Orientations

For an $\mathbb{R}$-vector space V, say two ordered bases $(e_1, \dots, e_n)$, $(\tilde{e}_1, \dots, \tilde{e}_n)$ are **consistently oriented** when the transition matrix $B = (B_i^j)$ with

$$e_i = B\,\tilde{e}_i$$

has $\det(B_i^j) > 0$.

relatively oriented — no canonical orientation

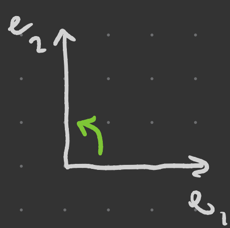
Define an equiv rel'n on ordered bases with $\underline{e} \sim \underline{e}' \iff$ consistently oriented $\iff B \in GL_n^+(\mathbb{R}) := \{A \in GL_n\mathbb{R} \mid \det A > 0\}$.

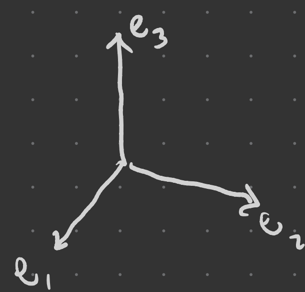
An **orientation** for V is an equivalence class of ordered bases.

Exc There are exactly two orientations on a given vector space.

- If V is oriented, call other ordered bases of V positively or negatively oriented when in / not in the given orientation.

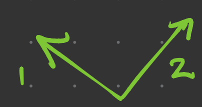
- Always give $\mathbb{R}^n$ the canonical orientation $[e_1, \dots, e_n]$.





"right-hand rule"

Q Is $((-1,1),(1,1))$ positively or negatively oriented?



Prop V a vector space of dim n. Each $\omega \in \Lambda^n V^*$ determines an orientation $\mathcal{O}_\omega$ of V as follows: if $n \geq 1$, then

$$\mathcal{O}_\omega = \{(E_1, \dots, E_n) \text{ ordered basis of } V \mid \omega(E_1, \dots, E_n) > 0\}$$

If $n = 0$, $\mathcal{O}_\omega$ is $+1$ if $\omega > 0$, -1 if $\omega < 0$. $\omega, \omega' \in \Lambda^n V^*$ determine the same or'n iff $\omega = \lambda \omega'$ for some $\lambda > 0$.

?

Pf If $B : V \longrightarrow V$ takes a basis E_j to $\tilde{E}_j$ then

$$\omega(\tilde{E}_1, \dots, \tilde{E}_n) = \omega(BE_1, \dots, BE_n)$$
$$= (\det B)\, \omega(E_1, \dots, E_n).$$

Thus $E, \tilde{E}$ are consistently or'd iff $\omega(E)$, $\omega(\tilde{E})$ have the same sign. $\square$

Upshot Choosing an or'n of V is equivalent to choosing a component of $\Lambda^n V^* \setminus \{0\}$.

Orientations of Manifolds

- A **pointwise orientation** of M is a choice of or'n of T_pM $\forall p \in M$.
- If (E_i) is a local frame for TM, call (E_i) (**positively**) **oriented** if $(E_1|_p, \ldots, E_n|_p)$ is a pos or'd basis of T_pM $\forall p \in U$.
- Call a ptwise or'n **continuous** if every pt of M is in the domain of an oriented local frame.
- An **orientation** of M is a cts ptwise or'n.

vs

(Möbius)

$\Lambda^2 T^*(\text{Möbius})$

Thm A smooth n-manifold is orientable (admits an orientation) iff the structure group of TM can be reduced to $GL_n^+\mathbb{R}$ iff $\Lambda^n T^*M$ admits a nonvanishing section $\omega \in \Omega^n(M)$ iff $\Lambda^n T^*M \cong M \times \mathbb{R}$ is trivial.

$$
\begin{tikzcd}
U_\alpha \cap U_\beta \arrow[r, "\tau_{\alpha\beta}"] \arrow[dr] & GL_n\mathbb{R} \\
& GL_n^+\mathbb{R} \arrow[u, phantom, sloped, "\subset"]
\end{tikzcd}
$$

Pf p.381 + Exc. □

Prop Products of orientable smooth mflds are orientable.

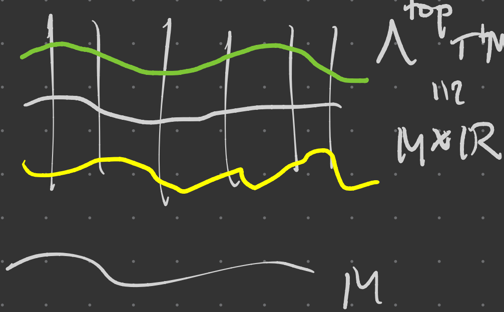
Pf For orientable sm mflds $M_1, \dots, M_k$, choose non-vanishing top dim'l forms $\omega_1, \dots, \omega_k$ on each. Then

$$\pi_1^* \omega_1 \wedge \cdots \wedge \pi_k^* \omega_k$$

is a nonvanishing top dim'l form on $M_1 \times \cdots \times M_k$. $\square$

Q How many orientations does $\underbrace{\text{an}}_{\text{conn'd}}$ orientable mfld have?

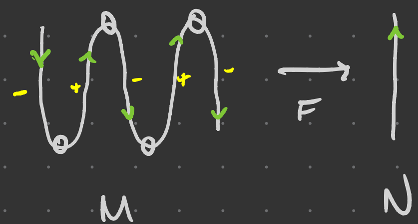
A 2.



Take M, N oriented sm mflds, $F: M \to N$ a local diffeo.

Say F is orientation-preserving if $\forall p \in M$, $dF_p : T_pM \xrightarrow{\cong} T_{p}N$ takes or'd bases to or'd bases; o/w F is orientation-reversing

pos pos



Prop Suppose N oriented, $F: M \to N$ local diffeo. Then M has a unique orientation s.t. F is orientation-preserving.

Pf Choose ptwise or'n of M s.t. each $dF_p : T_pM \to T_pN$ is or'n preserving. If ω is a smooth or'n of N, then

$F^*\omega$ is a smooth or'n form of M. $\square$

<u>Note</u> Parallelizable mflds (e.g. Lie groups) are orientable.

$$\begin{gathered} TM \text{ trivial} \\ \Updownarrow \\ T^*M \text{ trivial} \\ \Downarrow \\ \Lambda^{top} T^*M \text{ trivial} \end{gathered}$$

$$\omega \in \Omega^k(M) = \Gamma(\Lambda^k T^*M)$$

$$\begin{tikzcd} \Lambda^k T^*M \arrow[d] \\ M \arrow[u, bend right, "\omega"'] \end{tikzcd} \qquad \begin{tikzcd} \omega_p \\ p \arrow[u, mapsto] \end{tikzcd}$$

$\omega_p \longleftrightarrow$ alternating multilin form

$$(T_p^*M)^{\times k} \longrightarrow \mathbb{R}$$

Line integral: $\omega \in \Omega^1(M)$, $\gamma : [a,b] \longrightarrow M$

$$\int_\gamma \omega = \int_a^b \gamma^* \omega$$

$$\begin{tikzcd} {[a,b]} \arrow[r, "\gamma"] & M \\ {[c,d]} \arrow[u, "\cong"] \arrow[ur, "\tilde\gamma"'] & \end{tikzcd}$$

$$\| $$

$$-\int_{\tilde\gamma} \omega$$