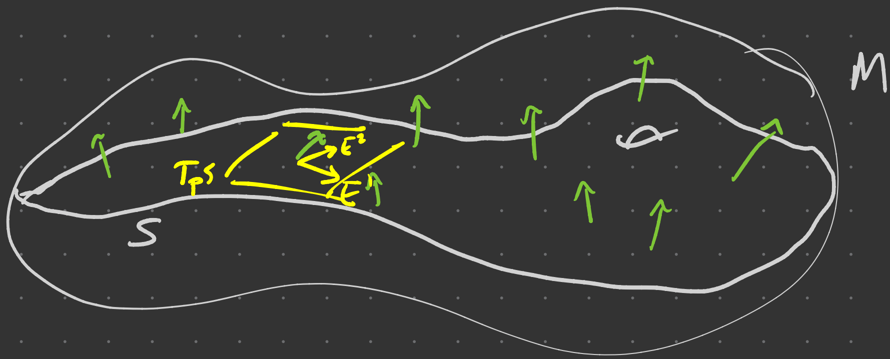
3.V.22

Orientations of Hypersurfaces

Thm M or'd smooth n-mfld w/ or w/o ∂, $S \subseteq M$ immersed hypersurface w/ or w/o ∂, N a vector field along S $(N: S \to TM)$ nowhere tangent to S.



Then S has a unique or'n s.t. $\forall p \in S$, $(E_1, \dots, E_{n-1})$ is an or'd basis for T_pS iff $(N_p, E_1, \dots, E_{n-1})$ is an or'd basis for T_pM.

If $\omega \in \Lambda^n M$ is an or'n form for M, then $i_S^*(N \lrcorner \omega)$ is an or'n form for S wrt this or'n.

Pf p.385 $\square$

E.g. $S^n \subseteq \mathbb{R}^{n+1}$ has nowhere tangent vector field $N = \sum x^i \frac{\partial}{\partial x^i}$

so S^n is orientable. Define the standard orientation of S^n to be the one provided by N.

Prop M or'd sm mfld, $S \subseteq M$ reg level set of $f: M \to \mathbb{R}$ smooth. Then S is orientable.

Pf $N = \operatorname{grad} f|_S$ works. (Here grad f is dual to df wrt choice of Riemannian structure on M.) $\square$

Boundary Orientations

Prop M an or'd sm n-mfld with ∂, $n \geq 1$. Then ∂M is orientable and all outward pointing vector fields on ∂M determine the same or'n on ∂M. (called the <u>induced</u> or <u>Stokes</u> orientation.)

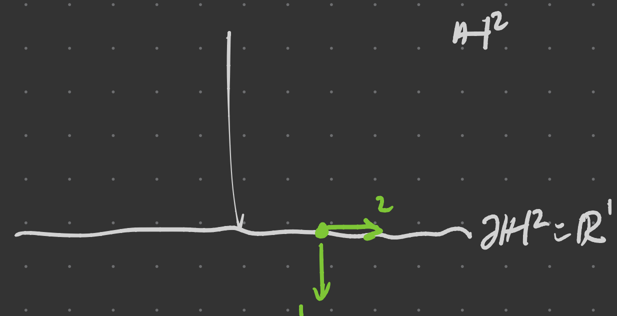
<u>Pf</u> For N a smooth outward-pointing vf along ∂M,
$\iota_{\partial M}^*(N \lrcorner \omega)$ orients ∂M (ω an or'n form of M).

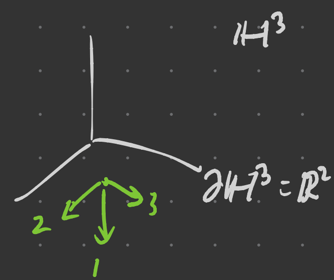
Independence from choice of N: p.386. $\square$

<u>Note</u> Std or'n on S^n = induced or'n on $\partial \bar{\mathbb{B}}^{n+1} = S^n$.

<u>E.g.</u> Induced or'n on $\partial \mathbb{H}^n = \mathbb{R}^{n-1}$ is pos or'd iff $\left[-\frac{\partial}{\partial x^n}, \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^{n-1}}\right]$

$= -\left[\frac{\partial}{\partial x^n}, \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^{n-1}}\right] = (-1)^n \left[\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right]$ is pos or'd for std or'n of $\mathbb{R}^n$.





Orientations and Covering Maps

Prop If $\pi: E \longrightarrow M$ is a smooth covering map and M is orientable, then E is also orientable.

$\varphi: M \xrightarrow[\text{diffeo}]{\text{local}} N$

Pf π is a local diffeo. $\square$

$\varphi^* \mathcal{O}_N$

G a Lie gp acting smoothly on a sm or'd mfld E.
Call the action orientation-preserving if L_g is orientation preserving $\forall g \in G$,

$x \longmapsto g \cdot x$

Thm E a conn'd or'd smooth mfld, $\pi: E \longrightarrow M$ smooth normal covering map. Then M is orientable iff action of $\mathrm{Aut}_\pi(E)$ on E is orientation-preserving.

Pf Let $\mathcal{O}_E$ = or'n of E.

($\Rightarrow$) M has two or'ns. Pick some $q \in E$. Let $\mathcal{O}_M$ be the or'n s.t. $d\pi_q : T_q E \to T_{\pi(q)} M$ preserves or'n. Since $\pi^* \mathcal{O}_M$ agrees with $\mathcal{O}_E$ at q, they must be the same or'n. Now take $\varphi \in \mathrm{Aut}_\pi(E)$:

$$\begin{tikzcd} E \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & E \arrow[dl, "\pi"] \\ & M & \end{tikzcd}$$

Since $\pi \circ \varphi = \pi$,

$$\varphi^* \mathcal{O}_E = \varphi^*(\pi^* \mathcal{O}_M) = (\pi \circ \varphi)^* \mathcal{O}_M = \pi^* \mathcal{O}_M = \mathcal{O}_E$$

so φ is or'n preserving.

($\Leftarrow$) $\mathrm{Aut}_\pi(E) \circlearrowright E$ or'n preserving. Pick $p \in M$. If $U \subseteq M$ is an evenly covered nbhd of p, then $\exists$ smooth section $\sigma : U \to E$ inducing or'n $\sigma^* \mathcal{O}_E$ on U.

By normality of π, $\mathrm{Aut}_\pi(E)$ acts transitively on fibers of π.

Thus if $\sigma_1 : U \to E$ is another smooth section we can take $\varphi \in \mathrm{Aut}_\pi(E)$ s.t. $\sigma_1(p) = \varphi(\sigma(p))$. $\varphi \circ \sigma$ is a smooth section agreeing with σ_1 at p, so $\sigma_1 = \varphi \circ \sigma$.

Since is is or'n preserving,

$$\sigma_1^* \mathcal{O}_E = \sigma^* \varphi^* \mathcal{O}_E = \sigma^* \mathcal{O}_E .$$

This allows us to globalize the or'n. $\square$

<u>TPS</u> For which n is $\mathbb{RP}^n$ orientable?

<u>A</u> $n=0$ or $n>0$ odd: $S^n \to \mathbb{RP}^n$ has antipodal map as automorphism, preserves or'n iff n odd.

E.g. (Möbius bundle)

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "q"] & E \arrow[d, "\pi"', "\text{Möbius bundle}"] \\ & S^1 \end{tikzcd}$$

$\mathrm{Aut}_q(\mathbb{R}^2) \cong \mathbb{Z}$ with $n \cdot (x,y) = (x+n, (-1)^n y)$

For n odd, $(x,y) \longmapsto n\cdot(x,y)$ pulls $dx \wedge dy$ back to $-dx \wedge dy$, so the action of $\mathrm{Aut}_q(\mathbb{R}^2)$ is not or'n preserving $\Rightarrow$ E is not orientable.

Orientation Covering

Call $\pi : N \longrightarrow M$ a generalized covering map if it satisfies all the covering map conditions except possibly connected N.

M = conn'd smooth n-mfld, $n > 0$

$$\hat{M} := \{ (p, \mathcal{O}_p) \mid p \in M,\ \mathcal{O}_p \text{ an or'n of } T_pM \}.$$

$$\begin{array}{ccc} \hat{M} & & (p, \mathcal{O}_p) \\ \hat{\pi} \downarrow & & \downarrow \\ M & & p \end{array}$$

2-1 map — the orientation cover of M.

Prop Can give $\hat{M}$ the structure of a smooth or'd mfld s.t.

(a) $\hat{\pi}: \hat{M} \to M$ is a gen'lized smooth cover map.

(b) conn'd open $U \subseteq M$ is evenly covered by $\hat{\pi}$ iff U is orientable

(c) if $U \subseteq M$ evenly covered, then or'ns of U = pullback or'n induced by local sections of $\hat{\pi}$ over U.

Pf pp. 394-395 $\square$

E.g. • If M is orientable, then $\hat{M} = M \times \mathbb{Z}/2\mathbb{Z}$.

• If n is even, $S^n = \widehat{\mathbb{RP}^n}$. • $SU(2) = \widehat{SO(3)}$

In gen'l, if M is non-orientable, $\hat{M}$ is conn'd and $\hat{\pi}$ is a 2-sheeted smooth covering map.

Thm M a conn'd smooth mfld. If $\pi_1 M$ has no subgp of index 2, then M is orientable.

(In particular, simply conn'd $\Rightarrow$ orientable

$\pi_1 M$ finite of odd order $\Rightarrow$ orientable

nonorientable $\Rightarrow$ $\exists\, H \leq \pi_1 G$ s.t. $[G:H] = 2$.)

Pf Suppose M nonorientable, consider universal and orientation covers:

$$\begin{tikzcd} \tilde{M} \arrow[d, "\tilde{\pi}"'] \arrow[dr] & \\ M & \hat{M} \arrow[l, "\hat{\pi}"] \end{tikzcd}$$

$\mathrm{Aut}_{\hat{\pi}}(\hat{M}) \cong C_2$, gen'd by

$(p, \mathcal{O}_p) \longmapsto (p, -\mathcal{O}_p)$.

If $H = \hat{\pi}_* \pi_1 \hat{M}$ then

$\mathrm{Aut}_{\hat{\pi}}(\hat{M}) \cong \pi_1 M / H$. $\square$