15. V. 23

Overview of compactly supported cohomology

pp. 452-457

Poincaré lemma with compact support

For $n \geq p \geq 1$, $\omega \in \Omega^p_c(\mathbb{R}^n) \cap Z^p(\mathbb{R}^n)$

[and if $p=n$ also suppose $\int_{\mathbb{R}^n} \omega = 0$],

$\exists\, \eta \in \Omega^{p-1}_c(\mathbb{R}^n)$ s.t. $d\eta = \omega$.

it's the compactly supported part that's new!

Play the same game but with $\Omega^n_c(M)$.

Gain well-defined integration functions

$Z^p = \ker d$

$B^p = \operatorname{im} d$

(Proof uses computation of $H^*_{dR}(\mathbb{R}^n \setminus pt)$)

Now use the cochain cpx of compactly supported diff'l forms $\Omega^\bullet_c(M)$ to define $H^*_c(M)$, the compactly supported de Rham cohomology of M:

$$H^p_c(M) = \ker\left(d : \Omega^p_c(M) \to \Omega^{p+1}_c(M)\right) \Big/ \operatorname{im}\left(d : \Omega^{p-1}_c(M) \to \Omega^p_c(M)\right).$$

Thm For $n \geq 1$, $H^p_c(\mathbb{R}^n) \cong \begin{cases} \mathbb{R} & \text{if } p = n \\ 0 & \text{o/w} \end{cases}$. (Lost the $\mathbb{R}$ in deg 0.)

Pf By compactly supported Poincaré lemma.

H^p_c is not functorial wrt all smooth maps; restrict to proper $F : M \to N$ to get $H^p_c(N) \xrightarrow{F^*} H^p_c(M)$.

Vaguity Compare with Grothendieck's six functor formalism.

For M oriented smooth n-mfld, get linear map

$$\int_M : \Omega^n_c(M) \longrightarrow \mathbb{R}.$$

If $\partial M = \emptyset$, then by Stokes' Thm, $\int_M B^p_c(M) = 0$

(i.e. $\int_M d\eta = \int_{\partial M} \eta = 0$) so $\int_M$ descends to $H^n_c(M)$:

$$\int_M : H^n_c(M) \longrightarrow \mathbb{R}$$

$$[\omega] \longmapsto \int_M \omega.$$

Thm If M is a conn'd or'd smooth n-mfld, then $\int_M : H^n_c(M) \cong \mathbb{R}$.

Key lemma If $\omega \in \Omega^n_c(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \omega = 0$, then $\omega = d\eta$ for some $\eta \in \Omega^{n-1}_c(\mathbb{R}^n)$. [Really Poincaré again!]

Pf for $n=2$ Have $\omega = f\, dx\wedge dy$. Define $g(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$.

By Fubini + $\int_{\mathbb{R}^2} \omega = 0$, we know $\int_{-\infty}^{\infty} g(x)\, dx = 0$. Define $G(x,y) = \varepsilon(y) g(x)$

for $\varepsilon(y)$ a bump fn w/ total area 1. Then set

$$\eta(x,y) = -\left(\int_{-\infty}^{y} [f(x,t) - G(x,t)]\, dt\right) dx$$

$$+ \left(\int_{-\infty}^{y} G(t,y)\, dt\right) dy \in \Omega^1_c(\mathbb{R}^2).$$

We get $d\eta = [f(x,y) - G(x,y)]\, dx\wedge dy + G(x,y)\, dx\wedge dy$

$= \omega \quad \square$

Pf Thm Must show $\int_M \omega = 0 \Rightarrow \omega = d\eta$. Take $\{U_i\}$ a finite open cover of supp ω with each $U_i \approx \mathbb{R}^n$. Take $\{f_i\}$ smooth POU subordinate to $\{U_i\}$ so $\int_M \omega = \sum_i \int_{U_i} f_i \omega$.

By Key Lemma, $[f_i \omega]_{U_i} = [\omega_i]_{U_i}$ where ω_i is supported on a small nbhd of a point $x_i \in M$ — i.e. ω_i is a bump n-form.

Take $U \approx \mathbb{R}^n$ and containing all the x_i $\Rightarrow$ $\sum \omega_i$ is compactly supported in U with $0 = \int_M \omega = \int_M \sum \omega_i = \int_{\underset{\mathbb{R}^n}{\overset{U}{\shortparallel}}} \sum \omega_i$

so $\sum \omega_i = d\eta$ for some $\eta \in \Omega_c^{n-1}(\mathbb{R}^n)$,

$f_i \omega = \omega_i + d\eta_i$ so $\omega = \sum f_i \omega = \sum \omega_i + d\eta_i = d\eta + \sum d\eta_i = d(\eta + \sum \eta_i)$.

Thm Suppose M is a conn'd n-mfld.

- If M is compact & orientable, then $H^n_{dR}(M) \cong \mathbb{R}$. ✓
- If M is noncompact & orientable, then $H^n_{dR}(M) = 0$
- If M is nonorientable, then $H^n_c(M) = H^n_{dR}(M) = 0$.

455-457.

Degree Theory

Suppose M, N compact conn'd or'd smooth n-mflds (same n!).

Then a smooth map $F: M \longrightarrow N$ induces

$$\begin{tikzcd} H^n_{dR}(N) \arrow[r, "F^*"] \arrow[d, "\int_N"', "\cong"] & H^n_{dR}(M) \arrow[d, "\int_M", "\cong"'] \\ \mathbb{R} \arrow[r, "k"'] & \mathbb{R} \end{tikzcd}$$

where $k = \int_M \circ F^* \circ \int_N^{-1}$ is multiplication by some real number k, i.e.

$$\int_M F^*\omega = k \int_N \omega \quad \forall \omega \in Z^n(M).$$

Thm The constant $k = k_F$ is an integer, and if $q \in N$ is a regular value of F, then $k = \sum_{x \in F^{-1}\{q\}} \operatorname{sgn}(x)$ where

$$\operatorname{sgn}(x) = \begin{cases} +1 & \text{if } dF_x \text{ is or'n preserving} \\ -1 & \text{if } dF_x \text{ is or'n reversing.} \end{cases}$$

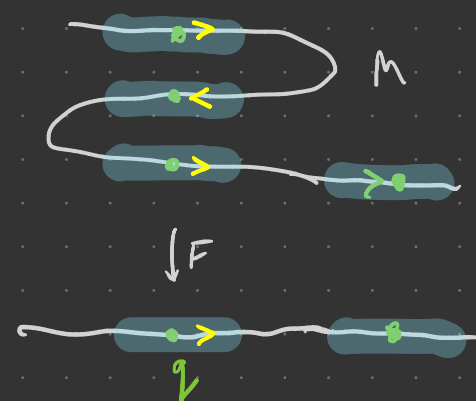
Defn Call $k = k_F =: \deg(F)$ the degree of F.

Pf It suffices to show $k = \sum\limits_{x \in F^{-1}\{q\}} \operatorname{sgn}(x)$. Take $q \in N$ a regular value of F. Then $F^{-1}\{q\}$ is finite. Suppose $F^{-1}\{q\} = \{x_1, \dots, x_n\} \neq \emptyset$.

By inverse function theorem, $\forall i \ \exists$ open $U_i \ni x_i$ s.t. $F : U_i \approx W_i \ni q$.

Shrinking the U_i if necessary, we may assume they are distinct.

Add'l point-set massaging: arrange for $q \in W \subseteq N$ open with $F^{-1} W = \coprod\limits_{i=1}^{m} V_i$ with $x_i \in V_i \subseteq M$ open, $F : V_i \approx W$.



Note that F is either or'n pres or rev on each V_i.

Let $\omega \in \Omega^n_c(W)$ with $\int_N \omega = \int_W \omega = 1$, so that $\int_M F^*\omega = k$.

We have $k = \int_M F^*\omega = \sum_{i=1}^m \underbrace{\int_{V_i} F^*\omega}_{= \pm \int_W \omega = \operatorname{sgn}(x_i)} = \sum_{i=1}^m \operatorname{sgn}(x_i)$.

Now suppose $F^{-1}\{q\} = \emptyset$. Take $\exists W \subseteq N \setminus F(M)$ open nbhd of q.

If $\omega \in \Omega^n_c(W)$, then $\int_M F^*\omega = 0$, so $k = 0 = \sum_{\emptyset} \operatorname{sgn}(x_i)$. $\square$

<u>Prop</u> M, N, P compact conn'd or'd smooth n-mflds, $M \xrightarrow{F} N \xrightarrow{G} P$ smooth

(a) $\deg(G \circ F) = \deg(G)\deg(F)$

(b) If F is a diffeo, then $\deg(F) = \pm 1$ (or'n pres vs rev).

(c) If $F_0 \simeq F_1 : M \to N$, then $\deg F_0 = \deg F_1$.

Pf (c)

$$\begin{tikzcd}
H^n_{dR}(N) \arrow[r, bend left, "F_0^*"] \arrow[r, bend right, "F_1^*"'] \arrow[r, phantom, "\|"] \arrow[d, "\int_N"'] & H^n_{dR}(M) \arrow[d, "\int_M"] \\
\mathbb{R} \arrow[r, bend left, "\deg F_0"] \arrow[r, bend right, "\deg F_1"'] \arrow[r, phantom, "\|"] & \mathbb{R}
\end{tikzcd}$$

$\square$

Recall that by Whitney approx'n, every cts $F: M \to N$ is htpic to a smooth map $M \to N$.

This allows us to define $\deg(F) := \deg(\text{any smooth map htpic to } F)$.

Fact $\deg : \pi_n(S^n) \xrightarrow{\cong} \mathbb{Z}$ for $n \geq 1$.

└ based htpy classes of pointed maps $S^n \longrightarrow S^n$

(Milnor, Topology from the differentiable viewpoint)

Digression Degree Theory in Motivic Homotopy

Fix a base field k. For an algebraic function

$$f : \mathbb{P}^1_k \longrightarrow \mathbb{P}^1_k$$

(i.e. rational function $f \in k(z)$) and p a regular value of f and k-point of $\mathbb{P}^1_k$, define

$$\deg^{\mathbb{A}^1}(f) := \sum_{q \in f^{-1}\{p\}} \langle \det Jf(q) \rangle \in GW(k).$$

" $\underline{\operatorname{sgn}(q)}$ "

Here $GW(k) := \left(\text{regular symm bilin forms}/k, \oplus, \otimes\right)^{gp}$

$$\langle a \rangle : k \times k \longrightarrow k$$
$$(x,y) \longmapsto axy.$$

$$\frac{V \otimes_k V \longrightarrow k}{V \xrightarrow{\cong} V^*}$$

replaces $\operatorname{sgn}(q) = \begin{cases} +1 & \det Jf(q) > 0 \\ -1 & \det Jf(q) < 0 \end{cases}$ for $k = \mathbb{R}$.

$GW(\mathbb{C}) = \mathbb{Z}$

$GW(\mathbb{R}) = \mathbb{Z} \oplus \mathbb{Z}$ (dim, sgn)

$= \mathbb{Z}[h]/(h^2 - 2h)$

Morel proves $\deg^{\mathbb{A}^1}$ is an $\mathbb{A}^1$-homotopy invariant and induces an iso

$$[\mathbb{P}^n/\mathbb{P}^{n-1}, \mathbb{P}^n/\mathbb{P}^{n-1}] \longrightarrow GW(k) \text{ for } n \geq 2.$$

- Kirsten Wickelgren
- Marc Levine