19.V.23

The de Rham Theorem

Lemma M a smooth n-mfld. Suppose $P(U)$ is a statement about open subsets of M, satisfying the following properties:

(1) $P(U)$ is true for $U \approx \mathbb{R}^n$

(2) $P(U), P(V), P(U \cap V) \implies P(U \cup V)$

(3) $\{U_\alpha\}$ disjoint, $P(U_\alpha)$ $\forall \alpha \implies P(\bigsqcup U_\alpha)$.

Then $P(M)$ is true.

pf Step 1 If $M \approx$ open subset of $\mathbb{R}^n$, then $P(M)$ is true.

WLOG, $M =$ open subset of $\mathbb{R}^n$. By (1) + (2) + induction, $P(U)$ is true for $U =$ union of finite # of convex open subsets b/c

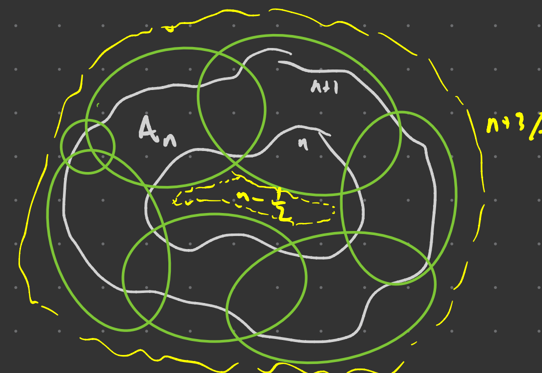
$$(U_1 \cup \cdots \cup U_n) \cap U_{n+1} = (U_1 \cap U_{n+1}) \cup \cdots \cup (U_n \cap U_{n+1}).$$

Let $f: M \longrightarrow [0, \infty)$ be an exhaustion. ($f^{-1}[0, c]$ compact for $c > 0$)

Set $A_n = f^{-1}[n, n+1]$, which is compact. Cover A_n by finitely many convex opens in $f^{-1}(n - \frac{1}{2}, n + \frac{3}{2})$ and let U_n be their union.

Then $A_n \subseteq U_n \subseteq f^{-1}(n - \frac{1}{2}, n + \frac{3}{2})$.

so the U_{even} are disjoint, as are the U_{odd}.



Since U_n = fin union convex opens,

$P(U_n)$ true $\forall n$. Set $U = \bigcup U_{2n}$, $V = \bigcup U_{2n+1}$. By (3), $P(U)$, $P(V)$ true. But $U \cap V = \left(\bigcup U_{2n}\right) \cap \left(\bigcup U_{2n+1}\right) = \coprod_{i,j} \underbrace{(U_{2i} \cap U_{2j+1})}_{\text{finite union convex opens}}$

Thus $P(U \cap V)$ also true. By (2), $P(M) = P(U \cup V)$ is true. ✓

Step 2 For the gen'l case, we may now substitute

(1') $P(U)$ is true for all $U \approx$ open subset of $\mathbb{R}^n$

for (1). Repeat the Step 1 proof replacing "convex open subset of $\mathbb{R}^n$" with "open subset of $\mathbb{R}^n$". $\square$

We are interested in the statement $P(U) =$ "$H^p_{dR}(U) \cong H^p(U;\mathbb{R})$".

But for what iso? For M a smooth mfld, $p \geq 0$, the de Rham homomorphism $\mathcal{I} : H^p_{dR}(M) \longrightarrow H^p(M;\mathbb{R})$ is given by

$$[\omega] \longmapsto \left([c] \in H_p(M) \cong H^\infty_p(M) \mapsto \int_{\tilde{c}} \omega \right)$$

$[\omega] \in Z^p(M) \subseteq \Omega^p(M)$

$\tilde{c}$ smooth p-cycle representing $[c]$

Here if $\sigma : \Delta_p \to M$ smooth, then $\int_\sigma \omega := \int_{\Delta_p} \sigma^* \omega$

domain of integration in $\mathbb{R}^p$.

and $\int_{\sum c_i \sigma_i} \omega := \sum c_i \int_\sigma \omega$.

<u>Stokes' Thm for Chains</u> If c is a smooth p-chain in a smooth mfld M and ω is a smooth $(p-1)$-form on M, then

$$\int_{\partial c} \omega = \int_c d\omega .$$

<u>Pf</u> Suffices to prove this for $c = \sigma : \Delta_p \to M$ smooth. By Stokes' (w/ corners),

$$\int_\sigma d\omega = \int_{\Delta_p} \sigma^* d\omega = \int_{\Delta_p} d\sigma^*\omega = \int_{\partial \Delta_p} \sigma^* \omega .$$

Now $$\int_{\partial\Delta_p} \sigma^* \omega = \sum_{i=0}^{p} (-1)^i \int_{\Delta^{p-1}} F_{i,p}^* \sigma^* \omega$$ [p. 481]

$$= \sum_{i=0}^{p} (-1)^i \int_{\Delta^{p-1}} (\sigma \circ F_{i,p})^* \omega$$

$$= \sum_{i=0}^{p} (-1)^i \int_{\sigma \circ F_{i,p}} \omega$$

$$= \int_{\partial \sigma} \omega$$

so we're done! $\square$

We can now check well-defn of the de Rham homomorphism

$$\mathcal{I} : H^p_{dR}(M) \longrightarrow H^p(M;\mathbb{R})$$

$$\begin{tikzcd} Z^p(M) \arrow[r] \arrow[d] & H^p(M;\mathbb{R}) \\ H^p_{dR}(M) \arrow[ur, dashed] & \end{tikzcd} \qquad [\omega] \longmapsto \left(\begin{array}{ccc} [c] & & H_p(M;\mathbb{R}) \\ \downarrow & & \downarrow \\ \int_{\tilde c} \omega & & \mathbb{R} \end{array} \right).$$

Indeed, if $\tilde c, \tilde c'$ smooth rep'g $[c]$, then $\tilde c - \tilde c' = \partial \tilde b$ for some smooth $(p+1)$-chain $\tilde b$. Thus

$$\int_{\tilde c} \omega - \int_{\tilde c'} \omega = \int_{\partial \tilde b} \omega = \int_{\tilde b} \cancelto{0}{d\omega} = 0.$$

Further, if $\omega = d\eta$ is exact, then $\int_{\tilde c} \omega = \int_{\tilde c} d\eta = \int_{\partial \tilde c} \eta = 0$ b/c $\partial \tilde c = 0$.

Set $P(U)$ = "$\mathscr{I}: H^p_{dR}(U) \xrightarrow{\cong} H^p(U;\mathbb{R})\ \forall p$". Suffices to show

(1) $P(U)$ is true for $U \approx \mathbb{R}^n$

(2) $P(U), P(V), P(U \cap V) \Longrightarrow P(U \cup V)$

(3) $\{U_\alpha\}$ disjoint, $P(U_\alpha)\ \forall \alpha \Longrightarrow P(\coprod U_\alpha)$.

(1) This is (essentially) the Poincaré lemma!

Both domain and codomain are $\mathbb{R}$ concentrated in deg 0.

If $\sigma: \Delta_0 \to M$ (smooth), f = constant fn 1 on M, then

$$\mathscr{I}[f][\sigma] = \int_{\Delta_0} \sigma^* f = (f \circ \sigma)(0) = 1$$

so $\mathscr{I}: H^0_{dR}(M) \xrightarrow{\cong} H^0(M;\mathbb{R})$. ✓

(2) Naturality of the chain level de Rham homomorphism + 5-lemma:

Also write $\mathcal{I}: \Omega^p(M) \longrightarrow C^p(M;\mathbb{R})$.

$$\omega \longmapsto \left(\begin{array}{c} c \\ \downarrow \\ \int_{\tilde{c}} \omega \end{array} \right) \qquad \begin{array}{c} C_p(M) \\ \downarrow \\ \mathbb{R} \end{array}$$

This induces a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \Omega^\bullet(U\cup V) & \longrightarrow & \Omega^\bullet(U)\oplus\Omega^\bullet(V) & \longrightarrow & \Omega^\bullet(U\cap V) & \longrightarrow & 0 \\ & & \downarrow \mathcal{I} & & \downarrow \mathcal{I}\oplus\mathcal{I} & & \downarrow \mathcal{I} & & \\ 0 & \to & C^\bullet(U\cup V;\mathbb{R}) & \longrightarrow & C^\bullet(U;\mathbb{R})\oplus C^\bullet(V;\mathbb{R}) & \longrightarrow & C^\bullet(U\cap V;\mathbb{R}) & \longrightarrow & 0 \end{array}$$

of chain complexes and hence a map of Mayer-Vietoris long exact sequences:

$$(\star)\quad \begin{tikzcd}[column sep=small]
\cdots \arrow[r] & H^p_{dR}(U\cup V) \arrow[r] \arrow[d, "\mathcal{J}"'] & H^p_{dR}(U)\oplus H^p_{dR}(V) \arrow[r] \arrow[d, "\mathcal{J}\oplus\mathcal{J}"'] & H^p_{dR}(U\cap V) \arrow[r] \arrow[d, "\mathcal{J}"'] & H^{p+1}_{dR}(U\cup V) \arrow[r] \arrow[d, "\mathcal{J}"'] & \cdots \\
\cdots \arrow[r] & H^p(U\cup V;\mathbb{R}) \arrow[r] & H^p(U;\mathbb{R})\oplus H^p(V;\mathbb{R}) \arrow[r] & H^p(U\cap V;\mathbb{R}) \arrow[r] & H^{p+1}(U\cap V;\mathbb{R}) \arrow[r] & \cdots
\end{tikzcd}$$

From homological algebra, we have the

Five lemma If

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d, "f_1"', "\cong"] & A_2 \arrow[r] \arrow[d, "f_2"', "\cong"] & A_3 \arrow[r] \arrow[d, "f_3"', "\cong"] & A_4 \arrow[r] \arrow[d, "f_4"', "\cong"] & A_5 \arrow[d, "f_5"', "\cong"] \\
B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5
\end{tikzcd}$$

is a commutative diagram of modules with exact rows and f_1, f_2, f_4, f_5 isos, then f_3 is an iso. (Prove it! $\square$)

By Five Lemma + ⊛, we see $P(U), P(V), P(U \cap V) \Rightarrow P(U \cup V)$. ✓

(3)

$$\begin{tikzcd} H^p_{dR}(\coprod U_j) \arrow[r, "(i_j^*)_j", "\cong"'] \arrow[d, "\mathcal{I}"'] & \prod H^p_{dR}(U_j) \arrow[d, "\mathcal{I}"] \\ H^p(\coprod U_j ; \mathbb{R}) \arrow[r, "\cong", "(i_j^*)_j"'] & \prod H^p(U_j ; \mathbb{R}) \end{tikzcd}$$

commutes so right $\mathcal{I}$ an iso $\Rightarrow$ left $\mathcal{I}$ an iso.

✓

$\square$

<u>Fact</u> $F : M \to N$ smooth then

$$\begin{tikzcd} H^p_{dR}(N) \arrow[r, "F^*"] \arrow[d, "\mathcal{I}"'] & H^p_{dR}(M) \arrow[d, "\mathcal{I}"] \\ H^p(N;\mathbb{R}) \arrow[r, "F^*"'] & H^p(M;\mathbb{R}) \end{tikzcd}$$

commutes.