5.V.23

Integration of differential forms

Goal For an oriented smooth n-mfld M, define

$$\int_M : \underbrace{\Omega^n_c(M)}_{\substack{\text{compactly} \\ \text{supported n-forms} \\ \text{on } M}} \longrightarrow \mathbb{R}$$

$$\omega \longmapsto \int_M \omega$$

Strategy ① $D \subseteq \mathbb{R}^n$ domain of integration (bdd with $\mu(\partial D) = 0$)
$\omega \in \Omega^n(D)$, define $\int_D \omega = \int_D f \, dV$ for $\omega = f \, dx^1 \wedge \cdots \wedge dx^n$

② $\omega \in \Omega^n(M)$ compactly supported in the domain of a single chart (U, φ) that is pos or neg or'd. Define

$$\int_M \omega = \pm \int_{\varphi(U)} (\varphi^{-1})^* \omega$$

sign according to or'n of chart

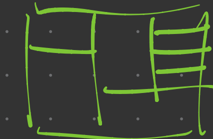
③ $\{U_i\}$ finite open cover of $\mathrm{supp}(\omega)$ for $\omega \in \Omega^n_c(M)$, $\{\psi_i\}$ subordinate smooth POU. Define

$$\int_M \omega = \sum_i \int_M \psi_i \omega$$

$$= \sum_i \pm \int_{\varphi(U)} (\varphi^{-1})^* (\psi_i \omega).$$

Throughout, show def'ns are independent of choices.

⓪ Review of multivariable integration.



- $f : R = [a_1, b_1] \times \cdots \times [a_n, b_n] \longrightarrow \mathbb{R}$
- $P = (P_1, \ldots, P_k)$ partition of R into small rectangles
- $U(f, P) = \sum_i \sup(f|_{P_i}) \operatorname{vol}(P_i)$

  $L(f, P) = \sum_i \inf(f|_{P_i}) \operatorname{vol}(P_i)$

- $f : R \longrightarrow \mathbb{R}$ is Riemann integrable if $\forall \varepsilon > 0$ $\exists$ partition P s.t. $U(f, P) - L(f, P) < \varepsilon$

- If $f : R \longrightarrow \mathbb{R}$ integrable, define

$$\int_R f = \int_R f \, dV = \int_R f \, dx^1 \cdots dx^n = \inf_P U(f, P) = \sup_P L(f, P).$$

- Above defn adapts to "domains of integration"

Thm If $f : D \to \mathbb{R}$ is cts w/ compact support, then f is integrable.

Change of Variables $U, V \subseteq \mathbb{R}^n$ open, $\varphi : U \to V$ diffeo.

Then $\int_V f = \int_U |\det J\varphi| (f \circ \varphi)$.

① $D \subseteq \mathbb{R}^n$, $\omega \in \Omega^n_c(D)$. Then $\omega = f\, dx^1 \wedge \cdots \wedge dx^n$.

Set $\int_D \omega := \int_D f$ i.e. $\int_D f\, dx^1 \wedge \cdots \wedge dx^n = \int_D f\, dx^1 \cdots dx^n$.

Prop $U, V \subseteq \mathbb{R}^n$ or $\mathbb{H}^n$ open, $G : U \to V$ smooth and or'n preserving or reversing diffeo. If $\omega \in \Omega^n_c(V)$ then

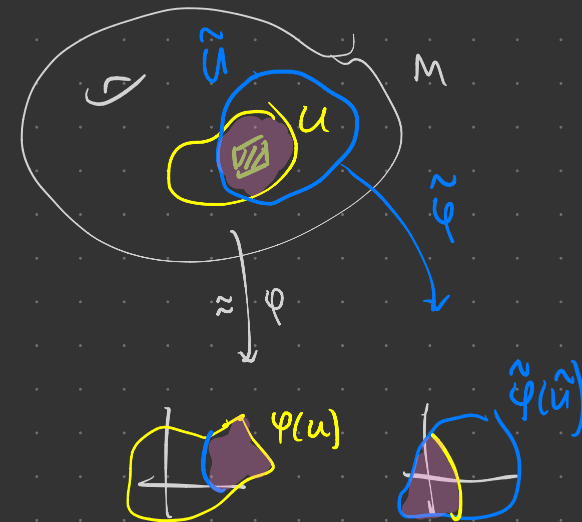
$$\int_U G^* \omega = \pm \int_V \omega.$$

Pf CoV. $\square$

② (U, φ) pos or neg or'd chart on M

$\omega \in \Omega^n_c(M)$, $\operatorname{supp}(\omega) \subseteq U$.

Define $\int_M \omega := \pm \int_{\varphi(U)} (\varphi^{-1})^* \omega$ $\overset{?}{=} \pm \int_{\tilde{\varphi}(\tilde{U})} (\tilde{\varphi}^{-1})^* \omega$



Prop This doesn't depend on choice of smooth chart with domain $\supseteq \operatorname{supp} \omega$.

Pf CoV. $\square$

③ $\{U_i\}$ finite open cover of $\operatorname{supp}(\omega)$ for $\omega \in \Omega^n_c(M)$,
$\{\psi_i\}$ subordinate smooth POU. Define

$$\int_M \omega = \sum_i \int_M \psi_i \omega$$

$$= \sum_i \pm \int_{\varphi(U)} (\varphi^{-1})^* (\psi_i \omega).$$

Prop This defn doesn't depend on U_i or ψ_i.

Pf $\{\tilde{U}_j\}$, $\{\tilde{\psi}_j\}$ another choice. For each i,

$$\int_M \psi_i \omega = \int_M \Big(\sum_j \tilde{\psi}_j\Big) \psi_i \omega = \sum_j \int_M \tilde{\psi}_j \psi_i \omega.$$

Thus $\displaystyle\sum_i \int_M \psi_i \omega = \sum_{i,j} \int_M \tilde{\psi}_j \psi_i \omega = \sum_j \int_M \tilde{\psi}_j \omega$

The same argument works for $\int_M \tilde{\varphi}_j \omega$ so both covers / POUs give same output. $\square$

If $\dim M = 0$, $\int_M f := \sum_{p \in M} \pm f(p)$.

sign according to or'n of pts

If $S \subseteq M$ or'd immersed k-dim'l submfld, $\omega \in \Omega^k(M)$ with $\iota_S^* \omega$ compactly supported, set $\int_S \omega := \int_S \iota_S^* \omega$.

In particular, $\int_{\partial M} \omega$ makes sense for $\omega \in \Omega^{n-1} M$.

w/ induced or'n

Prop (a) $\int_M : \Omega^n_c(M) \longrightarrow \mathbb{R}$ is linear $\int_M a\omega + \eta = \left(a \int_M \omega\right) + \int_M \eta$

$a \in \mathbb{R},\ \omega, \eta \in \Omega^n_c(M)$

(b) $\int_{-M} \omega = -\int_M \omega$

reversed or'n

(c) If $\omega \in \Omega^n(M)$ is a pos or'd or'n form, then $\int_M \omega > 0$.

(d) $F : N \longrightarrow M$ or'n preserving or reversing diffeo,

then $\int_M \omega = \pm \int_N F^* \omega$.

sign + for or'n pres, − for rev $\square$

Computation? Parametrize!

$D_1, \dots, D_k$ open domains of integration in $\mathbb{R}^n$

$F_i : \bar{D}_i \longrightarrow M$ s.t.

(i) $F_i|_{D_i}$ or'n preserving diffeo onto open $W_i \subseteq M$

(ii) $W_i \cap W_j = \emptyset$ for $i \neq j$

(iii) $\operatorname{supp} \omega \subseteq \bar{W}_1 \cup \dots \cup \bar{W}_k$.

Then $$\int_M \omega = \sum_{i=1}^{k} \int_{D_i} F_i^* \omega.$$

Pf CoV (pp. 408-409) $\square$

$0 < \phi < \pi$

$0 < \theta < 2\pi$

$(0, \pi) \times (0, 2\pi) \longrightarrow S^2$

$(\phi, \theta) \longmapsto \dots$