17.V.23

The de Rham Theorem $H^p_{dR}(M) \cong H^p_{sing}(M;\mathbb{R})$

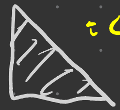
Recollections on singular homology

M a space.

$C_p(M) = \mathbb{Z}\{\underbrace{\sigma:\Delta_p \longrightarrow M}_{\text{singular } p\text{-simplex}} \text{ cts}\}$ = singular chain group of M in deg p

Δ_0 •

Δ_1 

Δ_2  $= \text{convex}\{0, e_1, e_2\}$

i-th face map in Δ_p:

$F_{i,p}:\Delta_{p-1} \longrightarrow \Delta_p$ includes as face opposite vertex i

$\Delta_p = \text{convex}\{e_0, e_1, \dots, e_p\}$

Boundary operator
$$\partial : C_p(M) \longrightarrow C_{p-1}(M) \qquad \text{(extended linearly).}$$
$$\sigma \longmapsto \sum_{i=0}^{p} (-1)^i \, \sigma \circ F_{i,p}$$

We have $\partial \circ \partial = 0$ so
$$C_\bullet(M) = \left(\cdots \xrightarrow{\partial} C_p(M) \xrightarrow{\partial} C_{p-1}(M) \xrightarrow{\partial} \cdots \right)$$
is a chain cpx, and the p-th singular homology group of M is
$$H_p(M) := Z_p(M) \big/ B_p(M) = \ker\left(\partial : C_p(M) \to C_{p-1}(M)\right) \big/ \mathrm{im}\left(\partial : C_{p+1}(M) \to C_p(M)\right).$$

Each cts $F : M \to N$ induces $F_* : C_\bullet(M) \longrightarrow C_\bullet(N)$ a chain map,
$$\sigma \longmapsto F \circ \sigma$$

$$\begin{tikzcd} \Delta_p \arrow[d, "\sigma"'] \arrow[dr, "F_*\sigma"] & \\ M \arrow[r, "F"'] & N \end{tikzcd}$$

so $H_p : \mathrm{Top} \longrightarrow \mathrm{Ab}$ is a functor.

$$\begin{tikzcd} M \arrow[d, "F"'] \arrow[r, mapsto] & H_p(M) \arrow[d, "F_*"] \\ N \arrow[r, mapsto] & H_p(N) \end{tikzcd}$$

Singular cohomology

Fix an Abelian group A and define

$$C^p(M;A) := \mathrm{Hom}_{\mathbb{Z}}(C_p(M), A)$$

singular p-cochains of M w/ coefficients in A

homomorphisms of (Abelian) groups

Then $d := \mathrm{Hom}(\partial, A) : C^p(M;A) \longrightarrow C^{p+1}(M;A)$ $\qquad \mathcal{C}(-,x)$

$$\mathrm{Hom}(\partial, A) = \partial^*$$

$$d(d\varphi) = d(\varphi\circ\partial) = \varphi\circ\cancel{\partial\circ\partial} = 0$$

$$\begin{tikzcd} C_p(M) \arrow[d, "\varphi"'] \\ A \end{tikzcd} \longmapsto \begin{tikzcd} C_{p+1}(M) \arrow[d, "d\varphi"'] \arrow[r, "\partial"] & C_p(M) \arrow[dl, "\varphi"] \\ A & \end{tikzcd}$$

satisfies $d\circ d = 0$ and makes $C^\bullet(M;A)$ a cochain complex.

The degree p singular cohomology of M with coefficients in A is $$H^p(M;A) := \frac{\ker\left(d : C^p(M;A) \to C^{p+1}(M;A)\right)}{\mathrm{im}\left(d : C^{p-1}(M;A) \to C^p(M;A)\right)}$$

For $f : M \to N$ cts, we get a chain map

$$C^\bullet(N;A) \xrightarrow{f^*} C^\bullet(M;A)$$

$$\begin{tikzcd}
C_p(N) \arrow[d, "\varphi"'] & \longmapsto & C_p(M) \arrow[d, "f^*\varphi"'] \arrow[r, "f_*"] & C_p(N) \arrow[dl, "\varphi"] \\
A & & A &
\end{tikzcd}$$

and thus $H^p(-;A) : \mathrm{Top}^{op} \longrightarrow \mathrm{Ab}$ is a functor.

$$\begin{tikzcd}
M \arrow[d, "f"'] \arrow[r, mapsto] & H^p(M;A) \\
N \arrow[r, mapsto] & H^p(N;A) \arrow[u, "f^*"']
\end{tikzcd}$$

By the universal coefficient theorem (homological algebra),

$\exists$ SES

$$0 \longrightarrow \underbrace{\mathrm{Ext}_{\mathbb{Z}}(H_{p-1}(M), A)}_{\substack{=0 \text{ for } A \text{ a field,} \\ \text{e.g. } A = \mathbb{R}}} \longrightarrow H^p(M; A) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(H_p(M); A) \to 0$$

So $H^p(M; \mathbb{R}) \cong \mathrm{Hom}_{\mathbb{Z}}(H_p(M), \mathbb{R}) \cong \mathbb{R}^{\text{free rk}(H_p(M))}$

Prop
- $H^*(pt; \mathbb{R}) \cong \mathbb{R}$ concentrated in degree 0
- $H^p(\coprod M_j; \mathbb{R}) \cong \prod_j H^p(M_j; \mathbb{R})$
- $H^p(\ ; \mathbb{R})$ is homotopy invariant.

Mayer-Vietoris for Cohomology For $U, V \subseteq M$ open with $U \cup V = M$, there is a LES

$$\cdots \xrightarrow{\partial^*} H^p(M;\mathbb{R}) \xrightarrow{k^* \oplus l^*} H^p(U;\mathbb{R}) \oplus H^p(V;\mathbb{R}) \xrightarrow{i^* - j^*} H^p(U \cap V;\mathbb{R}) \xrightarrow{\partial^*} H^{p+1}(M;\mathbb{R}) \to \cdots$$

$$\begin{tikzcd} U \cap V \arrow[r, "i"] \arrow[d, "j"'] \arrow[dr, phantom, "\square"] & U \arrow[d, "k"] \\ V \arrow[r, "l"'] & M \end{tikzcd}$$

Pf Mayer-Vietoris for singular homology +
Hom$(-,\mathbb{R})$ is **exact**: takes exact sequences to exact sequences $\square$

Why cohomology? $H^*(M;A) := \bigoplus_{p \geq 0} H^p(M;A)$ carries a product making it a graded ring! Two spaces with isomorphic

when A is a comm ring

homotopy groups might not have isomorphic cohomology rings, so cohomology is a more refined invariant.

<u>E.g.</u> $S^2 \vee S^4$ and $\mathbb{C}P^2$ have isomorphic (co)homology groups but nonisomorphic cohomology rings.

$H^* \cong \mathbb{Z} \oplus 0 \oplus \mathbb{Z} \oplus 0 \oplus \mathbb{Z} \oplus 0 \oplus 0 \oplus 0 \oplus 0$

- $H^*(S^2 \vee S^4; \mathbb{Z}) \cong \mathbb{Z}[x_2, x_4]/(x_2^2, x_4^2, x_2 x_4)$
- $H^*(\mathbb{C}P^2; \mathbb{Z}) \cong \mathbb{Z}[x_2]/(x_2^3)$

Ring structure on $H^*(M; A)$ is given by the <u>cup product</u>:

$$\smile : H^p(M; A) \times H^q(M; A) \longrightarrow H^{p+q}(M; A)$$

\smile

defined on cochains
$$
\begin{tikzcd}
C_p(M) \arrow[d, "\alpha"'] \\ A
\end{tikzcd}
,\quad
\begin{tikzcd}
C_q(M) \arrow[d, "\beta"'] \\ A
\end{tikzcd}
$$
by

$$
\begin{tikzcd}
C_{p+q}(M) \arrow[d, "\alpha \cup \beta"'] & \sigma : \Delta_{p+q} \to M \arrow[d, mapsto] \\
A & \alpha\big(\underbrace{\sigma \circ i_{0,1,\dots,p}}_{\sigma|_{\text{"p-th front face"}}}\big) \cdot \beta\big(\underbrace{\sigma \circ i_{p,p+1,\dots,p+q}}_{\sigma|_{\text{"q-th back face"}}}\big)
\end{tikzcd}
$$

<u>Fact</u> $d(\alpha \cup \beta) = d\alpha \cup \beta + (-1)^p \alpha \cup d\beta$ and this is what is needed for $\cup$ to descend to cohomology.

<u>Fact</u> $\cup$ is graded-commutative on H^* $(\alpha \cup \beta = (-1)^{pq} \beta \cup \alpha)$

but this is only true up to cochain homotopy

on $C^\bullet$ $\rightsquigarrow$ Steenrod squares!

How can we relate de Rham and singular (w/ $\mathbb{R}$ coeffs) cohomology?

$$\begin{tikzcd}
\Omega^\bullet \arrow[r] \arrow[d, squiggly] & C^\bullet_\infty(; \mathbb{R}) \arrow[d, squiggly] & C^\bullet(; \mathbb{R}) \arrow[l] \arrow[d, squiggly] \\
H^*_{dR} \arrow[r, "\cong"'] & H^*_\infty(; \mathbb{R}) & H^*(; \mathbb{R}) \arrow[l, "\cong"]
\end{tikzcd}$$

(smooth singular cohomology

Smooth singular homology

A smooth p-simplex is a smooth map $\sigma : \Delta_p \to M$, M a smooth mfld.

$C_p^\infty(M) = \mathbb{Z}\{\sigma : \Delta_p \to M \text{ smooth}\}$

$\partial(\text{smooth}) = \text{smooth}$ so get a sub-chain cpx $C_\bullet^\infty(M) \subseteq C_\bullet(M)$ with homology groups the smooth singular homology of M

$$H_*^\infty(M).$$

Thm $C_\bullet^\infty(M) \subseteq C_\bullet(M)$ induces $H_*^\infty(M) \cong H_*(M)$.

Pf Technical. pp. 474-480 Main idea: Whitney approximation. $\square$ ✓

Cauchy - Riemann

$f : \mathbb{C} \longrightarrow \mathbb{C}$ holomorphic : cpx diff'l

$$f'(z) := \underbrace{\lim_{h \to 0}}_{\text{in } \mathbb{C}} \frac{f(z+h) - f(z)}{h} \quad \text{exists } \forall z \in \mathbb{C}$$

$$f(z) = u(z) + i v(z), \qquad u, v : \mathbb{C} \longrightarrow \mathbb{R}$$

$$= u(x,y) + i v(x,y) \qquad x := \mathrm{Re}(z), \; y := \mathrm{Im}(z).$$

<u>Claim</u> Considered as a function $(u,v) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

nice things happen!

$f' : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is "$\mathbb{C}$-linear"

$$\mathbb{C} \cong \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in R^{2\times 2} \;\middle|\; a, b \in \mathbb{R} \right\}$$

$$f' = Jf = \begin{pmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{pmatrix}$$

$$\Longrightarrow \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

f holo iff
C-R eq'ns hold +
$u, v \in C^1$