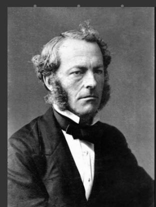
8. V. 23

Stokes' Thm Let M be an oriented smooth n-manifold with or without boundary, and let $\omega \in \Omega_c^{n-1}(M)$. Then



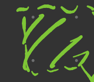
George Stokes
1819 - 1903

$$\boxed{\int_M d\omega = \int_{\partial M} \omega} .$$

Élie Cartan
1869 - 1951

Note
- ∂M has the induced orientation
- If $\partial M = \emptyset$, then RHS $= 0$.

Pf Take a finite open cover $\{U_\alpha\}$ of $\operatorname{supp}\omega$ with U_α diffeomorphic to

(i) $(0,1)\times\cdots\times(0,1)$ (interior points) or 

(ii) $(0,1]\times(0,1)\times\cdots\times(0,1)$ (boundary points). 

Let $\{\psi_\alpha\}$ be a smooth PoU subordinate to $\{U_\alpha\}$. Then

$$\int_{\partial M}\omega = \sum_\alpha \int_{\partial M}\psi_\alpha\omega = \sum_\alpha \int_{\partial U_\alpha}\psi_\alpha\omega.$$

We have $d(\psi_\alpha\omega) = (d\psi_\alpha)\wedge\omega + \psi_\alpha\, d\omega$ so

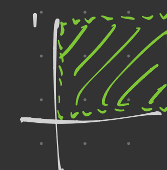
$$\sum_\alpha \int_{U_\alpha} d(\psi_\alpha\omega) = \int_M \Big(\cancelto{0}{d\big(\underbrace{\textstyle\sum_\alpha \psi_\alpha}_{1}\big)}\wedge\omega + \big(\underbrace{\textstyle\sum_\alpha \psi_\alpha}_{1}\big)\, d\omega \Big)$$

$$= \int_M d\omega.$$

Thus it suffices to show

$$\int_{\partial U_\alpha} \psi_\alpha \omega = \int_{U_\alpha} d(\psi_\alpha \omega)$$

for all α. I.e. we may assume ω is supported on one U_α.
For simplicity, consider the $n=2$ case (moral exercise: adapt to $n>2$). Write $\omega = f_1 dx^1 + f_2 dx^2$. If U_α is a boundary neighborhood, then



$$\int_{\partial U_\alpha} \omega = \int_{\partial U_\alpha} f_1 dx^1 + f_2 dx^2 = \int_0^1 f_2(1, x^2)\, dx^2$$

Meanwhile,

$$\int_{U_\alpha} d\omega = \int_{U_\alpha} \left(\frac{\partial f_2}{\partial x^1} - \frac{\partial f_1}{\partial x^2} \right) dx^1 \wedge dx^2$$

$$= \int_0^1 \left(\int_0^1 \frac{\partial f_2}{\partial x^1} \, dx^1 \right) dx^2 - \int_0^1 \left(\int_0^1 \frac{\partial f_1}{\partial x^2} \, dx^2 \right) dx^1$$

$$= \int_0^1 \left(f_2(1, x^2) - f_2(0, x^2) \right) dx^2$$

$$- \int_0^1 \left(f_1(x^1, 1) - f_1(x^1, 0) \right) dx^1$$

$$= \int_0^1 f_2(1, x^2) \, dx^2 .$$

Thus $\int_{\partial U_\alpha} \omega = \int_{U_\alpha} d\omega$ on boundary nbhds.

Now suppose U_α is an interior nbhd. Then $\partial U_\alpha = \emptyset$

so $\int_{\partial U_\alpha} \omega = 0$. Meanwhile,

$$\int_{U_\alpha} d\omega = \int_0^1 \left(f_2(\cancel{1}, x^2) - f_2(\cancel{0}, x^2)\right) dx^2$$

$$- \int_0^1 \left(f_1(\cancel{x^1}, 1) - f_1(\cancel{x^1}, 0)\right) dx^1$$

$$= 0$$

since $\operatorname{supp}(\omega) \subseteq U_\alpha$. This concludes the proof! $\square$

Note We can recover the fundamental thm for line integrals from Stokes'. Suppose $\gamma : [a,b] \to M$ is a smooth embedding, $S = \gamma[a,b]$ $\subseteq M$ submfld w/ boundary. Orient S so that γ is or'n preserving. Then for $f \in C^\infty(M)$,

$$\int_\gamma df = \int_{[a,b]} \gamma^* df = \int_S df = \int_{\partial S} f = f(\gamma(b)) - f(\gamma(a)).$$

Cor If M is or'd smooth mfld w/o ∂, then $\forall \omega \in \Omega_c^{n-1}(M)$,

$$\int_M d\omega = 0.$$

Cor If $d\omega = 0$ (i.e. ω closed), then $\int_{\partial M} \omega = 0$.

Cor M smooth w/ or w/o ∂. $S \subseteq M$ or'd compact smooth k-dim'l submfld w/o ∂, ω a closed ($d\omega = 0$) k-form on M s.t. $\int_S \omega \neq 0$, then

(a) ω is not exact on M, and

(b) $S \neq \partial N$ for any or'd compact smooth $N \subseteq M$.

E.g. $\omega = \frac{1}{x^2+y^2}(x\,dy - y\,dx)$ has $\int_{S^1} \omega \neq 0$, so

ω is not exact (no f s.t. $df = \omega$) and S^1 is not the

boundary of a compact submfld (w/ or w/o ∂) of $\mathbb{R}^2 \setminus 0$.

Cor (Green's Thm) $D \subseteq \mathbb{R}^2$ compact reguler domain, $P, Q \in C^\infty(D)$.

Then
$$\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = \int_{\partial D} P\,dx + Q\,dy.$$

Note Stokes' Thm also holds on mflds w/ corners (pp. 415-421).

Planimeters For $D \subseteq \mathbb{R}^2$ a compact regular domain,

$$\operatorname{area}(D) = \int_D dx \wedge dy .$$

Call $P, Q \in C^\infty(D)$ planimetric when $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$.

(Equivalently, $d(P\,dx + Q\,dy) = dx \wedge dy$.)

For planimetric P, Q, Green's Thm implies

$$\operatorname{area}(D) = \int_{\partial D} P\,dx + Q\,dy .$$

$Q = 2x$

TPS Find a planimetric pair P, Q smooth on $\mathbb{R}^2$.

$P = y$

$Q = x$, $P = 0$

TPS Prove that this device accurately computes area!

Polar Planimeter

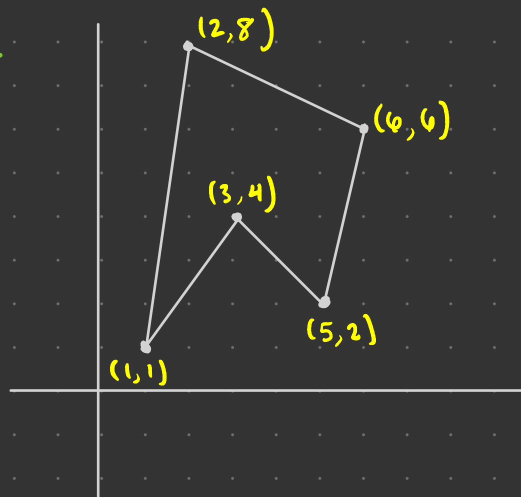
TPS Use Green's Thm to prove the surveyor's area formula:

If the vxs of a simple polygon, listed counterclockwise around the perimeter, are $(x_0, y_0), \dots, (x_{n-1}, y_{n-1})$, then the area of the polygon is

$$\text{Area} = \frac{1}{2}\sum_{i=0}^{n} \det\begin{pmatrix} x_i & x_{i+1} \\ y_i & y_{i+1} \end{pmatrix}$$

where $(x_n, y_n) := (x_0, y_0)$.

E.g.



$$\begin{array}{cc|r} 1 & 1 & \\ & & 1 \\ 3 & 4 & \\ & & -14 \\ 5 & 2 & \\ & & 18 \\ 6 & 6 & \\ & & 36 \\ 2 & 8 & \\ & & -6 \\ 1 & 1 & \\ \hline & & 35 \end{array} \implies \text{Area} = \frac{35}{2}$$

"shoelace algorithm"