12.V.23

$$\boxed{\mathcal{L}_V du = d(\mathcal{L}_V u)}$$

IOU

Pf For $X \in \mathfrak{X}(M)$, $f \in C^\infty(M)$, direct calculation with limits gives

$$(\mathcal{L}_V(df))(X) = \lim_{t\to 0} \frac{1}{t}\left(((\theta_t^*(df))(X) - df(X)\right)$$

$$= \frac{d}{dt}\Big|_{t=0} X(f\circ\theta_t)$$

$$= XVf.$$

Meanwhile, $\mathcal{L}_V f = Vf$ so

$$(d(\mathcal{L}_V f))(X) = d(Vf)(X) = XVf$$

so these are equal. □

Upshot This completes the proof of Cartan's magic fmla,

$$\mathcal{L}_V \omega = V \lrcorner (d\omega) + d(V \lrcorner \omega),$$

i.e. if $i_V = V\lrcorner(\ )$, then

$$\boxed{\mathcal{L}_V = d\circ i_V + i_V \circ d}.$$

Homotopy invariance

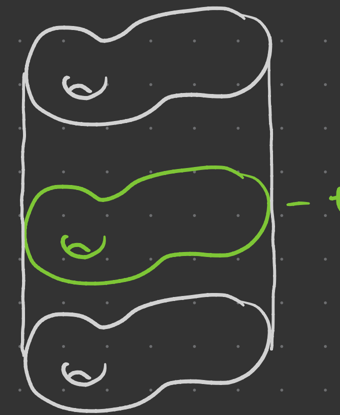
Given $F, G : M \to N$ smooth maps, a collection of linear maps $h : \Omega^p(N) \longrightarrow \Omega^{p-1}(M)$ s.t. $d(h\omega) + h(d\omega) = G^*\omega - F^*\omega \quad \forall \omega$ is called a cochain homotopy between F^* and G^*.

Prop If $\exists$ cochain htpy b/w F^* and G^*, then $F^* = G^* : H^p_{dR}(N) \to H^p_{dR}(M)$ for all p.

Pf If $\omega \in Z^p(M)$, then $G^*\omega - F^*\omega = d(h\omega) + h(\cancel{d\omega})$ $\stackrel{0}{}$

$$\Longrightarrow [G^*\omega] = [F^*\omega]. \quad \square$$

For $t \in [0,1]$, let $i_t : M \longrightarrow M \times [0,1]$
$x \longmapsto (x, t)$.



<u>Lemma</u> $\exists$ cochain homotopy between

$$i_0^*, i_1^* : \Omega^\bullet(\underbrace{M \times [0,1]}_{N}) \longrightarrow \Omega^\bullet(M).$$

<u>Pf</u> Let S be the vector field on $M \times \mathbb{R}_s$ given by

$S_{(q,s)} = (0, \frac{\partial}{\partial s}\big|_s)$. For $\omega \in \Omega^p(M \times I)$, define $h\omega \in \Omega^{p-1}(M)$

by $$h\omega = \int_0^1 i_t^*(S \lrcorner\, \omega)\, dt,$$

i.e. $$(h\omega)_q = \int_0^1 i_t^*\left((S \lrcorner\, \omega)_{(q,t)}\right) dt.$$

function of t with values in $\Lambda^{p-1} T_q^* M$

elt of $\Lambda^{p-1} T_q^* M$

May differentiate under the integral sign in local coords, so

$$d(h\omega) = \int_0^1 d(i_t^*(S \lrcorner \omega))\, dt.$$

Thus

$$h(d\omega) + d(h\omega) = \int_0^1 \left(i_t^*(S \lrcorner d\omega) + d(i_t^*(S \lrcorner \omega)) \right) dt$$

$$= \int_0^1 \left(i_t^*(S \lrcorner d\omega) + i_t^* d(S \lrcorner \omega) \right) dt$$

$$= \int_0^1 i_t^* (\mathcal{L}_S \omega)\, dt \qquad \text{(magic)}.$$

Flow of S on $M \times \mathbb{R}$ is $\theta_t(q,s) = (q, t+s)$ so S is complete.

Have $i_t = \theta_t \circ i_0$, so

$$i_t^*(\mathcal{L}_S \omega) = i_0^*(\theta_t^*(\mathcal{L}_S \omega))$$

$$= i_0^*\left(\frac{d}{dt}(\theta_t^* \omega)\right) \qquad \text{Prop 12.36}$$

$$= \frac{d}{dt} i_0^*(\theta_t^* \omega)$$

$$= \frac{d}{dt} i_t^* \omega.$$

By FTC, $h(d\omega) + d(h\omega) = i_1^*\omega - i_0^*\omega$, as desired. $\square$

Prop Homotopic smooth maps induce the same maps on H^p_{dR}.

Pf If $F \simeq G : M \to N$ then $\exists$ smooth htpy $H : M \times I \to N$ with $F = H \circ i_0$, $G = H \circ i_1$. Thus

$$F^* = i_0^* \circ H^* \underset{\text{lemma}}{=} i_1^* \circ H^* = G^* . \qquad \square$$

Thm (Homotopy invariance of de Rham cohomology)

If M, N are htpy equivalent smooth mflds w/ or w/o ∂, then $H^*_{dR}(M) \cong H^*_{dR}(N)$ (with iso induced by any smooth htpy equiv $M \to N$).

Pf Whitney approx'n + above proposition. $\square$

Cor de Rham cohom factors

$$\begin{tikzcd} \mathrm{Diff} \arrow[r, "H^*_{dR}"] \arrow[d] & \mathrm{Vect}_{\mathbb{R}} \\ \mathrm{Top} \arrow[ur] \arrow[d] & \\ \mathrm{Hot} \arrow[uur] & \end{tikzcd}$$

In particular, H^*_{dR} is a smooth, topological, and homotopy invariant.

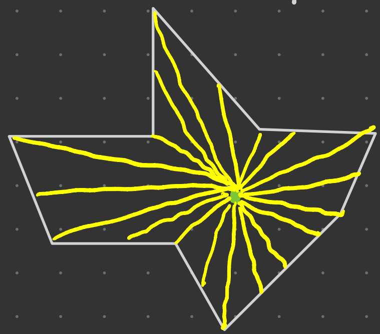
Note H^*_{dR} cannot distinguish smooth structures on the same underlying top'l manifold.

Computations via htpy invariance

Thm If M is a contractible smooth mfld w/ or w/o ∂, then

$H^*_{dR}(M) = \mathbb{R}$ concentrated in degree 0. $\square$

Note • This proves the Poincaré lemma for star-shaped open subsets of $\mathbb{R}^n$.



• Every closed form is locally exact!

H^1_{dR} & π_1

Define $\int : H^1_{dR}(M) \times \pi_1(M,q) \longrightarrow \mathbb{R}$

$$([\omega],[\gamma]) \longmapsto \int_{\tilde{\gamma}} \omega$$

where $\tilde{\gamma}$ is a pw smooth curve representing the path class of γ. Then define

$\mathbb{R}$-vs by pointwise $+, \cdot$ in codomain

$$\Phi : H^1_{dR}(M) \longrightarrow Gp(\pi_1(M,q), \mathbb{R})$$

$$[\omega] \longmapsto \left([\gamma] \longmapsto \int_{\tilde{\gamma}} \omega \right).$$

$[\gamma] = [\gamma']$

$\gamma \cdot \gamma'^{(-)} \simeq *$

$\int_{\tilde{\gamma} \simeq *} \omega = 0$

Thm For M conn'd smooth, $q \in M$,

Φ is a well-defined injective linear map.

Later, we will see that Φ is an isomorphism.

Pf Sketch (p 448) Well defined since $[\omega] = [\omega']$

$$\Rightarrow \omega - \omega' = df \Rightarrow \int_{\tilde{\gamma}} \omega - \int_{\tilde{\gamma}} \omega' = \int_{\tilde{\gamma}} df = f(q) - f(q) = 0.$$

For injectivity, check $\Phi[\omega] = 0 \rightarrow \omega$ is conservative. $\square$

Mayer-Vietoris

$U, V \subseteq M$ open

$U \cup V = M$

$$\begin{tikzcd} U \cap V \arrow[r, hook, "i"] \arrow[d, hook, "j"'] \arrow[dr, phantom, "\square"] & U \arrow[d, hook, "k"] \\ V \arrow[r, hook, "l"'] & M \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \Omega^\bullet(M) \arrow[r, "k^*"] \arrow[d, "l^*"'] & \Omega^\bullet(U) \arrow[d, "i^*"] \\ \Omega^\bullet(V) \arrow[r, "j^*"'] & \Omega^\bullet(U \cap V) \end{tikzcd}$$

$\rightsquigarrow$ SES of chain complexes

$$0 \longrightarrow \Omega^\bullet(M) \xrightarrow{k^* \oplus l^*} \Omega^\bullet(U) \oplus \Omega^\bullet(V) \xrightarrow{i^* - j^*} \Omega^\bullet(U \cap V) \longrightarrow 0.$$

- For $n \geq 2$, $H^p_{dR}(\mathbb{R}^n \setminus pt) \cong \begin{cases} \mathbb{R} & \text{if } p = 0 \text{ or } n-1 \\ 0 & \text{o/w}. \end{cases}$

The only non-formal part is checking $\dim H^1_{dR}(S^1) = 1$.

Know ≥ 1 since $\int_{S^1} \omega \neq 0$ for any orientation form.

Since $\mathrm{Hom}(\pi_1(S^1), \mathbb{R}] = \mathbb{R} \hookleftarrow H^1_{dR}(S^1)$, get dim 1.

In gen'l, $H^n_{dR}(S^n)$ has basis the cohom class of any smooth or'n form for S^n.