

V	q	G
k^1	ax^2	(a)
k^2	$x^2 + y^2$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
k^2	xy	$\begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$
k^2	$x^2 - y^2$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
k^2	$x^2 - 4xy + 3y^2$	$\begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix}$
k^2	$x^2 + 2xy + y^2$	$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$
k^3	$2xy + 4xz + 2yz$	$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$

Note that the second-to-last form is not regular since $\det \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 0$. We can also observe that xy and $x^2 - y^2$ are isometric[13].

[13] 🏋 Exercise! The identity $x^2 - y^2 = (x + y)(x - y)$ is certainly relevant.

Crucially, every sbf is *diagonalizable*. That is, there exists a basis for which the associated Gram matrix is diagonal. If a sbf is diagonal with Gram matrix $\operatorname{diag}(a_1, \ldots, a_n)$, then we denote it

$$\langle a_1, \ldots, a_n \rangle.$$

Note that the associated quadratic form is $a_1x_1^2 + \cdots + a_nx_n^2$. The proof of this result is inductive, leveraging the *representation criterion*: if a sbf b satisfies $b(v, v) = a$ for some $v \in V$, $a \in k$, then b is isometric to the orthogonal sum of $\langle a \rangle$ and some other sbf b'. We will not prove this result in these notes, but orthogonal sum is defined in the next subsection.

3.2 *Operations*

We can combine sbf's with via *orthogonal sum* and *tensor product*. Let b_1 and b_2 be k-sbf's on vector spaces V_1 and V_2, respectively. Their orthogonal sum is given by the rule

$$\begin{aligned} b_1 \perp b_2 \colon (V_1 \oplus V_2) \times (V_1 \oplus V_2) &\longrightarrow k \\ ((v_1, v_2), (w_1, w_2)) &\longmapsto b_1(v_1, w_1) + b_2(v_2, w_2) \end{aligned}$$

and their tensor product is defined by

$$\begin{aligned} b_1 \otimes b_2 \colon (V_1 \otimes V_2) \times (V_1 \otimes V_2) &\longrightarrow k \\ (v_1 \otimes v_2, w_1 \otimes w_2) &\longmapsto b_1(v_1, w_1) b_2(v_2, w_2). \end{aligned}$$

On Gram matrices, these operations correspond to block sum and Kronecker product:

$$G_{b_1 \perp b_2} = \begin{pmatrix} G_{b_1} & 0 \\ 0 & G_{b_2} \end{pmatrix}, \qquad G_{b_1 \otimes b_2} = \begin{pmatrix} a_{11} G_{b_2} & \cdots & a_{1n} G_{b_2} \\ \vdots & \ddots & \vdots \\ a_{n1} G_{b_2} & \cdots & a_{nn} G_{b_2} \end{pmatrix}$$

where $G_{b_1} = (a_{ij})$. On diagonal forms, we have

$$\begin{aligned} \langle a_1, \ldots, a_n \rangle \perp \langle b_1, \ldots, b_m \rangle &= \langle a_1, \ldots, a_n, b_1, \ldots, b_m \rangle \\ \langle a_1, \ldots, a_n \rangle \otimes \langle b_1, \ldots, b_m \rangle &= \langle a_1 b_1, \ldots, a_1 b_m, \ldots, a_n b_1, \ldots, a_n b_m \rangle. \end{aligned}$$

Isometry classes $\mathcal{S}(k)$ of regular k-sbf's together with the operations $\perp$ and $\otimes$ form a *commutative semiring*. Semirings are sometimes called *rigs* because they are "rings without negatives." In our case, $(\mathcal{S}(k), \perp)$ is a commutative monoid with identity (the unique 0-dimensional sbf 0), $(\mathcal{S}(k), \otimes)$ is a commutativie monoid with identity $\langle 1 \rangle$, $\otimes$ distributes over $\perp$, and multiplication by 0 annihilates $\mathcal{S}(k)$. In fact, $(\mathcal{S}(k), \perp)$ is especially nice since it is cancellative in the sense that

$$b \perp d = c \perp d \implies b = c.$$

Given any monoid $(M, \cdot)$, we may form the *group completion* M^{gp} of M in the following fashion: Let $\underline{M} = \{\underline{m} \mid m \in M\}$ and set

$$M^{\mathrm{gp}} := \langle \underline{M} \mid \underline{1} = 1, \underline{mn} = \underline{m} \cdot \underline{n} \text{ for all } m, n \in M \rangle.$$

(The right-hand side is the standard notation for the presentation of a group.) Then the monoid homomorphism $M \to M^{\mathrm{gp}}$, $m \mapsto \underline{m}$ witnesses that $(\ )^{\mathrm{gp}}$ is left adjoint to the forgetful functor from groups to monoids, which is a fancy way of saying that whenever G is a group and $M \to G$ is a homomorphism, there is a unique homomorphism $M^{\mathrm{gp}} \to G$ making the diagram

$$\begin{tikzcd} M \arrow[r] \arrow[d] & G \\ M^{\mathrm{gp}} \arrow[ur] & \end{tikzcd}$$

commute.

While the above definition works, it is mostly unusable for general monoids. Life is much simpler for a commutative monoid $(M, +)$. In this case,

(a) every element of M^{gp} can be expressed as $\underline{m} - \underline{n}$ for some $m, n \in M$;

(b) if $m, n \in M$, then $\underline{m} = \underline{n} \in M^{\mathrm{gp}}$ if and only if $m + \ell = n + \ell$ for some $\ell \in M$;

(c) the set underlying the group completion M^{gp} is the set-theoretic quotient of M by the equivalence relation generated by $(m, n) \sim (m', n')$ whenever there exists $\ell \in M$ such that $m + n' + \ell = m' + n + \ell$.

Condition (c) is even simpler for cancellative commutative monoids, in which case

$$M^{\mathrm{gp}} = M \times M/(m, n) \sim (m', n') \text{ for } m + n' = m' + n.$$

Relatedly, the homomorphism $M \to M^{\mathrm{gp}}$ is injective if and only if M is cancellative.

Example 3.3. We have $\mathbb{Z} = (\mathbb{N}, +)^{\mathrm{gp}}$ and $\mathbb{Q}^{\times} = (\mathbb{Z} \setminus \{0\}, \cdot)^{\mathrm{gp}}$.

When we apply group completion to the additive monoid of a (commutative) semiring $(M, +, \cdot)$, we get a (commutative) ring by defining

$$(m, n) \cdot (m', n') = (mm' + nn', mn' + nm').$$

This formula will become intuitive as soon as the reader expands the product $(m - n)(m' - n')$.

Definition 3.4. The *Grothendieck–Witt ring* of a field k is

$$\mathrm{GW}(k) := (\mathcal{S}(k), \perp, \otimes)^{\mathrm{gp}}.$$

When working with $\mathrm{GW}(k)$, we will elide the distinction between a sbf and its isometry class, and we will write $+$ for $\perp$ and $\cdot$ (or concatenation) for $\otimes$.

3.3 Presentation and examples

Diagonalization allows us to only consider diagonal forms when working in $\mathrm{GW}(k)$ — indeed, every isometry class contains a diagonal representative. But these representative are not unique! For instance, we may reorder the diagonal coefficients, and we also have $\langle ab^2 \rangle \cong \langle a \rangle$. The following theorem gives a full set of relations amongst diagonal forms in $\mathrm{GW}(k)$:

Theorem 3.5. *The ring* $\mathrm{GW}(k)$ *is generated (as a ring) by the unary forms* $\langle a\rangle$ *for* $a \in k^\times$ *subject to the following relations:*

(a) $\langle a\rangle = \langle ab^2\rangle$ *for* $a, b \in k^\times$,

(b) $\langle a\rangle\langle b\rangle = \langle ab\rangle$ *for* $a, b \in k^\times$,

(c) $\langle a\rangle + \langle b\rangle = \langle a+b\rangle + \langle ab(a+b)\rangle$ *for* $a, b \in k^\times$ *and* $a + b \neq 0$.[14]

[14] The expression on the right-hand side is symmetric in a, b, so this implies that addition is commutative.

From these, a not-so-easy exercise reveals that

$$\langle a, -a\rangle = \langle 1, -1\rangle$$

for all $a \in k^\times$. We write

$$h := \langle 1, -1\rangle$$

for this so-called *hyperbolic form*.

⚠ The hyperbolic form plays an outsized role in the algebraic theory of quadratic forms. As we proceed, keep in mind that $h = \langle 1, -1\rangle = x^2 - y^2 \cong xy$ and that $q \otimes h \cong (\operatorname{rank} q)h$ for all q.

Example 3.6. Write $k^\boxtimes := \{a^2 \mid a \in k^\times\}$ for the set of nonzero squares in k. We call k *quadratically closed* when $k^\boxtimes = k^\times$. In this case, $\langle a\rangle = \langle 1\rangle$ for all $a \in k^\times$, and it follows that

$$\begin{aligned} \mathbb{Z} &\longrightarrow \mathrm{GW}(k) \\ n &\longmapsto n\langle 1\rangle \end{aligned}$$

is an isomorphism. In particular, $\mathrm{GW}(\mathbb{C}) \cong \mathbb{Z}$ via the rank homomorphism.

Most authors write $(k^\times)^2$ for the squares in $k^\times$, but I worry that this might be confused with $k^\times \cdot k^\times = \{ab \mid a, b \in k^\times\} = k^\times$. I hope you'll excuse — or even enjoy — the notational foible $k^\boxtimes$.

Example 3.7. Recall that the isometry class of a real quadratic form is completely determined by its Sylvester type (n_0, n_+, n_-). When the form is regular, $n_0 = 0$ and $n_+ + n_-$ equals the rank of the form. Thus rank and signature $(n_+ - n_-)$ provide a set of complete invariants of regular quadratic forms over $\mathbb{R}$ and

$$\mathrm{GW}(\mathbb{R}) \cong \{(n, s) \in \mathbb{Z} \times \mathbb{Z} \mid n + s \equiv 0 \pmod 2\}$$

where the right-hand side is thought of as a subring of the product ring $\mathbb{Z} \times \mathbb{Z}$. We may also write

$$\mathrm{GW}(\mathbb{R}) \cong \mathbb{Z}[h]/(h^2 - 2h)$$

where h is the hyperbolic form.

Example 3.8. For a finite field k,

$$\mathrm{GW}(k) \cong \mathbb{Z} \times k^\times/k^\boxtimes$$

via rank and discriminant. (*Discriminant* is the determinant of the Gram matrix up to square classes.) Of course, $k^\times/k^\boxtimes \cong \mathbb{Z}/2\mathbb{Z}$ since $k^\times$ is cyclic of order $|k| - 1$.

Example 3.9. For $k = \mathbb{Q}$, the field of rational numbers, we have

$$\mathrm{GW}(\mathbb{Q}) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}^{\infty}$$

as an additive group, where the final term represents a countably infinite direct sum of $\mathbb{Z}/2\mathbb{Z}$'s. Projected onto the first two factors, the isomorphism is rank and signature, while the rest of the factors record "discriminants of residues" for finite rational primes.

Example 3.10. The *Witt ring* of a field is

$$\mathrm{W}(k) := \mathrm{GW}(k)/(h) = \mathrm{GW}(k)/\mathbb{Z}h.$$

(We have $(h) = \mathbb{Z}h$ since $q \cdot h = \mathrm{rank}(q)h$ for all q.) But this is an anachronistic definition! Elements of $\mathrm{W}(k)$ are in bijective correspondence with isometry classes of *anisotropic* quadratic forms: q such that $q(v) = 0 \implies v = 0$. Witt's decomposition theorem says that every quadratic form is uniquely expressible (up to isometry) as the orthogonal sum of (1) a totally isotropic form, (2) an anisotropic form, and (3) a sum of hyperbolic forms. Thus Witt was able to define $\mathrm{W}(k)$ by adding and multiplying anisotropic forms and then taking the anisotropic part of the outcome.

Regardless of the definition we choose, we get the following computations:

» $\mathrm{W}(\mathbb{C}) \cong \mathbb{Z}/2\mathbb{Z}$ via parity of rank,

» $\mathrm{W}(\mathbb{R}) \cong \mathbb{Z}$ via signature,

» for k finite,

$$\mathrm{W}(k) \cong \begin{cases} \mathbb{Z}/4\mathbb{Z} & \text{if } |k| \equiv 3 \pmod 4, \\ \mathbb{F}_2[k^{\times}/k^{\boxtimes}] & \text{if } |k| \equiv 1 \pmod 4, \end{cases}$$

To this author, it remains deeply mysterious that the abstract isomorphism type of $\mathrm{W}(k)$ can distinguish some finite fields, while $\mathrm{GW}(k)$ cannot.

» for $k = \mathbb{Q}$,

$$\mathrm{W}(\mathbb{Q}) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \bigoplus_{p>2} \mathrm{W}(\mathbb{F}_p).$$

3.4 *The fundamental ideal*

The *fundamental ideal* $\mathrm{GI} = \mathrm{GI}(k)$ of $\mathrm{GW}(k)$ is the kernel of the rank homomorphism:

$$\mathrm{GI} := \ker(\mathrm{rank}\colon\ \mathrm{GW}(k) \longrightarrow \mathbb{Z}).$$

The image of $\mathrm{GI}(\mathsf{k})$ under the quotient homomorphism $\mathrm{GW}(\mathsf{k}) \to \mathrm{W}(\mathsf{k})$ is called $\mathrm{I}(k)$, the fundamental ideal of $\mathrm{W}(\mathsf{k})$. We will interpret this ideal through the lens of the following commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & \mathbb{Z}h \arrow[r, "\cong"] \arrow[d] & 2\mathbb{Z} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{GI}(\mathsf{k}) \arrow[r] \arrow[d, "\cong"'] & \mathrm{GW}(\mathsf{k}) \arrow[r, "\mathrm{rank}"'] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{I}(\mathsf{k}) \arrow[r] \arrow[d] & \mathrm{W}(\mathsf{k}) \arrow[r, "\mathrm{rank}_0"'] \arrow[d] & \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{$\aleph$}$$

in which the rows and columns are exact. (Here $\mathrm{rank}_0(q + \mathbb{Z}h) = \mathrm{rank}(q) + 2\mathbb{Z}$ is well-defined.) There are two observations hiding in the diagram. First, the restriction of the quotient homomorphism $\mathrm{GW}(k) \to \mathrm{W}(k)$ to $\mathrm{GI}(k)$ is an isomorphism onto $\mathrm{I}(\mathsf{k})$. This follows because $\mathrm{GI}(\mathsf{k}) \cap \mathbb{Z}h = 0$. Second, an anisotropic form q is in $\mathrm{I}(\mathsf{k})$ if and only if its dimension is 0; this requires a proof, but we won't rehearse it here.

Diagram ($\aleph$) informs us that the GI-adic filtration of $\mathrm{GW}(k)$ and I-adic filtration of $\mathrm{GW}(k)$ are identical in positive degrees. Famously, the Milnor conjecture (proved by Voevodsky, Orlov–Vishik–Voevodsky, Weibel, Rost, *et al*) states that

$$\mathrm{I}^n \,/\, \mathrm{I}^{n+1} \cong K_n^M(k)/2 \cong H^n_{\text{ét}}(k; \mathbb{F}_2).$$

We are writing I^n for $\mathrm{I}(k)^n$.

Here the second term is mod 2 Milnor K-theory, and the third is is étale cohomology with coefficients in $\mathbb{F}_2$; we will not define either of these objects presently.

3.5 Transfers

We will need one more construction in our study of Milnor forms, namely a transfer map $\mathrm{GW}(L) \to \mathrm{GW}(k)$ for L/k a separable field extension. Recall from field theory the additive transfer map $\mathrm{tr}_{L/k} \colon L \to k$ where $\mathrm{tr}_{L/k}(a)$ is the trace of the k-linear multiply-by-a map $m_a \colon L \to L$. One has

$$\mathrm{tr}_{L/k}(a) = \sum_{\sigma \in \mathrm{Aut}(L/k)} \sigma(a).$$

Given an L-sbf $b \colon V \times V \to L$, we may postcompose with $\mathrm{tr}_{L/k}$ to get a k-sbf

$$\mathrm{tr}_{L/k}(b) := \mathrm{tr}_{L/k} \circ b.$$

(We are harmlessly overloading the notation $\mathrm{tr}_{L/k}$ here.) This yields a homomorphism of additive groups

$$\mathrm{tr}_{L/k}\colon \mathrm{GW}(L) \to \mathrm{GW}(k)$$

called the *transfer homomomorphism*.

Transfers are wrong-way maps, and we have skipped over the more natural right-way maps, namely the extension[15] (ring) homomorphism

$$\mathrm{ext}_{L/k}\colon \mathrm{GW}(k) \to \mathrm{GW}(L)$$

which takes a k-quadratic form (viewed as a homogeneous degree 2 polynomial over k) and thinks of it as a polynomial over $L \supseteq k$. On k-sbf's, this corresponds to extending scalars, $b \mapsto b \otimes_k L$.

Transfer and extension enjoy a number of nice properties (like an analogue of Frobenius reciprocity) which are encapsulated by calling GW a *Mackey functor*. Again, the details are beyond the scope of these notes, but the interested reader can read Bachmann's paper.[16]

[15] Because the literature is infested with algebraic geometry, this is typically called restriction, but we will not do so here.

[16] Bachmann, T. (2021). Motivic Tambara functors. *Math. Z.*, 297(3-4):1825–1852

4 *Motivic degree*

We now come to the construction at the heart of our exploration. The classical Milnor number can be defined in terms of local topological degree of ∇f, and we will define Milnor forms in the same way but using *local motivic degree*, $\deg_p^{\mathbb{A}^1}$, which is valued in $\mathrm{GW}(k)$. We will give a purely algebraic definition of $\deg_p^{\mathbb{A}^1}$ in a moment, but first should say something about its proper context, *motivic* (or $\mathbb{A}^1$-) *homotopy theory*.

4.1 *Motivic homotopy theory*

Motivic homotopy theory is a homotopy theory of smooth schemes in which the affine line $\mathbb{A}^1$ plays the role of the unit interval $[0,1]$ in classical topology. Instead of incanting a terminological spell,[17] we will content ourselves with a phenomenological description of this subject. The category of motivic spaces Spc_k is an enlargement of the category of smooth k-schemes that allows both simplicial and colimit[18] constructions useful in algebraic topology. It contains both smooth schemes (via the Yoneda embedding) and simplicial sets[19] (via the constant presheaf functor).

Spheres play a crucial role in algebraic topology, serving to probe spaces and detect holes. In motivic homotopy, there is a bigraded family of spheres. There are the simplicial spheres,

$$S^{n,0} := \underbrace{S^1 \wedge \cdots \wedge S^1}_{n \text{ copies}},$$

[17] *The $\mathbb{A}^1$-localization of the Nisnevich localization of the (simplicial) model category of simplicial presheaves on the category of smooth separated finite type schemes over k.*

[18] When $X \subseteq Y$ is an embedding of motivic spaces, we may form the homotopy quotient Y/X in Spc_k. Beware that if X and Y are smooth schemes, it is no longer the case that Y/X is representable.

[19] Beyond their formal definition, simplicial sets should be thought of as a combinatorial standin for (nice) topological spaces.

We start to use smash products $\wedge$ here. For pointed simplicial sets (X, p) and (Y, q), we have $X \wedge Y = X \times Y/(X \times q \cup p \times Y$. For pointed simplicial presheaves, we perform this operation section-wise. We won't belabor the chosen basepoints in the discussion below.

and the geometric spheres,

$$S^{n,n} := \underbrace{(\mathbb{A}^1 \smallsetminus 0) \wedge \cdots \wedge (\mathbb{A}^1 \smallsetminus 0)}_{n \text{ copies}}.$$

We can also intermix the two classes of spheres to get

$$S^{m,n} := \underbrace{S^1 \wedge \cdots \wedge S^1}_{m-n \text{ copies}} \wedge \underbrace{(\mathbb{A}^1 \smallsetminus 0) \wedge \cdots \wedge (\mathbb{A}^1 \smallsetminus 0)}_{n \text{ copies}} = S^{m-n,0} \wedge S^{n,n}.$$

The grading here is chosen so that m counts the total number of spheres, while n counts the number of geometric spheres.

The homotopy category of Spc_k witnesses the contractibility of $\mathbb{A}^1$ (along with some simplicial and "Nisnevich local" data). Given motivic spaces X and Y, the homotopy classes of maps $X \to Y$ are denoted

$$[X, Y]_{\mathbb{A}^1}.$$

We're being a little sloppy with based vs. unbased homotopy classes here. In general, our computations will happen in the based category.

4.2 *Local and global motivic degrees*

We are interested in maps $g\colon \mathbb{A}^n \to \mathbb{A}^n$, but $\mathbb{A}^n \simeq *$, so $[\mathbb{A}^n, \mathbb{A}^n]_{\mathbb{A}^1} = *$. Suppose $g(p) = q$. In order to get a meaningful local invariant of g near p, consider the induced map

$$\overline{g}\colon \mathbb{A}^n/(\mathbb{A}^n \smallsetminus p) \longrightarrow \mathbb{A}^n/(\mathbb{A}^n \smallsetminus q).$$

The quotient $\mathbb{A}^n/(\mathbb{A}^n \smallsetminus p)$ is *not* a two-point space. Rather, we should think of it as more akin to $\mathbb{R}^n/(\mathbb{R}^n \smallsetminus B) \cong S^n$ where B is an open ball. Indeed,

$$\mathbb{A}^n/(\mathbb{A}^n \smallsetminus p) \simeq S^{2n,n}.$$

Thus the homotopy class of $\overline{g}$ may be considered in $[S^{2n,n}, S^{2n,n}]_{\mathbb{A}^1}$. The following theorem is one of the main results of Morel's book.[20]

[20] Morel, F. (2012). $\mathbb{A}^1$*-algebraic topology over a field*, volume 2052 of *Lecture Notes in Mathematics*. Springer, Heidelberg

Theorem 4.1 (Morel)**.** *There is a motivic degree map*

$$\deg^{\mathbb{A}^1}\colon [S^{2n,n}, S^{2n,n}]_{\mathbb{A}^1} \longrightarrow \mathrm{GW}(k)$$

which is an isomorphism for $n \geq 2$.

Recall that the classical degree of a map $S^n \to S^n$ may be computed by summing signs of Jacobian determinants over the fiber of a regular value. Morel's insight was to record the square class of the Jacobian determinant as a unary quadratic form, and then add these values up in $\mathrm{GW}(k)$. More precisely, for $\overline{f}\colon S^{2n,n} \to S^{2n,n}$ induced by $f\colon \mathbb{A}^n \to \mathbb{A}^n$ and q a regular value of f,

We write $k(p)$ for the residue field of p.

$$\deg^{\mathbb{A}^1}(\overline{f}) := \sum_{p \in f^{-1}q} \mathrm{tr}_{k(p)/k}\langle \det Jf(p)\rangle.$$

In the case of $g\colon (\mathbb{A}^n, p) \to (\mathbb{A}^n, q)$, we define the *local motivic degree* of g at p to be

$$\deg_p^{\mathbb{A}^1}(g) := \deg^{\mathbb{A}^1}(\overline{g}) \in \mathrm{GW}(k).$$

With the definitions sorted and organized as we've done here, it follows directly that

$$\deg^{\mathbb{A}^1}(f) = \sum_{p \in f^{-1}q} \deg_p^{\mathbb{A}^1}(f).$$

Motivic degree extends topological degree on both complex and real points. Indeed, if $k \subseteq \mathbb{R}$, then we may extend the functors of complex and real points to motivic k-spaces, and we get the following commutative diagram:

$$\begin{tikzcd} {[S^n, S^n]} \arrow[d, "\deg"'] & {[S^{2n,n}, S^{2n,n}]_{\mathbb{A}^1}} \arrow[l, "\mathbb{R}\text{-points}"'] \arrow[r, "\mathbb{C}\text{-points}"] \arrow[d, "\deg^{\mathbb{A}^1}"] & {[S^{2n}, S^{2n}]} \arrow[d, "\deg"] \\ \mathbb{Z} & \mathrm{GW}(k) \arrow[l, "\mathrm{sgn}"] \arrow[r, "\mathrm{rank}"'] & \mathbb{Z}. \end{tikzcd} \tag{4.2}$$

Example 4.3. Suppose $f\colon \mathbb{A}^n \to \mathbb{A}^n$ has an isolated simple zero at p with $\det Jf(p) \neq 0$. Then

$$\deg^{\mathbb{A}^1}(f) = \deg_p^{\mathbb{A}^1}(f) = \langle \det Jf(p) \rangle.$$

For instance, if f is a linear transformation $x \mapsto Ax$, then

$$\deg^{\mathbb{A}^1}(f) = \langle \det A \rangle.$$

Example 4.4. If $f\colon \mathbb{A}^1 \to \mathbb{A}^1,\ x \mapsto x^2$, then 0 is *not* a regular value, so we cannot compute the motivic degree of f as its local motivic degree at 0. Instead, 1 is a regular value with $f^{-1}1 = \{\pm 1\}$ and

$$\deg^{\mathbb{A}^1}(f) = \deg_1^{\mathbb{A}^1}(f) + \deg_{-1}^{\mathbb{A}^1}(f) = \langle 2 \rangle + \langle -2 \rangle = h.$$

When f is not regular at p, we need other techniques to compute its local degree at p. By work of Kass–Wickelgren, the Eisenbud–Levine/Khimshiashvilli form provides just such a tool.

4.3 *Eisenbud–Levine/Khimshiashvilli forms*

Well before the invention of motivic homotopy theory, Eisenbud–Levine[21] and Khimshiashvilli[22] independently defined a symmetric bilinear form with signature recovering the local topological degree when $k = \mathbb{R}$. Their definition is completely algebraic and may be made over an arbitrary field k. Kass–Wickelgren[23] and Brazelton–Burklund–McKean–Montoro–Opie[24] prove that this form's image in

[21] Eisenbud, D. and Levine, H. I. (1977). An algebraic formula for the degree of a C^∞ map germ. *Ann. of Math. (2)*, 106(1):19–44

[22] Himšiašvili, G. N. (1977). The local degree of a smooth mapping. *Sakharth. SSR Mecn. Akad. Moambe*, 85(2):309–312

[23] Kass, J. L. and Wickelgren, K. (2019). The class of Eisenbud-Khimshiashvili-Levine is the local $\mathbf{A}^1$-Brouwer degree. *Duke Math. J.*, 168(3):429–469

[24] Brazelton, T., Burklund, R., McKean, S., Montoro, M., and Opie, M. (2021). The trace of the local $\mathbb{A}^1$-degree. *Homology Homotopy Appl.*, 23(1):243–255

$\mathrm{GW}(k)$ is equal to the local motivic degree. Presently, we will define the EL/K-form[25] and make its relationship with $\deg_p^{\mathbb{A}^1}(f)$ precise.

Suppose $f\colon \mathbb{A}^n \to \mathbb{A}^n$ and that $f(p) = q$ for k-rational points p, q.[26] The *local ring* of f at p is

$$Q_p(f) := \mathcal{O}_{\mathbb{A}^n,p}/(f-q).$$

Here $\mathcal{O}_{\mathbb{A}^n} = k[x_1,\ldots,x_n]$ is the coordinate ring of $\mathbb{A}^n$, and the subscript p indicates localization at the maximal ideal $(x-p) = (x_1 - p_1,\ldots,x_n - p_n)$. The quotient is by $(f-q) = (f_1 - q_1,\ldots,f_n - q_n)$. If p is an isolated solution of $f(x) = q$, then $Q_p(f)$ is finite-dimensional as a k-vector space. We should think of $Q_p(f)$ as a k-algebra that records what $V(f-q)$ looks like in an infinitesimal neighborhood of p.

[25] In other references, this is abbreviated to EKL-form.

[26] In the subsequent subsection, we will see how to deal with non-rational points.

Example 4.5. If $f\colon \mathbb{A}^1 \to \mathbb{A}^1$ with $f(x) = x^2$, then

$$Q_0(f) = k[x]_{(x)}/(x^2) \cong k[x]/(x^2).$$

Definition 4.6. Assume that $\operatorname{char} k \nmid \dim_k Q_p(f)$. Then the *EL/K-form* $\omega^{\mathrm{EL/K}} = \omega_p^{\mathrm{EL/K}}(f)$ of f at p is the isometry class of the sbf given by

$$\begin{aligned} \omega^{\mathrm{EL/K}}\colon Q_p(f) \times Q_p(f) &\longrightarrow k \\ (a,b) &\longmapsto \eta(ab) \end{aligned}$$

where $\eta\colon Q_p(f) \to k$ is any k-linear map satisfying $\eta(\det Jf) = \dim_k Q_p(f)$.

Recall that Jf is the Jacobian matrix $\left(\frac{\partial f_i}{\partial x_j}\right)$.

To this author, it remains a surprise and tremendous delight that the isometry class of $\omega^{\mathrm{EL/K}}$ is independent of the choice of η (as long as $\eta(\det Jf) = \dim_k Q_p(f)$). Proving this result requires a significant detour into the world of Gorenstein duality that we will not undertake.[27] When $\operatorname{char} k$ divides the dimension of $Q_p(f)$, there is still a remedy: one works with the "distinguished socle element" E instead of $\det Jf$ and demands $\eta(E) = 1$.[28]

[27] The original paper of Eisenbud–Levine is readable and fascinating if you have a background in commutative algebra.

[28] This relies on some difficult work of Scheja–Storch. When the characteristic assumption is satisfied, $\det Jf = (\dim_k Q_p(f))E \in Q_p(f)$.

Example 4.7. Following up on Example 4.5, note that the Jacobian determinant of the squaring map is $2x$, so we may choose η such that $\eta(1) = 0$ and $\eta(2x) = 2$. Using $1, 2x$ as an ordered basis of $Q_0(x^2)$, we see that $\omega^{\mathrm{EL/K}}$ has Gram matrix

$$\begin{pmatrix} \eta(1) & \eta(2x) \\ \eta(2x) & \eta(4x^2) \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix},$$

where the bottom right entry is 0 because $4x^2 = 0 \in Q_0(x^2) \cong k[x]/(x^2)$. This form is hyperbolic, agreeing with Example 4.4.

Example 4.8. Consider the function $f\colon \mathbb{A}^1 \to \mathbb{A}^1$ with $f(x) = ax^n$ where $a \in k^\times$. Then $Q_0(f) \cong k[x]/(x^n)$ with dimension n, and

This and the following example are inspired by the paper of Quick–Strand–Wilson.

$\det Jf = nax^{n-1}$. Endow $Q_0(f)$ with ordered basis $1, x, x^2, \ldots, x^{n-1}$ and define η to take the value 0 on each of these except

$$\eta(x^{n-1}) = a^{-1}.$$

Then $\eta(nax^{n-1}) = naa^{-1} = n$, as required, and the Gram matrix for $\omega^{\mathrm{EL/K}}$ is anti-diagonal of the form

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & a^{-1} \\ 0 & 0 & \cdots & a^{-1} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & a^{-1} & \cdots & 0 & 0 \\ a^{-1} & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

We derive that

$$\omega^{\mathrm{EL/K}} = \begin{cases} \frac{n}{2}h & \text{if } n \text{ is even,} \\ \frac{n-1}{2}h + \langle a \rangle & \text{if } n \text{ is odd.} \end{cases}$$

Example 4.9. Fix constants $a, b \in k^\times$ and consider the map $f\colon \mathbb{A}^2 \to \mathbb{A}^2$ with components $f_1 = xy$ and $f_2 = -ax^2 + by^2$. We have $Q_0(f) \cong k[x,y]/(xy, -ax^2 + by^2)$ with ordered basis $1, y, y^2, x$. Furthermore,

$$Jf = \begin{pmatrix} y & x \\ -2ax & 2by \end{pmatrix}$$

with determinant

$$\det Jf = 2(ax^2 + by^2) \in k[x,y].$$

Reducing this modulo $(xy, -ax^2 + by^2)$ we get

$$\det Jf = 4by^2 \in Q_0(f).$$

We may thus define η by

$$\begin{aligned} \eta(1) &= 0 \\ \eta(y) &= 0 \\ \eta(y^2) &= b^{-1} \\ \eta(x) &= 0. \end{aligned}$$

With respect to this η, the Gram matrix for $\omega^{\mathrm{EL/K}}$ is

$$\begin{pmatrix} \eta(1) & \eta(y) & \eta(y^2) & \eta(x) \\ \eta(y) & \eta(y^2) & \eta(xy^2) & \eta(xy) \\ \eta(y^2) & \eta(y^3) & \eta(y^4) & \eta(xy^2) \\ \eta(x) & \eta(xy) & \eta(xy^2) & \eta(x^2) \end{pmatrix} = \begin{pmatrix} \eta(1) & \eta(y) & \eta(y^2) & \eta(x) \\ \eta(y) & \eta(y^2) & \eta(0) & \eta(0) \\ \eta(y^2) & \eta(0) & \eta(0) & \eta(0) \\ \eta(x) & \eta(0) & \eta(0) & \eta(a^{-1}by^2) \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & b^{-1} & 0 \\ 0 & b^{-1} & 0 & 0 \\ b^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & a^{-1} \end{pmatrix}.$$

Note, *e.g.*, that $y^3 = ab^{-1}x^2y = 0 \in Q_0(f)$ since $by^2 = ax^2$ and $xy = 0$ in that ring.

Thus
$$\omega^{\mathrm{EL/K}} = \langle a, b, 1, -1\rangle = \langle a, b\rangle + h \in \mathrm{GW}(k).$$

4.4 *Traces and general points*

Everything in the preceding section is accurate, but only works when p is a rational point of $\mathbb{A}^n$, *i.e.*, a k-point of $\mathbb{A}^n$. From the functor of points viewpoint on algebraic geometry, we know that we will care about L-points as well where L is a field extension of k. Recall that a point p of $\mathbb{A}^n = \operatorname{Spec} k[x_1, \dots, x_n]$ is a prime ideal. The residue field of p is
$$k(p) = k[x_1, \dots, x_n]_p / p$$
and we call p a $k(p)$-point.

When $k(p)$ is a finite separable extention of k, we get the following formula for the local degree of $f\colon \mathbb{A}^n \to \mathbb{A}^n$ at p:

Note that finite separable points are closed points. Though not obvious, local $\mathbb{A}^1$-degree at the generic point (0) recovers global degree.

$$\deg_p^{\mathbb{A}^1}(f) = \mathrm{tr}_{k(p)/k} \deg_p^{\mathbb{A}^1}(f \otimes k(p)) = \mathrm{tr}_{k(p)/k}(\omega_p^{\mathrm{EL/K}}(f \otimes k(p))).$$

In other words, we can compute the local degree of f considered as an n-tuple of $k(p)$-polynomials in n variables, and then apply trace to get the local degree over k.

5 *Milnor forms*

We are now prepared to define and explore the *Milnor forms* of hypersurface singularities over a field k. Recall that one of the expressions for the Milnor number of a function germ $f\colon \mathbb{C}^n \to \mathbb{C}$ at p is

Milnor forms are typically called motivic or $\mathbb{A}^1$-Milnor numbers in the literature. I am not quite ready to call a quadratic form a number, and thus prefer this name.

$$\mu_p(f) = \deg_p(\nabla f).$$

We proceed by direct analogy using the motivic technology we have developed.

Definition 5.1. Let $f\colon \mathbb{A}^n \to \mathbb{A}^1$ be an algebraic map and suppose $V(f)$ has an isolated singularity at p. The *Milnor form* of $V(f)$ at p is

$$\mu_p^{\mathbb{A}^1}(f) = \deg_p^{\mathbb{A}^1}(\nabla f).$$

Note that we may consider $\nabla f = (\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$ as an algebraic map $\mathbb{A}^n \to \mathbb{A}^n$. By the work of the previous section, when p is a rational point we can also express $\mu_p^{\mathbb{A}^1}(f)$ as an EL/K-form:

$$\mu_p^{\mathbb{A}^1}(f) = \omega_p^{\mathrm{EL/K}}(\nabla f) \in \mathrm{GW}(k).$$

Let's unpack this a bit. Since $p \in \mathrm{Sing}(f)$, we have $\nabla f(p) = 0$. Thus

$$Q_p(\nabla f) = k[x_1, \ldots, x_n]_p/(\partial f/\partial x_1, \ldots, \partial f/\partial x_n).$$

Now observe that

$$J(\nabla f) = Hf = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{i,j},$$

the Hessian of second partial derivatives of f. Thus the EL/K-form of ∇f on $Q_p(\nabla f)$ is given by $(a,b) \mapsto \eta(ab)$ where η is a k-linear map $Q_p(\nabla f) \to k$ such that

$$\eta(\det Hf) = \dim_k Q_p(\nabla f).$$

Note that if $k \subseteq \mathbb{C}$, then (4.2) implies that

$$\operatorname{rank} \mu_p^{\mathbb{A}^1}(f) = \mu_p(f \otimes \mathbb{C}) \tag{5.2}$$

where the right-hand side is the classical Milnor number of f considered as a polynomial with $\mathbb{C}$ coefficients.

Example 5.3. We begin with the cusp $f(x,y) = x^3 - y^2$ over a field k with characteristic not dividing 6. As initial data, we compute

$$\begin{aligned} \nabla f &= (3x^2, -2y), \\ Hf &= \begin{pmatrix} 6x & 0 \\ 0 & -2 \end{pmatrix}, \\ \det Hf &= -12x. \end{aligned}$$

Consider the ordered basis $1, x$ of $Q_0(\nabla f) \cong k[x,y]/(x^2, y)$. Define a k-linear map $\eta\colon Q_0(\nabla f) \to k$ by $\eta(1) = 0, \eta(x) = \frac{-1}{6}$. Then the EL/K-form of ∇f at 0 is

$$\begin{pmatrix} 0 & -1/6 \\ -1/6 & 0 \end{pmatrix}.$$

Thus

$$\mu_0^{\mathbb{A}^1}(x^3 - y^2) = h.$$

The rank of h is 2, which matches our computation of $\mu_0(x^3 - y^2)$ from Example 2.5 and thus verifies this case of (5.2).

Example 5.4. There is only one singularity simpler than a cusp, namely a *node*. These are singularities p at which $Hf(p)$ is nonsingular. In this case, p is a regular point of ∇f and we may compute $\mu_p^{\mathbb{A}^1}(f)$ without recourse to the EL/K-form. Instead, for p rational we have

$$\mu_p^{\mathbb{A}^1}(f) = \deg_p^{\mathbb{A}^1}(f) = \langle \det Hf(p) \rangle$$

and for p a finite separable point,

$$\mu_p^{\mathbb{A}^1}(f) = \mathrm{tr}_{k(p)/k} \langle \det Hf(p) \rangle.$$

Without loss of generality, suppose $p = 0$. In dimension 2, we can perform a change of coordinates so that the singularity is of the form

$$f(x,y) = ax^2 + by^2 + \text{higher order terms}$$

for some $a, b \in k^\times$. In this case,

$$\det Hf(0) = 4ab$$

and we conclude that

$$\mu_0^{\mathbb{A}^1}(f) = \langle 4ab \rangle = \langle ab \rangle.$$

In the arbitrary dimension case,

$$f(x_1, \ldots, x_n) = a_1 x_1^2 + \cdots + a_n x_n^2 + \text{higher order terms}$$

and

$$\mu_0^{\mathbb{A}^1}(f) = \langle 2^n a_1 \cdots a_n \rangle = \begin{cases} \langle a_1 \cdots a_n \rangle & \text{if } n \text{ is even,} \\ \langle 2a_1 \cdots a_n \rangle & \text{if } n \text{ is odd.} \end{cases}$$

Computation of (Gram matrices of) Milnor forms can be automated via work of Pauli.[29] To conclude this section, we present a table of Milnor forms extracted from *loc. cit*. The first seven entries are the ADE plane curve singularities, and the ones thereafter are du Val singularities of some surfaces in $\mathbb{A}^3$. All computations hold over $\mathbb{Q}$ and thus for any characteristic 0 field.

[29] Pauli, S. (2020). Computing $\mathbb{A}^1$-Euler numbers with Macaulay2. arXiv:2003.01775v3

name	equation f	$\mu_0^{\mathbb{A}^1}(f)$
A_n, n odd	x^2+y^{n+1}	$\frac{n-1}{2}h+\langle 2(n+1)\rangle$
A_n, n even	x^2+y^{n+1}	$\frac{n}{2}h$
D_n, n even	$y(x^2+y^{n-2})$	$\frac{n-2}{2}h+\langle -2, 2(n-1)\rangle$
D_n, n odd	$y(x^2+y^{n-2})$	$\frac{n-1}{2}h+\langle -2\rangle$
E_6	x^3+y^4	$3h$
E_7	$x(x^2+y^3)$	$3h+\langle -3\rangle$
E_8	x^3+y^5	$4h$
E_{12}	$x^7+y^3+z^2$	$6h$
Z_{11}	$x^5+xy^3+z^2$	$5h+\langle -6\rangle$
Q_{10}	$x^4+y^3+xz^2$	$5h$
E_{13}	$x^5y+y^3+z^2$	$6h+\langle -10\rangle$
Z_{12}	$x^4y+xy^3+z^2$	$5h+\langle -22\rangle+\langle -66\rangle$

6 *Zooming out and blowing up*

At this point we have summarized — or at least previewed — most of the major results on Milnor forms. Where do we go from here? Before presenting a research program related to resolution of singularities, we state a few open problems on motivic degree and Milnor forms.

6.1 *A smörgåsbord of open problems*

Here are a few ideas for research problems that are approachable (or at least state-able) given the background we've acquired up to now:

(1) Compute more Milnor forms in specific examples and use the perturbation theory results of Kass–Wickelgren[30] (extended by Pauli–Wickelgren[31]) to place constraints on the nodes into which they can bifurcate. The idea is that for generic $a_1, \ldots, a_n \in k$ and all $t \in k$, the equation

$$f(x) - a_1x_1 - \cdots - a_nx_n = t \tag{6.1}$$

only has nodal fibers. Kass–Wickelgren–Pauli prove that

$$\sum_{p\in \mathrm{Sing}(f)} \mu_p^{\mathbb{A}^1}(f)$$

equals the sum of the Milnor forms of the nodes of (6.1). In this fashion, Milnor forms constrain the types of nodes that can appear when the singularity is perturbed.

[30] Kass, J. L. and Wickelgren, K. (2019). The class of Eisenbud-Khimshiashvili-Levine is the local $\mathbf{A}^1$-Brouwer degree. *Duke Math. J.*, 168(3):429–469

[31] Pauli, S. and Wickelgren, K. (2021). Applications to $\mathbb{A}^1$-enumerative geometry of the $\mathbb{A}^1$-degree. *Res. Math. Sci.*, 8(2):Paper No. 24, 29

(2) Classify the Grothendieck–Witt classes that can be realized as (local) motivic degrees. Up to rank 7, this has been done by Quick–Strand–Wilson.[32] *Loc. cit.* also proves that every EL/K-form of rank at least 2 contains h as a direct summand, which partially explains the prevalance of hyperbolic terms in our table of Milnor forms. Whether Milnor forms are more constrained than general EL/K-forms has not been addressed in the literature to the best of my knowledge. (Pauli's tables for du Val singularities all have anisotropic part of rank at most 2!) What if the class of maps/singularities is constrained in some fashion? What if a particular field or class of fields (finite, local, formally real, *etc.*) is fixed?

[32] Quick, G., Strand, T., and Wilson, G. M. (2021). Representability of the local motivic Brouwer degree. arXiv:2011.04046v2

(3) Relatedly, what can be said about various quadratic form invariants associated with Milnor forms? We know that rank matches the classical Milnor number and that signature matches real Milnor numbers, but what about discriminant? Stiefel–Whitney classes?

(4) Classically, curve singularities are often studied via their associated Puiseux series and Newton polygons. It is in fact possible to deduce the Milnor number of a plane curve from this data (see Wall[33] Exercise 6.7.2]). Is there an analogous statement for Milnor forms?

[33] Wall, C. T. C. (2004). *Singular points of plane curves*, volume 63 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge

(5) (⚠ Hard!) Recall that, in the classical case,

$$\chi(\overline{F}_\theta) = 1 + (-1)^{n-1}\mu_p(f) \tag{6.2}$$

where $\overline{F}_\theta$ is the Milnor fiber of f at p. Determine the relationship between Milnor forms and $\mathbb{A}^1$-Euler characteristic of Milnor fibers. Beware, though, that there is no "Milnor fiber" in this context; rather, work of Denef and Loeser provides a way to think of $\chi^{\mathbb{A}^1}$(Milnor fiber) as the difference of two Euler characteristics. Progress in special cases has been made by Levine–Pepin Lehalleur–Srinivas and Azouri.[34] In particular, one can derive from their work an enrichment of the the following theorem of Milnor–Orlik[35]: If $f \in \mathbb{C}[x_1, \ldots, x_n]$ is homogeneous of degree m with an isolated singularity at 0, then

$$\mu_p(f) = (m-1)^n.$$

Beware that there are some nonintuitive zero-dimensional correction factors in the motivic case!

[34] Levine, M., Pepin Lehalleur, S., and Srinivas, V. (2021). Euler characteristics of homogeneous and weighted-homogeneous hypersurfaces. arXiv:2101.00482v2; and Azouri, R. (2021). The quadratic Euler characteristic of nearby cycles and a generalized conductor formula. arXiv:2101.02686v1

[35] Milnor, J. and Orlik, P. (1970). Isolated singularities defined by weighted homogeneous polynomials. *Topology*, 9:385–393

(6) As a consequence of (6.2) and the inclusion-exclusion formula for Euler characteristic, one can deduce that

$$\mu_p(fg) = \mu_p(f) + \mu_p(g) + 2(V(f)\cdot V(g))_p - 1$$

for $f, g \in k[x,y]$ and $(V(f) \cdot V(g))_p$ the local intersection number of $V(f)$ and $V(g)$ at p. Does this formula admit a quadratic refinement? Perhaps $h \deg_p^{\mathbb{A}^1}(f,g)$ could replace $2(V(f) \cdot V(g))_p$. An experimental approach may shed light on the situation. In the exercises, you will discover that the term -1 will need to be replaced by some -1-dimensional class in $\mathrm{GW}(k)$.

6.2 *Blowups and resolutions*

An etymological note: Blowing up is a process of enlargement, not one of destruction. Think about blowing up a photograph, not a bomb.

Our present aim is to explore how Milnor forms transform under blowup and resolution of singularities. We restrict our attention to plane curve singularities in order to simplify the picture and access a wealth of beautiful classical material.

Let's begin by blowing up $\mathbb{A}^2$ at the origin. The idea is to replace 0 with the space of lines passing through it (namely $\mathbb{P}^1$) without disturbing the rest of $\mathbb{A}^2$. We accomplish this with

$$\mathrm{B}\ell_0\mathbb{A}^2 := \{(x,\ell) \in \mathbb{A}^2 \times \mathbb{P}^1 \mid x \in \ell\}.$$

Note that if $x \neq 0$, then there is exactly one $\ell \in \mathbb{P}^1$ containing x, but the origin is in every element of $\mathbb{P}^1$. The blowup is equipped with a natural projection

$$\begin{aligned} \pi\colon \mathrm{B}\ell_0\mathbb{A}^2 &\longrightarrow \mathbb{A}^2 \\ (x,\ell) &\longmapsto x \end{aligned}$$

which is an isomorphism over $\mathbb{A}^2 \setminus 0$, while $\pi^{-1}0 = 0 \times \mathbb{P}^1$. We call $E := \pi^{-1}0$ the exceptional divisor.

There are two natural coordinate charts on $\mathrm{B}\ell_0\mathbb{A}^2$, namely

$$\begin{aligned} U_0 := \{((x,y),[z_0 : z_1]) \mid z_0 \neq 0, (x,y) \in [z_0 : z_1]\} &\xrightarrow{\cong} \mathbb{A}^2 \\ ((x,y),[z_0 : z_1]) &\mapsto (u_0 = x, v_0 = z_1/z_0) \end{aligned}$$

and

$$\begin{aligned} U_1 := \{((x,y),[z_0 : z_1]) \mid z_1 \neq 0, (x,y) \in [z_0 : z_1]\} &\xrightarrow{\cong} \mathbb{A}^2 \\ ((x,y),[z_0 : z_1]) &\mapsto (u_1 = z_0/z_1, v_1 = y). \end{aligned}$$

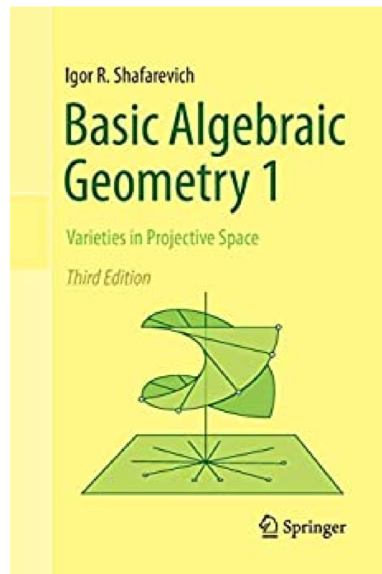

Figure 8: The cover of Shafarevich's *Basic Algebraic Geometry I* nicely illustrates the real points of the blowup construction. Note that the top and bottom portions of the picture upstairs are identified.

In terms of these coordinates, we have

$$\begin{aligned} \pi|_{U_0}(u_0, v_0) &= (u_0, u_0 v_0), \\ \pi|_{U_1}(u_1, v_1) &= (u_1 v_1, v_1). \end{aligned}$$

(According to Brieskorn–Knörrer,[36] "This description of the mapping by a quadratic transformation should be remembered without fail.") Also note that the exceptional divisor $E = \pi^{-1}0$ is easy to track in these coordinates: it corresponds to $u_0 = 0$ in U_0 and $v_1 = 0$ in U_1.

[36] Brieskorn, E. and Knörrer, H. (1986). *Plane algebraic curves*. Modern Birkhäuser Classics. Birkhäuser/Springer Basel AG, Basel. Translated from the German original by John Stillwell, [2012] reprint of the 1986 edition

Now suppose that $V = V(f) \subseteq \mathbb{A}^2$ is some hypersurface. The *strict transform* $\widetilde{V}$ of V is the closure of $\pi^{-1}V \smallsetminus E$ in $\mathrm{B}\ell_0\mathbb{A}^2$. We have

$$\pi^{-1}V \cap U_0 = V(f(u_0, u_0v_0)),$$
$$\pi^{-1}V \cap U_1 = V(f(u_1v_1, v_1)).$$

If $0 \in V$, then

$$f(u_0, u_0v_0) = u_0^m f_0^{(1)}(u_0, v_0),$$
$$f(u_1v_1, v_1) = v_1^n f_1^{(1)}(u_1, v_1)$$

and $\widetilde{V}$ has equation $f_0^{(1)}$ in U_0, $f_1^{(1)}$ in U_1.

Example 6.3. Let us return to our beloved cusp $V = V(f)$ with $f(x,y) = x^3 - y^2$. The equation for $\pi^{-1}V \cap U_0$ is

$$u_0^3 - (u_0v_0)^2 = u_0^2(u_0 - v_0^2) = 0$$

so $f_0^{(1)}(u_0, v_0) = u_0 - v_0^2$ with zero locus a smooth parabola. The equation for $\pi^{-1}V \cap U_1$ is

$$(u_1v_1)^3 - v_1^2 = v_1^2(u_1^3v_1 - 1) = 0$$

so $f_1^{(1)}(u_0, v_0) = u_1^3v_1 - 1$ with smooth zero locus not intersecting the exceptional divisor.

We see, then, that $\widetilde{V}$ is smooth. Since $\pi\colon \mathrm{B}\ell_0\mathbb{A}^2 \to \mathbb{A}^2$ is an isomorphism away from E, $\pi|_{\widetilde{V}}\colon \widetilde{V} \to V$ is an isomorphism away from the unique point in $\pi^{-1}V \cap E$. This means that $\pi|_{\widetilde{V}}$ is a proper birational map with domain a smooth variety. In other words, $\widetilde{V} \to V$ is a *resolution of singularities*.

Figure 9: Andreu Alfaro's sculpture *Lebenskraft*, a real-life blowup. Photo: Carl McTague.

There are two things to note now: First, there was nothing special about the origin or $\mathbb{A}^2$ in our description of blowing up. We may blow up any algebraic surface at any point and similarly replace the point with a copy of $\mathbb{P}^1$ recording slopes through that point. Second, it is generally the case that singularities become simpler when we blow up:

Theorem 6.4. *The singularities of any plane algebraic curve may be resolved by a finite sequence of blowups. Moreover, after potentially applying some additional blowups, the proper preimage of the curve will meet the system of exceptional divisors transversely.*

The proof of this theorem is classical and requires a careful analysis of how Newton polygons transform under blowup.

The following theorem exhibits a very nice relationship between the Milnor number of a plane curve singularity and its blowup:

Theorem 6.5. *Let $V \subseteq \mathbb{A}^2_{\mathbb{C}}$ be a plane curve with isolated singularity at p of multiplicity d, where V has r different tangent lines at p. Then*

$$\mu_p(V) = 1 + d(d-1) - r + \sum_{x \in \mathrm{Sing}(\widetilde{V} \cap E)} \mu_x(\widetilde{V}).$$

Question 6.6. *Does Theorem 6.5 have an analogue when V is an arbitrary plane curve and μ_p is replaced by $\mu_p^{\mathbb{A}^1}$?*

My thesis student, Usman Hafeez,[37] studied this problem by computing

$$\Delta_p(f) := \mu_p^{\mathbb{A}^1}(f) - \sum_{x \in \mathrm{Sing}(\widetilde{V} \cap E)} \mu_x^{\mathbb{A}^1}(\widetilde{f})$$

for a number of examples. For $f = x^n + y^m$, he found

$$\Delta_0(f) = \begin{cases} \frac{n(n-1)}{2} h & \text{if } n \text{ is odd,} \\ \frac{n(n-1)}{2} h & \text{if } n \text{ is even and } m \text{ is odd,} \\ \frac{n(n-1)}{2} h + \langle mn \rangle - \langle n(m-n) \rangle & \text{if } n, m \text{ are both even.} \end{cases}$$

For D_n singularities, Hafeez computed

$$\Delta_0(D_n) = \begin{cases} 2h + \langle -2, 2(n-1) \rangle - \langle 2(n-4) \rangle & \text{if } n \text{ is even,} \\ 2h + \langle -2 \rangle & \text{if } n \text{ is odd.} \end{cases}$$

Both calculations indicate that $\frac{d(d-1)}{2}h$ might be the correct replacement for the $d(d-1)$ term in Theorem 6.5, but the correction term for $1 - r$ remains mysterious.

There is a beautiful classical theory of resolution of singularities for plane curves in terms of particular decorated trees and "multiplicity systems." A positive solution to Question 6.6 would provide an iterative method for the computation of Milnor forms.

[37] Hafeez, M. U. (2021). $\mathbb{A}^1$-milnor numbers. Undergraduate thesis, Reed College

Exercises

(1) Apply the second derivative test to classify the local extrema of the function $(x, y) \mapsto \cos(x)\sin(y)$.

(2) Let $f(x, y) = x^3 - y^2$. Check that $K_{0,\varepsilon}(f)$ lies on a torus of the form

$$\{(x, y) \in \mathbb{C}^2 \mid \|x\| = \xi,\ \|y\| = \eta\}$$

for some $\xi, \eta > 0$. Then go one step further and explicitly parametrize $K_{0,\varepsilon}(f)$.

(3) Choose several singular curves and compute their Milnor numbers algebraically (in terms of the local algebra $\mathbb{C}[x, y]_p/(\nabla f)$) and topologically (in terms of Euler characteristic or local degree).

(4) Find a singular curve for which the dimension of $\mathbb{C}[x,y]_p/(\nabla f)$ differs from that of $\mathbb{C}[x,y]/(\nabla f)$.

(5) Demonstrate the correspondence between {quadratic forms + ordered bases} and degree 2 homogeneous polynomials.

(6) Let k be a field of characteristic different from 2, and let V be a finite-dimensional k-vector space. Show that nondegenerate quadratic forms on V, nondegenerate symmetric bilinear forms on V, and self-dual isomorphisms $V \to V^*$ are all "the same." (Here *self-dual* means that the dual map $(V^*)^* \to V^*$ is equal to the original after composing with the natural isomorphism $V \cong (V^*)^*$). Equivalently, the matrix for $V \to V^*$ with respect to some/any basis of V and dual basis of V^* is symmetric.)

(7) Prove that sbf's b and b' on V are isometric if and only if their Gram matrices are congruent:

$$G_{b'} = A^\top G_b A.$$

(8) Check that xy and $x^2 - y^2$ are isometric.

(9) Use the relations of Theorem 3.5 to prove that $\langle a, -a\rangle$ is hyperbolic for all $a \in k^\times$.

(10) Compute $\operatorname{tr}_{\mathbb{C}/\mathbb{R}}\langle 1\rangle_{\mathbb{C}}$. What about $\operatorname{tr}_{L/k}\langle 1\rangle_L$ for L a quadratic extension $L = k(\sqrt{\alpha})$?

(11) Prove (or recall) the following two facts about field traces for a finite extension L/k:

(a) For $\alpha \in L$,

$$\operatorname{tr}_{L/k}(\alpha) = [L : k(\alpha)]\operatorname{tr}_{k(\alpha)/k}(\alpha).$$

(b) If L/k is not separable,[38] then $\operatorname{tr}_{L/k} = 0$.

[38] This means that the minimal polynomial of every $\alpha \in L$ has no repeated roots in any extension; equivalently, the formal derivatives of minimal polynomials are nonzero.

(12) Use the standard gluing diagram (*i.e.*, pushout square)

$$\begin{tikzcd} \mathbb{A}^1 \smallsetminus 0 \arrow[r] \arrow[d] & \mathbb{A}^1 \arrow[d] \\ \mathbb{A}^1 \arrow[r] & \mathbb{P}^1 \end{tikzcd}$$

to show that $\mathbb{P}^1 \simeq S^{2,1}$. (*Hint*: It is always the case that the "homotopy pushout" of $* \leftarrow X \rightarrow *$ is the suspension $\Sigma X = S^{1,0} \wedge X$ of X.)

(13) The standard identification of complex numbers with 2×2 real matrices via

$$a + bi \longmapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

induces a homomorphism

$$\mathrm{GL}_n(\mathbb{C}) \longrightarrow \mathrm{GL}_{2n}(\mathbb{R}).$$

Show that every matrix in the image of this map has positive determinant. (*Hint*: There is a slick topological proof via connectivity.) How is this related to the fact that every function $S^{2n} \to S^{2n}$ induced by an algebraic map $\mathbb{A}^n \to \mathbb{A}^n$ has positive degree?

(14) Redo Example 4.7 but using the functional $\eta' \colon Q_0(x^2) \to k$ given by $\eta'(1) = 1, \eta'(2x) = 2$. You should ultimately get the same isometry class!

(15) Compute the $\mathbb{A}^1$-degree of the map $\mathbb{P}^1 \to \mathbb{P}^1,\ x \mapsto \frac{x^2-1}{x}$.[39]

(16) Let $f(x) = A(x)/B(x)$ be a rational function viewed as an endomorphism of $\mathbb{P}^1$. Suppose $A = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ and $B = b_{n-1}x^{n-1} + \cdots + b_0$ with no common zeros. Then

$$\frac{A(x)B(y) - A(y)B(x)}{x - y}$$

is a polynomial (check this!) and can be written as $\sum_{1 \le i,j \le n} c_{ij} x^{i-1} y^{j-1}$ for $c_{ij} \in k$. By work of C. Cazanave, the matrix (c_{ij}) is symmetric and nondegenerate and the corresponding element of $\mathrm{GW}(k)$ is equal to $\deg^{\mathbb{A}^1}(f)$. Use Cazanave's formula to compute the $\mathbb{A}^1$-degree of $x \mapsto x^2$ and of $x \mapsto \frac{x^2-1}{x}$.

(17) Let $f \colon \mathbb{A}^n \to \mathbb{A}^n$ and $g \colon \mathbb{A}^m \to \mathbb{A}^m$ be algebraic maps. Show that the map of motivic spheres induced by $f \times g \colon \mathbb{A}^{n+m} \to \mathbb{A}^{n+m}$ has motivic degree equal to

$$\deg^{\mathbb{A}^1}(f) \deg^{\mathbb{A}^1}(g).$$

(18) Let $f, g \colon \mathbb{A}^n \to \mathbb{A}^n$ be algebraic maps with isolated zeros at the origin. The *chain rule* for local motivic degree states that

$$\omega_0^{\mathrm{EL/K}}(f \circ g) = \omega_0^{\mathrm{EL/K}}(f)\omega_0^{\mathrm{EL/K}}(g) \in \mathrm{GW}(k).$$

Use the following steps (Quick–Strand–Wilson[40] Theorem 26) to prove the chain rule:

(a) Define $\tilde{f} = f \times \mathrm{id} \colon \mathbb{A}^{2n} \to \mathbb{A}^{2n}$ and $\tilde{g} = g \times \mathrm{id} \colon \mathbb{A}^{2n} \to \mathbb{A}^{2n}$. Show/observe that

$$\omega_0^{\mathrm{EL/K}}(\tilde{f} \circ \tilde{g}) = \omega_0^{\mathrm{EL/K}}(f \circ g).$$

(b) Let $L \colon \mathbb{A}^n \to \mathbb{A}^n$ be a unipotent linear transformation. Prove that

$$\omega_0^{\mathrm{EL/K}}(f \circ L \circ g) = \omega_0^{\mathrm{EL/K}}(f \circ g).$$

(This is the hard part! See Knight–Swaminathan–Tseng[41] Lemma 12.)

[39] The projective line has charts isomorphic to $\mathbb{A}^1$, so you are entitled to use the same techniques as for maps $\mathbb{A}^1 \to \mathbb{A}^1$.

[40] Quick, G., Strand, T., and Wilson, G. M. (2021). Representability of the local motivic Brouwer degree. arXiv:2011.04046v2

[41] Knight, J., Swaminathan, A. A., and Tseng, D. (2021). On the EKL-degree of a Weyl cover. *J. Algebra*, 565:64–81

(c) By infixing germane choices of unipotent maps, show that

$$\omega_0^{\mathrm{EL/K}}(\tilde{f} \circ \tilde{g}) = \omega_0^{\mathrm{EL/K}}(g \times f).$$

Use the previous exercise to derive the chain of equalities

$$\omega_0^{\mathrm{EL/K}}(f \circ g) = \omega_0^{\mathrm{EL/K}}(\tilde{f} \circ \tilde{g}) = \omega_0^{\mathrm{EL/K}}(g \times f) = \omega_0^{\mathrm{EL/K}}(g)\omega_0^{\mathrm{EL/K}}(f)$$

as desired.

(19) Compute the Milnor form of the A_n singularity $x^2 + y^{n+1}$.

(20) Compute the Milnor form of the D_n singularity $y(x^2 + y^{n-2})$.

(21) Use the fact that $D_n = yA_{n-3}$ to test the conjecture from smörgåsbord problem (6).

(22) Download Macaulay2 and implement Pauli's algorithm for motivic degrees and Milnor forms.[42] Start running experiments on the problems proposed in Section 6 (or your own).

(23) Fix the SageMath code available at this link so that it will successfully compute local motivic degrees. (Sage isn't good at localizations. Perhaps we can get around this using saturation?)

[42] Pauli, S. (2020). Computing $\mathbb{A}^1$-Euler numbers with Macaulay2. arXiv:2003.01775v3

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