22. V. 23

Riemannian metrics

M = smooth mfld w/ or w/o ∂.

A Riemannian metric $g = \langle \, , \rangle$ on M is a positive definite symmetric bilinear form (i.e. inner product) $g_p : T_pM \times T_pM \to \mathbb{R}$ which is smooth in $p \in M$.

- I.e., $g \in \Gamma(\mathrm{Sym}^2 TM)$ and $g_p(v,v) = \langle v, v\rangle_p \geq 0 \ \forall v \in T_pM$

$$\mathrm{Sym}^2 TM \subset TM \otimes TM$$

with $\langle v, v \rangle_p = 0$ iff $v = 0$.

$$\begin{tikzcd} \mathrm{Sym}^2 TM \arrow[d] \\ M \arrow[u, bend right, dashed, "g \text{ smooth}"'] \end{tikzcd}$$

- Locally, $g = \sum g_{ij}\, dx^i \otimes dx^j$ for (g_{ij}) a symm pos def matrix of smooth functions.

$(V, \langle\,,\rangle)$ inner prod space

V: $\mathbb{R}$-vs

$\langle\,,\rangle$: symm pos def bilin form

Given V a basis $e_1, \ldots, e_n$

$\langle e_i, e_j \rangle = g_{ij}$

(g_{ij}) Gram matrix of $\langle\,,\rangle$ wrt $e_1, \ldots, e_n$

$$\left\langle \sum c_i e_i, \sum d_j e_j \right\rangle = (c_1 \cdots c_n)\,(g_{ij}) \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}$$

E.g. The Euclidean metric $\bar{g}$ on $\mathbb{R}^n$ is given by

$$g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \delta_{ij} ,$$

so $g = \sum dx^i \otimes dx^i$. Gives standard dot product.

E.g. (M, g), $(\tilde{M}, \tilde{g})$ Riemannian mflds, then

$M \times \tilde{M}$ has Riemannian structure $\hat{g} = g \oplus \tilde{g}$.

Locally, $\hat{g} = \begin{pmatrix} g_{ij} & \\ & \tilde{g}_{ij} \end{pmatrix}$.

Prop Every smooth mfld (w/ or w/o ∂) admits a Riemannian metric.

Pf Let $\{U_\alpha\}$ be an open cover of M so that each $U_\alpha \cong \mathbb{R}^n$.
On each U_α, take the std Euclidean metric $\bar{g}_\alpha$.
For $\{\psi_\alpha\}$ smooth POU sub to $\{U_\alpha\}$, $\sum \psi_\alpha g_\alpha$ works. $\square$

Mflds often admit different metrics w/ wildly different properties. For instance, embed $\mathbb{R}^2 \hookrightarrow \mathbb{R}^3$ in different ways and pull back $\bar{g}$ on $\mathbb{R}^3$ to $\mathbb{R}^2$.

Defn On a Riemannian mfld (M, g),

- **length** of $v \in T_pM$ is $\langle v, v\rangle_p^{1/2} =: |v|_g$
- **angle** b/w $v, w \in T_pM \setminus 0$ is the unique $\theta \in [0,\pi]$ s.t. $\cos\theta = \dfrac{\langle v, w\rangle_p}{|v|_g |w|_g}$
- $v, w \in T_pM$ are **orthogonal** when $\langle v, w\rangle_p = 0$.

Nota

Gram-Schmidt can be applied smoothly, and mflds admit local frames, so Riemannian mflds admit local orthonormal frames.

Pullback metrics

If M, N smooth mflds (w/or w/o ∂), g a Riemannian metric on N, then $F^* g \in \Gamma(\mathrm{Sym}^2 TM)$. If $F^* g$ is pos def, then it's a Riemannian metric on M.

Have $(F^* g)_p(v, w) = g_{F(p)}(dF_p v, dF_p w)$.

Thus $(F^* g)_p(v, v) = g_{F(p)}(dF_p v, dF_p v) \geq 0$

and semi-definite iff dF_p injective $\forall p$. Thus:

Prop F^*g is a Riemannian metric on M iff F is an immersion. $\square$

May use this to induce metrics on submflds, e.g. S^n with round metric is $(S^n, \iota^*\bar{g})$ for $\iota : S^n \hookrightarrow \mathbb{R}^{n+1}$.

Isometries and flatness

(M, g), $(\tilde{M}, \tilde{g})$ Riemannian mflds

A smooth map $F : M \longrightarrow \tilde{M}$ is an isometry when it is a diffeomorphism s.t. $F^*\tilde{g} = g$.

Note
- Isometries preserve length, angle, orthogonality.
- Local isometry can also be useful: $\forall p \in M$ $\exists$ nbhd U of p s.t. $F|_U$ is an isometry $U \longrightarrow$ open of $\tilde{M}$.

A Riemannian n-mfld (M, g) is <u>flat</u> when (M, g) is locally isometric to $(\mathbb{R}^n, \bar{g})$.

Riem surface scalar curvature K

<u>Exc</u> S^2 with round metric is not flat.

$$\int_M K \, dA = 2\pi \chi(M)$$

Gauss-Bonnet (+)

<u>Normal bundle</u>

Previously saw normal bundle of $M \subseteq \mathbb{R}^n$ — secretly used std metric on $\mathbb{R}^n$.

Consider (M, g) + submfld $S \subseteq M$. Call for $p \in S$ $v \in T_pM$ <u>normal</u> to S when $\langle v, w \rangle_p = 0$ $\forall w \in T_pS$. The <u>normal space</u> to S at p is $N_pS := \{ v \in T_pM \mid \langle v, w \rangle_p = 0 \ \forall w \in T_pS \} \leq T_pM$.

This a smooth vector bundle over S via

$$\begin{tikzcd} NS \arrow[r, hook] \arrow[d, "\pi"'] & TM \arrow[d, "\pi"] \\ S \arrow[r, hook] & M \end{tikzcd}$$

If $\dim M = n$, $\dim S = k$, then NS is a smooth rank $n-k$ sub-vector bundle of $TM|_S$.

E.g.
Let $\mathbb{H} = \mathbb{H}^2_+$ and endow with the metric

$$g = \frac{dx \otimes dx + dy \otimes dy}{y^2} = \frac{1}{y^2} \iota^* \bar{g} .$$

This the Poincaré half-plane model of hyperbolic space.

Distance (M, g) Rie mfld, $\gamma : [a,b] \to M$ pw smooth curve.

The length of γ is $\quad L_g(\gamma) := \int_a^b |\gamma'(t)|_g \, dt$.

Exc (p. 338) Length is independent of reparametrization.

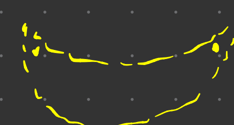
The <u>distance</u> b/w $p, q \in (M, g)$ is

$$d_g(p,q) := \inf_{\gamma : p \rightsquigarrow q} L_g(\gamma)$$

where the infimum is taken over all pw smooth paths p to q.

Thm For (M, g) a connected Riemannian mfld, (M, d_g) is a metric space.

Pf pp. 339–340. □



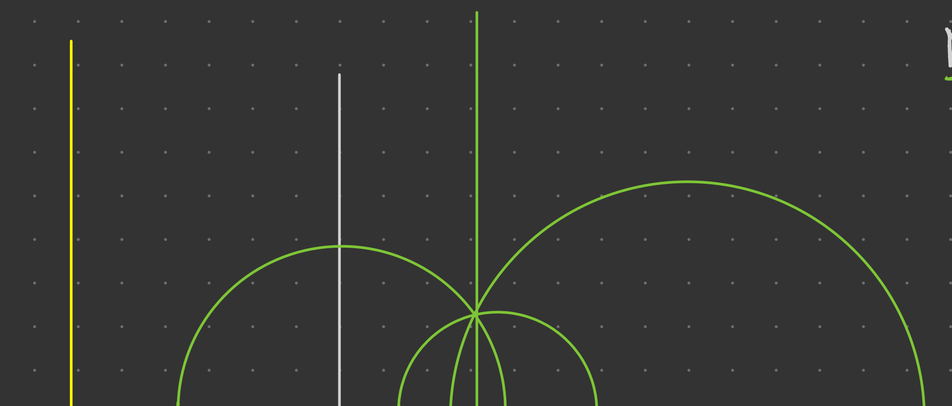
Cor Every smooth mfld w/ or w/o ∂ is metrizable.

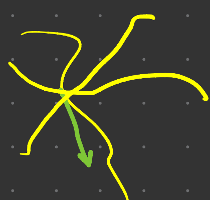
For $\mathbb{H}$,

$$d((x_1,y_1),(x_2,y_2)) = 2 \operatorname{arcsinh} \frac{\left((x_2-x_1)^2+(y_2-y_1)^2\right)^{1/2}}{2\sqrt{y_1 y_2}}.$$

Geodesics are curves that locally minimize distance. $p \in U$ s.t. $q \in U \cap \operatorname{im}(\gamma)$ then $d(p,q) = L_g(\gamma|_{..})$

In $\mathbb{H}$, these are semi-circles $\perp$ $\partial \mathbb{H}$.





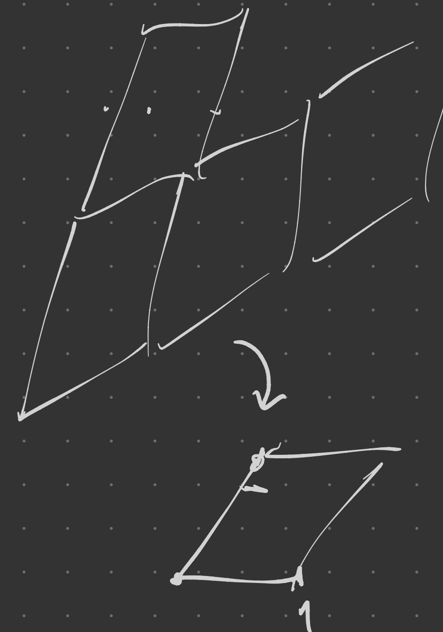
Note Parallel postulate fails in $\mathbb{H}$, but Euclid's other axioms hold.

$PSL_2\mathbb{R} \circlearrowright \mathbb{H}$ by isometries

$PSL_2\mathbb{R} = SL_2\mathbb{R}/\{\pm I\}$

$$\mathrm{Isom}(\mathbb{H}) = PSL_2\mathbb{R} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

$$\mathrm{Isom}(\mathbb{H}) = \{\varphi : \mathbb{H} \to \mathbb{H} \text{ isometry}\}$$



$$\{\text{lattices in } \mathbb{R}^2\} / \begin{matrix}\text{homothety} \\ \text{rotation}\end{matrix}$$

$$\cong \big\downarrow$$

$$\mathbb{H} / PSL_2\mathbb{Z}$$

$$\mathbb{Z}\{1, z\} = \mathbb{Z}\{1, Az\} \quad \forall A \in SL_2\mathbb{Z}$$

$$\{\text{lattices in } \mathbb{R}^2\} / \text{homothety} \cong UT(\mathbb{H}/PSL_2\mathbb{Z})$$