(• HW extn: Tu 30 May 1PM

• practice probs released Tu)

26.V.23

Quotient Manifolds

G a Lie group acting smoothly on a smooth mfld M.

M/G quotient space

Q When is M/G a top'l mfld admitting a smooth structure s.t. $\pi : M \longrightarrow M/G$ is a smooth submersion?

A When G acts freely and properly on M.

$g \cdot p = p$ iff $g = e$

$G \times M \longrightarrow M \times M$ is proper

$(g, p) \longmapsto (g \cdot p, p)$

(guarantees M/G H'ff.)

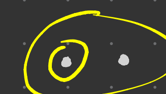
Recall $X \longrightarrow Y$ proper when preimages of compacts are compact.

E.g. $GL_n\mathbb{R} \circlearrowright \mathbb{R}^n$ via $A \cdot x = \underbrace{Ax}_{\text{matrix mult'n}}$

Since $A \cdot \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ $\forall A$, this action is <u>not</u> free.

Further, if $x \neq y \in \mathbb{R}^n \setminus 0$, then $\exists\, A$ s.t. $Ax = y$, so

$\mathbb{R}^n / GL_n\mathbb{R} = \left\{ \mathbb{R}^n \setminus \{0\},\ \{0\} \right\}$ with open sets $\emptyset, \{\mathbb{R}^n \setminus \{0\}\}, \mathbb{R}^n / GL_n\mathbb{R}$



Not a top'l mfld!

E.g. $C_4 \circlearrowright \mathbb{R}^2$ by $\pi/2$ rotation

fixed!

"cone point"

Not a <u>smooth</u> mfld.

E.g. $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $\mathbb{R} \circlearrowright \mathbb{T}^2 = S^1 \times S^1$ by

$$t \cdot (z, w) = (e^{2\pi i t} z, e^{2\pi i \alpha t} w).$$

This is a free action with dense orbits

$\Rightarrow$ only saturated opens of $\mathbb{T}^2$ are $\emptyset$, $\mathbb{T}^2$

$\Rightarrow$ $\mathbb{T}^2/\mathbb{R}$ has the trivial topology so not a top'l mfld

Exc Check that this action is not proper but is smooth.

Lemma 21.1 For any cts action of a top'l gp G on a space X, the quotient map $X \longrightarrow X/G$ is open.

Prop 21.4 If a Lie gp G acts ctsly + properly on a mfld M, then M/G is Hausdorff.

Characterization of Proper Actions (21.5) M mfld, G Lie gp acting ctsly on M. TFAE:

(a) The action is proper

(b) $(p_i), (g_i)$ sequences of M, G s.t. $(p_i), (g_i \cdot p_i)$ converge, then (g_i) converges

(c) $\forall K \subseteq M$ compact, $G_K := \{g \in G \mid (g \cdot K) \cap K \neq \emptyset\}$ is compact. $\square$

N.B. If $K = \{p\}$, then $G_K = G_p$ is the isotropy subgp of p.

Cor Every cts action by a compact Lie group is proper $\square$

Prop $\theta : G \times M \to M$ proper smooth action of Lie gp G on a smooth mfld M. $\forall p \in M$, $\theta^{(p)} : G \to M$, $g \mapsto g \cdot p$ is proper thus $G \cdot p = \Theta^{(p)}(G)$ is closed in M. If additionally $G_p = \{e\}$, then $\Theta^{(p)}$ is a smooth embedding with $G \cdot p \subseteq M$ a properly embedded submfld.

$\blacksquare$

<u>Quotient</u> <u>Mfld</u> <u>Thm</u> $G \circlearrowright M$ smooth, free, proper action of a Lie gp on a smooth mfld. Then M/G is a top'l mfld of dim $\dim M - \dim G$, and has a unique smooth structure s.t. $\pi : M \longrightarrow M/G$ is a smooth submersion.

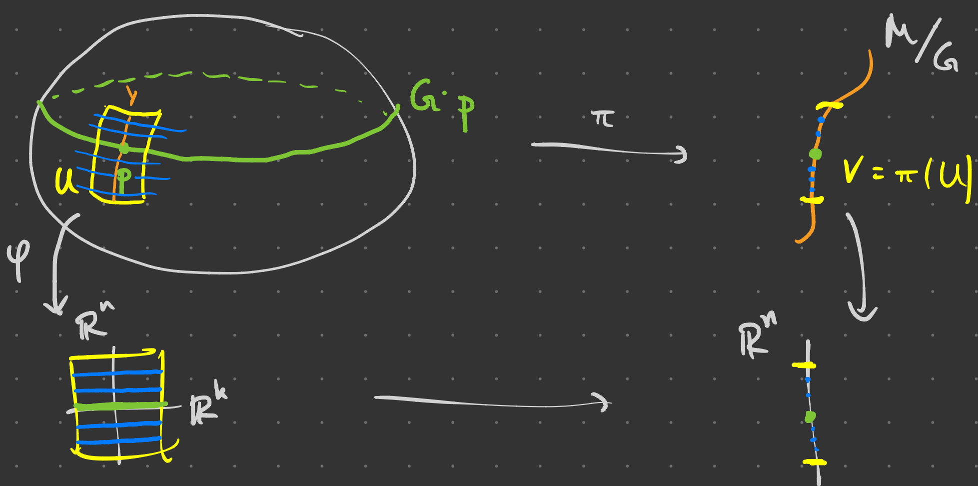
<u>Pf</u> Uniqueness of smooth structure follows from quotient property of smooth submersions:

$$\begin{tikzcd} M \arrow[d, "\pi"'] \arrow[dr, "\pi"] & \\ (M/G)_1 \arrow[r, "id"'] & (M/G)_2 \end{tikzcd}$$

.

Call a smooth chart (U, φ) for M adapted to the G-action when it's a cubical chart w/ coord fns $(x^1, \dots, x^k, y^1, \dots, y^n)$ for $k = \dim G$, $m = \dim M$, $n = m - k$ s.t. $\forall p \in M$,

$$G \cdot p \cap U = \begin{cases} \emptyset \\ \text{single slice of the form } (y^1, \dots, y^n) = (c^1, \dots, c^n). \end{cases}$$



Claim $\forall p \in M$ $\exists$ adapted chart centered at p.

Assume this for now.

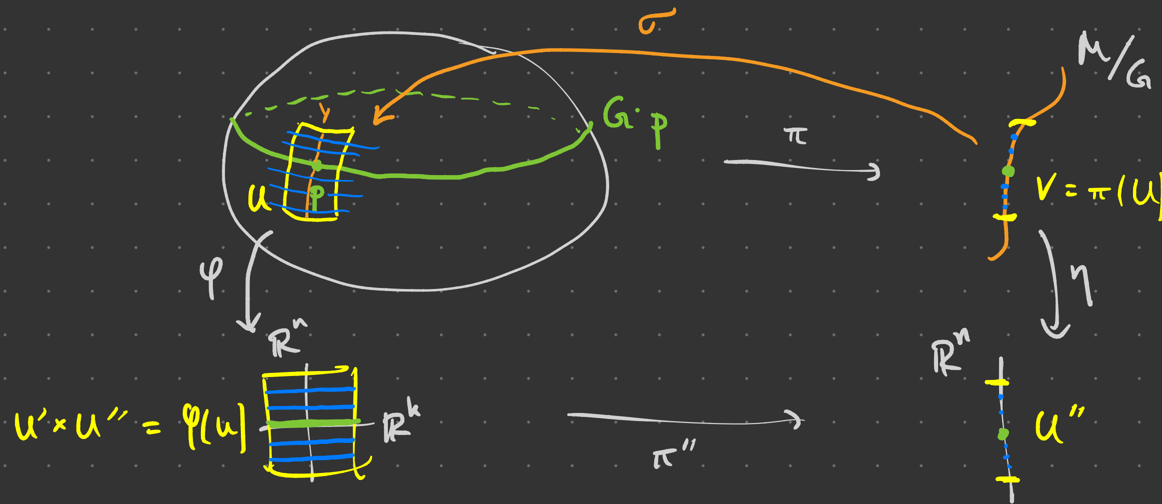
M/G is H'ff by 21.4. Since π is open by 21.1, countable basis $\{B_i\}$ of M becomes countable basis $\{\pi(B_i)\}$ for M/G; thus M/G is second countable.

For loc Euclidean, let $q = \pi(p)$ be an arbitrary pt of M/G, (U, φ) an adapted chart for M centered at p with

$$\varphi(U) = U' \times U'' \quad \text{for } U' \subseteq \mathbb{R}^k,\ U'' \subseteq \mathbb{R}^n \text{ open cubes.}$$

Set $V = \pi(U)$, open b/c π open.

Let $Y = \{x^1 = \cdots = x^k = 0\}$. Then $\pi|_Y : Y \longrightarrow V$ is a homeo.



Let $\sigma = (\pi|_Y)^{-1} : V \longrightarrow U$; it's a local section of π

Define $\eta : V \longrightarrow U''$ i.e. $\eta = \underbrace{\pi''\circ\varphi}_{\text{homeo } Y\to U''}\circ\underbrace{\sigma}_{\text{homeo onto } Y}$ which is a homeo

$[(x,y)] \longmapsto y$

$\Longrightarrow$ M/G is loc Euclidean and thus a top'l mfld.

Remains to show M/G has a smooth str s.t. π is a smooth submersion.

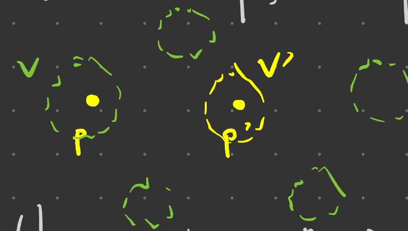
Use atlas $\{(V,\eta)\}$. Then $\pi : (x,y) \longmapsto y$ locally, so it's a smooth submersion as long as transition maps are smooth (exc / p. 547). $\square$

Covering mflds

Lemma 21.11 Suppose a discrete Lie group Γ acts ctsly and freely on a mfld E. The action is proper iff the following conditions hold:

(i) $\forall p \in E$ $\exists$ nbhd $U \ni p$ s.t. $\forall g \in \Gamma \setminus \{e\}$, $(g \cdot U) \cap U = \emptyset$

(ii) If $p' \notin \Gamma \cdot p$, then $\exists$ nbhds $V \ni p$, $V' \ni p'$ s.t. $(g \cdot V) \cap V' = \emptyset$ for all $g \in \Gamma$. $\square$



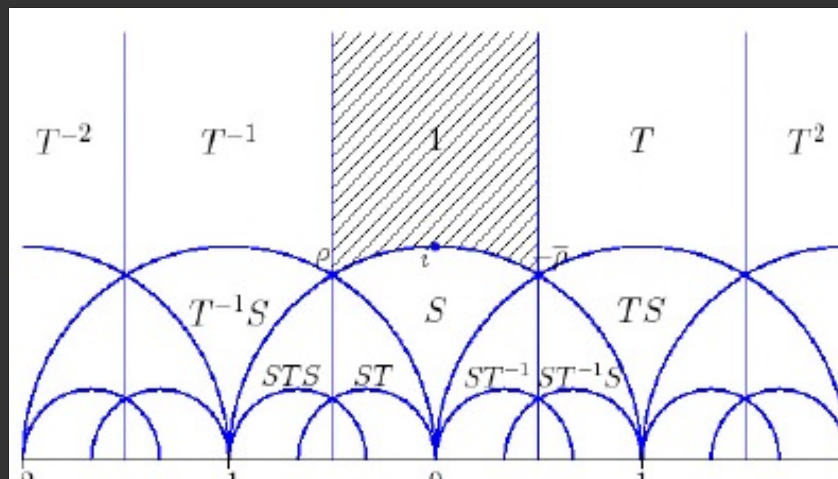
Prop 21.12 $\pi : E \to M$ smooth covering map. Then $\mathrm{Aut}_\pi(E)$ w/ discrete top acts smoothly, freely, and properly on E.

Thm 21.13 E conn'd smooth mfld, Γ discrete Lie group acting smoothly freely properly on E. Then E/Γ has a unique smooth structure s.t. $\pi : E \to E/\Gamma$ is a smooth normal covering map.

Γ acts transitively on fibers

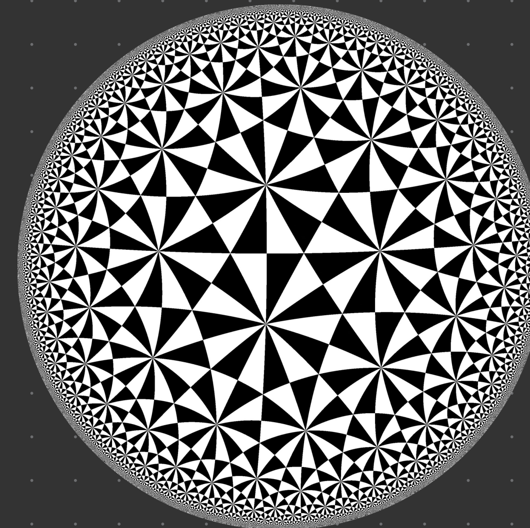
Fact: Every discrete subgroup of $PSL_2\mathbb{R}$ acts smoothly, properly, freely on $\mathbb{H}$.

Discrete $\Gamma \le PSL_2\mathbb{R}$ are called Fuchsian groups.

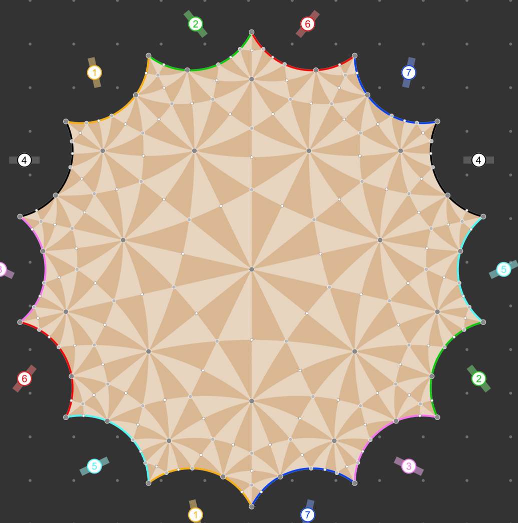


(2,3,7)-triangle gp generated by reflections over Δ w/ angles

$\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{7} \rightsquigarrow \Gamma \leq PSL_2\mathbb{R}$



$\mathbb{H}/\Gamma$ =: Klein quartic



Riemann surface of genus 3 with automorphism gp

$PSL_2\mathbb{F}_7$ of order 168 —

- max'l aut for genus 3
- 2nd smallest non-Abelian simple gp

"Nice" $X \subseteq M \circlearrowleft G$ s.t. $\forall p \in M$ $\exists$ one or two points of $G \cdot p$ inside X and if 2 both on ∂X