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Differential Forms on Manifolds

Recall: Given a rank n vb $\begin{array}{c} E \\ \downarrow \pi \\ M \end{array}$ and a rep'n $\rho : GL_n\mathbb{R} \to GL(V)$

we have a twisted vb $E \times_\rho V$ with cocycle data

$$\begin{tikzcd}
\mathcal{U}_{\alpha\beta} \arrow[rr, "\tau_{\alpha\beta}^{E \times_\rho V}"] \arrow[dr, "\tau_{\alpha\beta}^{E}"'] & & GL(V) \\
& GL_n\mathbb{R} \arrow[ur, "\rho"'] &
\end{tikzcd}$$

Consider

$$\rho_0 : GL_n\mathbb{R} \longrightarrow GL\left((\mathbb{R}^n)^{\otimes k}\right), \quad A \longmapsto A^{\otimes k}$$

$$\rho_1 : GL_n\mathbb{R} \longrightarrow GL\left(\underbrace{(\mathbb{R}^n)^{\wedge k}}_{\Lambda^k \mathbb{R}^n}\right), \quad A \longmapsto A^{\wedge k}$$

Twisting by ρ_0, ρ_1 results in $T^k E := E \times_{\rho_0} (\mathbb{R}^n)^{\otimes k}$

$$\Lambda^k E := E \times_{\rho_1} \underset{\Lambda^k \mathbb{R}^n}{\cancel{(\mathbb{R}^n)^{\wedge k}}} .$$

In particular, get $\Lambda^k T^*M$ with fibers

$$(\Lambda^k T^*M)_p = \Lambda^k T_p^* M .$$

A <u>differential</u> <u>k-form</u> (or just <u>k-form</u>) is a section of $\Lambda^k T^*M$:

$$\boxed{\Omega^k(M) := \Gamma(\Lambda^k T^*M)}$$

$$\begin{tikzcd} \Lambda^k T^*M \arrow[d] \\ M \arrow[u, bend right, dashed, "\omega"'] \end{tikzcd}$$

$\omega_p \in \Lambda^k T_p^* M$

Have a mult'n $\Omega^k(M) \times \Omega^l(M) \longrightarrow \Omega^{k+l}(M)$

$$(\omega, \eta) \longmapsto \omega \wedge \eta$$

given by pointwise wedge: $(\omega\wedge\eta)_p = \omega_p \wedge \eta_p$.

Then $\Omega^*(M) := \bigoplus_{k=0}^{n} \Omega^k(M)$ is an associative graded comm $\mathbb{R}$-algebra.

<u>Note</u> • In smooth coords (x^i),

$$\omega = \sum_{I=(i_1<i_2<\cdots<i_k)} \omega_I \, dx^{i_1}\wedge\cdots\wedge dx^{i_k} =: \sum \omega_I \, dx^I$$

• $\Omega^0(M) = C^\infty(M)$

$\Omega^1(M) = \mathfrak{X}^*(M)$

We have the expected functoriality:

$$F: M \longrightarrow N \text{ smooth} \rightsquigarrow F^*: \Omega^*(N) \longrightarrow \Omega^*(M)$$
$$\omega \longmapsto F^*\omega$$

a graded $\mathbb{R}$-alg hom

($\mathbb{R}$-linear +

$F^*(\omega \wedge \eta) = (F^*\omega) \wedge (F^*\eta)$)

where $(F^*\omega)_p (v_1, \ldots, v_k)$

$= \omega_{F(p)} (dF_p(v_1), \ldots, dF_p(v_k))$

In local coords,

$$F^*\left(\sum \omega_I \, dy^{i_1} \wedge \cdots \wedge dy^{i_k}\right) = \sum{}' (\omega_I \circ F) \, d(y^{i_1} \circ F) \wedge \cdots \wedge d(y^{i_k} \circ F).$$

E.g. $F: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ $\quad \omega = y \, dx \wedge dz + x \, dy \wedge dz \in \Omega^2(\mathbb{R}^3)$.

$(u,v) \longmapsto (u, v, u^2 - v^2)$

Then $F^*\omega = v \, du \wedge d(u^2 - v^2) + u \, dv \wedge d(u^2 - v^2)$

$$= v\, du \wedge (2u\, du - 2v\, dv) + u\, dv \wedge (2u\, du - 2v\, dv)$$

$$= v(-2v)\, du \wedge dv + (2u^2)\, dv \wedge du$$

$$= -2(u^2 + v^2)\, du \wedge dv$$

<u>E.g.</u> $\omega = dx \wedge dy$ $\quad (r, \theta) \longmapsto (r\cos\theta, r\sin\theta)$

$$dx \wedge dy = d(r\cos\theta) \wedge d(r\sin\theta)$$

$$= (r(-\sin\theta)\, d\theta + \cos\theta\, dr) \wedge (r\cos\theta\, d\theta + \sin\theta\, dr)$$

$$= r\, dr \wedge d\theta.$$

<u>Prop</u> (Pullback of top-degree forms)

$F: M \longrightarrow N$ smooth map of n-mflds, $(x^i), (y^i)$ smooth coords on $U \subseteq M$, $V \subseteq N$. Then for $u \in C^\infty(V)$, on $U \cap F^{-1}V$ we have

$$\boxed{F^*(u\, dy^1\wedge\cdots\wedge dy^n) = (u\circ F)(\det JF)\, dx^1\wedge\cdots\wedge dx^n}$$

Pf Have $F^*(u\, dy^1\wedge\cdots\wedge dy^n) = (u\circ F)\, dF^1\wedge\cdots\wedge dF^n$

for $F=(F^1,\dots,F^n)$ on $U\cap F^{-1}V$. Now

$$dF^1\wedge\cdots\wedge dF^n = \left(\sum \frac{\partial F^1}{\partial x^i}dx^i\right)\wedge\cdots\wedge\left(\sum\frac{\partial F^n}{\partial x^i}dx^i\right)$$

$$\sum_{\sigma\in\mathfrak{S}_n}\bigwedge_{j=1}^{n}\frac{\partial F^j}{\partial x^{\sigma(j)}}dx^{\sigma(j)}$$

$$= \sum_\sigma\left(\prod_{j=1}^n \frac{\partial F^j}{\partial x^{\sigma(j)}}\right)dx^{\sigma(1)}\wedge\cdots\wedge dx^{\sigma(n)}$$

$$= \sum_\sigma \operatorname{sgn}(\sigma)\prod_{j=1}^n\frac{\partial F^j}{\partial x^{\sigma(j)}}\, dx^1\wedge\cdots\wedge dx^n$$

$$= (\det JF)\, dx^1\wedge\cdots\wedge dx^n$$

so we are done. $\square$

Cor If $(U, (x^i))$ and $(\tilde{U}, (\tilde{x}^i))$ are overlapping coord charts, then on $U \cap \tilde{U}$,

$$d\tilde{x}^1 \wedge \cdots \wedge d\tilde{x}^n = \det\left(\frac{\partial \tilde{x}^j}{\partial x^i}\right) dx^1 \wedge \cdots \wedge dx^n. \quad \square$$

Exterior derivatives

Recall

$$\begin{array}{cccc} d: & C^\infty(M) & \longrightarrow & \mathfrak{X}^*(M). \\ & \| & & \| \\ & \Omega^0(M) & & \Omega^1(M) \end{array}$$

Extend to a map $d: \Omega^k(M) \longrightarrow \Omega^{k+1}(M) \quad \forall k \geq 0$

via the rule $$d\Big(\sum \omega_I \, dx^I\Big) = \sum \underbrace{d\omega_I}_{} \wedge dx^I \qquad (*)$$

ω_I is a smooth function, so we know how to do this

STOP A priori, this only makes sense locally on coord patches. How do we ensure it extends to M?

Lemma On $\mathbb{R}^n$, $d : \Omega^k(\mathbb{R}^n) \longrightarrow \Omega^{k+1}(\mathbb{R}^n)$ is

(a) $\mathbb{R}$-linear

(b) If $\omega \in \Omega^k(\mathbb{R}^n)$, $\eta \in \Omega^l(\mathbb{R}^n)$ then
$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$$

(c) $d \circ d = 0$

(d) $U \subseteq \mathbb{R}^n$ or $\mathbb{H}^n$ open, $V \subseteq \mathbb{R}^m$ or $\mathbb{H}^m$ open,

$F: U \to V$ smooth, $\omega \in \Omega^k(V)$, then

$$F^*(d\omega) = d(F^*\omega).$$

Pf (a), (b), (d): Computations from defns (p. 364–365)

(c): First check on 0-form $u \in \Omega^0(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$:

$$\begin{aligned}
d(du) &= d\left(\sum_j \frac{\partial u}{\partial x^j}\, dx^j\right) \\
&= \sum_{i,j} \frac{\partial^2 u}{\partial x^i \partial x^j}\, dx^i \wedge dx^j \\
&= \sum_{i<j} \left(\frac{\partial^2 u}{\partial x^i \partial x^j} - \frac{\partial^2 u}{\partial x^j \partial x^i} \right) dx^i \wedge dx^j \\
&= 0.
\end{aligned}$$

✓

For the general case,

$$d(d\omega) = d\left(\sum_J d\omega_J \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k}\right)$$

$$= \sum_J \underbrace{\cancel{d(d\omega_J)}}_{=0} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k}$$

$$+ \sum_J \sum_{i=1}^{k} (-1)^i \, d\omega_J \wedge dx^{j_1} \wedge \cdots \wedge \underbrace{\cancel{d(dx^{j_i})}}_{0} \wedge \cdots \wedge dx^{j_k}$$

$$= 0. \quad \checkmark \qquad \square$$

Thm (Existence & uniqueness of exterior differentiation)

M smooth mfld w/ or w/o ∂. There are unique operators

$d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$, $k \geq 0$, called exterior differentiation, satisfying the following properties:

(i) d is $\mathbb{R}$-linear

(ii) if $\omega \in \Omega^k(M)$, $\eta \in \Omega^l(M)$ then

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$$

(iii) $d \circ d = 0$

(iv) if $f \in \Omega^0(M) = C^\infty(M)$, df is the differential of f.

Furthermore, in any smooth chart, d is given by (*).

Pf Idea Existence: For smooth chart (U, φ), $\omega \in \Omega^k(M)$,

set $d\omega = \varphi^* d(\varphi^{-1*}\omega)$. Use naturality of (local) d wrt pullback to prove well-defn.

Uniqueness: p. 366 — bump fns.

Note The (global) exterior derivative is also natural wrt pullback:

$F: M \longrightarrow N$ smooth then $F^*(d\omega) = d(F^*\omega)$, i.e.

$$\begin{tikzcd}
\Omega^k(M) \arrow[d, "d"'] & \Omega^k(N) \arrow[l, "F^*"'] \arrow[d, "d"] \\
\Omega^{k+1}(M) & \Omega^{k+1}(N) \arrow[l, "F^*"]
\end{tikzcd}$$

commutes.

Examples in $\mathbb{R}^3$

$$\omega = P\,dx + Q\,dy + R\,dz \in \Omega^1(\mathbb{R}^3), \quad P, Q, R \in C^\infty(M)$$

$$d\omega = d(P\,dx) + d(Q\,dy) + d(R\,dz)$$

$$= \left(\frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy + \frac{\partial P}{\partial z}dz\right)\wedge dx + \left(\qquad\right)dy + \left(\quad\right)dz$$

$$= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\wedge dy + \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)dx\wedge dz$$

$$+ \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)dy\wedge dz$$

$$\eta = u\, dx\wedge dy + v\, dx\wedge dz + w\, dy\wedge dz \in \Omega^2(\mathbb{R}^3)$$

$$d\eta = \left(\frac{\partial u}{\partial z} - \frac{\partial v}{\partial y} + \frac{\partial w}{\partial x} \right) dx\wedge dy\wedge dz$$

<u>Upshot</u>

$$\begin{tikzcd}
C^\infty(\mathbb{R}^3) \arrow[r, "\mathrm{grad}"] \arrow[d, "\mathrm{id}"'] & \mathfrak{X}(\mathbb{R}^3) \arrow[r, "\mathrm{curl}"] \arrow[d, "\flat", "\cong"'] & \mathfrak{X}(\mathbb{R}^3) \arrow[r, "\mathrm{div}"] \arrow[d, "\beta", "\cong"'] & C^\infty(\mathbb{R}^3) \arrow[d, "*", "\cong"'] \\
\Omega^0(\mathbb{R}^3) \arrow[r, "d"'] & \Omega^1(\mathbb{R}^3) \arrow[r, "d"'] & \Omega^2(\mathbb{R}^3) \arrow[r, "d"'] & \Omega^3(\mathbb{R}^3)
\end{tikzcd}$$

commutes