# de Rham cohomology

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**Recall** Exterior derivative

$$d : \Omega^p(M) \longrightarrow \Omega^{p+1}(M)$$

satisfies $d \circ d = 0$.

Thus every exact form ($\omega = d\eta$ for some $\eta$) is closed ($d\omega = 0$).

But closed forms are not necessarily exact:

$$\omega = \frac{x\,dy - y\,dx}{x^2 + y^2} \in \Omega^1(\mathbb{R}^2 \setminus 0).$$

de Rham cohomology measures the extent to which closed forms can fail to be exact.

**Defn** For $M$ a smooth mfld w/ or w/o $\partial$ and $p \in \mathbb{N}$,

set
$$Z^p(M) := \ker\left(d : \Omega^p(M) \longrightarrow \Omega^{p+1}(M)\right) = \{\text{closed } p\text{-forms on } M\}$$
$$\cup\!\!\shortmid$$
$$B^p(M) = \operatorname{im}\left(d : \Omega^{p-1}(M) \to \Omega^p(M)\right) = \{\text{exact } p\text{-forms on } M\}$$

where $\Omega^p(M) = 0$ for $p<0$ or $p>n=\dim M$.

The de Rham cohomology of $M$ in degree $p$ is the $\mathbb{R}$-vs

$$H^p_{dR}(M) := Z^p(M) / B^p(M).$$

I.e. we have a cochain complex

$$\Omega^\bullet(M) = \left(\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \longrightarrow \cdots\right)$$

and $H^p_{dR}(M) = H^p(\Omega^\bullet(M))$.

Note
- $H^p_{dR}(M)$ is concentrated in degrees $0 \le p \le \dim M$,

  i.e. $H^p_{dR}(M) = 0$ outside this range.

- $H^1_{dR}(\mathbb{R}^2 \setminus 0) \neq 0$.

We will show
- $H^p_{dR}$ is a functor $\Longrightarrow$ diffeo invt.
- In fact, $H^p_{dR}$ is homotopy invariant.
- $H^p_{dR}$ satisfies Mayer-Vietoris
- de Rham Thm: $H^p_{dR}(M) \cong \underbrace{H^p_{sing}(M; \mathbb{R})}_{\text{singular cohomology with coefficients in } \mathbb{R}}$.

**Notation** For $\omega \in Z^p(M)$, let $[\omega] := \omega + B^p(M) \in H^p_{dR}(M)$.

When $[\omega] = [\omega']$, call $\omega, \omega'$ **cohomologous**.

**Functoriality**

$F : M \longrightarrow N$ smooth induces a map of chain complexes

$F^* : \Omega^\bullet(N) \longrightarrow \Omega^\bullet(M)$ :

$$
\begin{tikzcd}
\Omega^0(N) \arrow[r, "d"] \arrow[d, "F^*"'] & \Omega^1(N) \arrow[r, "d"] \arrow[d, "F^*"'] & \Omega^2(N) \arrow[r, "d"] \arrow[d, "F^*"'] & \cdots \\
\Omega^0(M) \arrow[r, "d"'] & \Omega^1(N) \arrow[r, "d"'] & \Omega^2(N) \arrow[r, "d"'] & \cdots
\end{tikzcd}
$$

and thus

$$\begin{tikzcd}
\Omega^p(M) & \Omega^p(N) \arrow[l, "F^*"'] \\
Z^p(M) \arrow[u, phantom, "\cup" sloped] & Z^p(N) \arrow[l, dashed] \arrow[u, phantom, "\cup" sloped] \\
B^p(M) \arrow[u, phantom, "\cup" sloped] & B^p(N) \arrow[l, dashed] \arrow[u, phantom, "\cup" sloped]
\end{tikzcd}$$

induces a well-defined linear map

$$F^* : H^p_{dR}(N) \longrightarrow H^p_{dR}(M)$$
$$[\omega] \longmapsto [F^*\omega] .$$

Furthermore, if $M \xrightarrow{F} N \xrightarrow{G} P$ then $(G \circ F)^* = F^* \circ G^*$ and

$\mathrm{id}_M^* = \mathrm{id}_{H^p_{dR}(M)}$ , so $H^p_{dR}$ is a functor

$$\mathrm{Diff}^{op} \longrightarrow \mathrm{Vect}_{\mathbb{R}}$$

$$\begin{tikzcd}
M \arrow[r, mapsto] \arrow[d, "F"'] & H^p_{dR}(M) \\
{} \arrow[r, mapsto] & {} \\
N \arrow[r, mapsto] & H^p_{dR}(N) \arrow[uu, "F^*"']
\end{tikzcd}$$

Prop If $M = \coprod M_j$ for $\{M_j\}$ a countable collection of smooth mflds w/ or w/o $\partial$, then the inclusion maps $\iota_j : M_j \hookrightarrow M$ induce

$$H^p_{dR}(M) \xrightarrow[\cong]{(\iota_j^*)} \prod H^p_{dR}(M_j).$$

Pf In fact, $\Omega^\bullet(M) \xrightarrow{(\iota_j^*)} \prod \Omega^\bullet(M_j)$ is already an iso. $\square$

$$\omega \longmapsto (\omega|_{M_j})$$

Prop $H^0_{dR}(M) = \ker\left(d : \underset{\substack{C^\infty(M) \\ \cong}}{\Omega^0(M)} \longrightarrow \Omega^1(M)\right)$ is the $\mathbb{R}$-vs of locally constant functions on $M$, so

$$H^0_{dR}(M) \cong \mathbb{R}^{|\pi_0(M)|}. \quad \square$$

Prop $H^*_{dR}(\coprod_n pt) \cong \mathbb{R}^n$ concentrated in degree 0 (n countable). $\square$

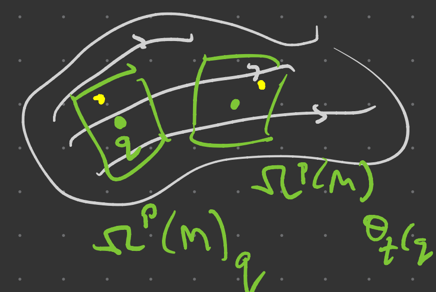
# Cartan's Magic Formula

pp. 372-373

To prove homotopy invariance, we'll need a fact about Lie derivatives that we skipped.

Take $V \in \mathfrak{X}(M)$ generating flow $\theta$. For $\omega \in \Omega^p(M)$, the Lie derivative of $\omega$ wrt $V$ is $\mathcal{L}_V \omega \in \Omega^p(M)$ given by

$$(\mathcal{L}_V \omega)_q = \frac{d}{dt}\Big|_{t=0} (\theta_t^* \omega)_q .$$




**Prop** $\mathcal{L}_V(\omega \wedge \eta) = (\mathcal{L}_V \omega) \wedge \eta + \omega \wedge (\mathcal{L}_V \eta)$

**Pf** Exercise. $\square$

**Thm** (Cartan's magic formula) $\forall V \in \mathfrak{X}(M)$, $\omega \in \Omega^p(M)$,

$$\mathcal{L}_V \omega = V \lrcorner (d\omega) + d(V \lrcorner \omega).$$

**Recall** $V \lrcorner \omega \in \Omega^{p-1}(M)$ is determined by

$$(V_1, \ldots, V_{p-1}) \longmapsto \omega(V, V_1, \ldots, V_{p-1}).$$

**Pf of Thm** Proceed by induction on $p$. If $f \in \Omega^0(M) = C^\infty(M)$, then

$$V \lrcorner (df) + d(\underbrace{V \lrcorner f}_{\in\, \Omega^{-1}(M) = 0}) = V \lrcorner df = df(V) = Vf = \mathcal{L}_V f. \checkmark$$

Now let $p \geq 1$ and assume the magic fmla holds for forms of degree $< p$. For $\omega \in \Omega^p(M)$, we have

$$\omega = \sum \omega_I \, dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

in local coordinates. Each term is of the form $du \wedge \beta$ for $u = x^{i_1}$, $\beta = \omega_I \, dx^{i_2} \wedge \cdots \wedge dx^{i_k}$. We have $\boxed{\mathcal{L}_V du = d(\mathcal{L}_V u)}$ IOU $= d(Vu)$, so

$$\mathcal{L}_V(du \wedge \beta) = (\mathcal{L}_V du \wedge \beta) + du \wedge (\mathcal{L}_V \beta)$$

$$= d(Vu) \wedge \beta + du \wedge \big(\underbrace{V \lrcorner d\beta + d(V \lrcorner \beta)}_{\text{induction hypothesis}}\big)$$

Meanwhile,

$$V \lrcorner d(du \wedge \beta) + d(V \lrcorner (du \wedge \beta)) \quad \text{Leibniz for } \lrcorner$$

$$= V \lrcorner (-du \wedge d\beta) + d\big((Vu)\beta - du \wedge (V \lrcorner \beta)\big)$$

$$= \cancel{-(Vu)\,d\beta} + du \wedge (V \lrcorner\, d\beta) + \underline{d(Vu) \wedge \beta + \cancel{(Vu) \wedge d\beta}}$$

$$+ \underline{du \wedge d(V \lrcorner\, \beta)}$$

$$= d(Vu) \wedge \beta + du \wedge \big( V \lrcorner\, \beta + d(V \lrcorner\, \beta) \big)$$

$$= \mathcal{L}_V (du \wedge \beta) . \qquad \square$$

Cor $\quad \mathcal{L}_V(d\omega) = d(\mathcal{L}_V \omega)$.

Pf By magic,

$$\mathcal{L}_V(d\omega) = V \lrcorner\, \cancel{d(d\omega)} + d(V \lrcorner\, d\omega) = d(V \lrcorner\, d\omega) ,$$

$$d(\mathcal{L}_V \omega) = d(V \lrcorner\, d\omega) + \cancel{d(d(V \lrcorner\, \omega))} = d(V \lrcorner\, d\omega) . \qquad \square$$

## Homotopy invariance

Given $F, G : M \to N$ smooth maps, a collection of linear maps $h : \Omega^p(N) \longrightarrow \Omega^{p-1}(M)$ s.t. $d(h\omega) + h(d\omega) = G^*\omega - F^*\omega \quad \forall \omega$ is called a cochain homotopy between $F^*$ and $G^*$.

**Prop** If $\exists$ cochain htpy b/w $F^*$ and $G^*$, then $F^* = G^* : H^p_{dR}(N) \to H^p_{dR}(M)$ for all $p$.

**Pf** If $\omega \in Z^p(M)$, then $G^*\omega - F^*\omega = d(h\omega) + h(\cancel{d\omega})$

$$\Rightarrow [G^*\omega] = [F^*\omega]. \qquad \square$$