26. IV. 23

Tensor bundles, tensor fields

$$\left.\begin{aligned} T^k T^* M &:= (T^* M)^{\otimes k} = \bigcup_{p \in M} (T_p^* M)^{\otimes k} \\ T^k TM &:= (TM)^{\otimes k} = \bigcup_{p \in M} (T_p M)^{\otimes k} \\ T^{(k,l)} TM &:= (TM)^{\otimes k} \otimes (T^* M)^{\otimes l} \end{aligned}\right\} \text{tensor bundles}$$

Sections of tensor bundles = tensor fields

Special case: $\mathcal{T}^k(M) := \Gamma(T^k T^* M)$

Note $\mathcal{T}^k(M)$ is a $C^\infty(M)$-module:

$A \in \mathcal{T}^k(M)$, $f \in C^\infty(M)$, $(fA)_p = f(p) A_p$.

Now for $A \in \mathcal{T}^k(M)$, $X_1, \dots, X_k \in \mathfrak{X}(M)$ vector fields,

we have $A(X_1, \dots, X_k) : M \longrightarrow \mathbb{R}$

$$p \longmapsto A_p(X_1|_p, \dots, X_k|_p)$$

a smooth function. (Exc)

Thus $A : \mathfrak{X}(M)^{\times k} \longrightarrow C^\infty(M)$ which is $C^\infty(M)$-multilinear

<u>Lemma</u> The assignment $\mathcal{T}^k(M) \longrightarrow \{ C^\infty(M)\text{-multilinear maps } \mathfrak{X}(M)^k \longrightarrow C^\infty(M) \}$

is bijective.

<u>Pf</u> pp. 318-319. $\square$

$$(V^*)^{\otimes k} \xrightarrow{\text{bij}} \{ \text{multilin maps } V^{\times k} \longrightarrow \mathbb{R} \}$$

Pullback of tensor fields

$F: M \longrightarrow N$ smooth

$\alpha \in T^k(T^*_{F(p)} N)$

$$\text{“}(T_p^* M)^{\otimes k} \cong (T_p M^{\otimes k})^*\text{”}$$

Define $dF_p^* \alpha \in T^k(T_p^* M)$ via

$$dF_p^* \alpha (v_1, \dots, v_k) = \alpha(dF_p(v_1), \dots, dF_p(v_k))$$

for any $v_1, \dots, v_k \in T_p M$.

For $A \in \mathcal{T}^k(N)$, define $F^* A \in \mathcal{T}^k(M)$ by

$$(F^* A)_p = dF_p^*(A_{F(p)})$$

If $v_1, \ldots, v_k \in T_pM$, we have

$$(F^*A)_p(v_1, \ldots, v_k) = A_{F(p)}\left(dF_p(v_1), \ldots, dF_p(v_k)\right).$$

Prop $M \xrightarrow{F} N \xrightarrow{G} P$ smooth, A, B covariant tensor fields on N, $f \in C^\infty(N)$ then

(a) $F^*(fB) = (f \circ F)\, F^*B$

(b) $F^*(A \otimes B) = F^*A \otimes F^*B$

(c) $F^*(A + B) = F^*A + F^*B$

(d) F^*B is smooth if B is smooth

(e) $(G \circ F)^* B = F^*(G^*B)$

(f) $\mathrm{id}_N^* B = B$

Now back to linear algebra.....

Tensor & Exterior Algebras

The tensor algebra of a F-vector space V is

$$T(V) := \underbrace{F}_{0} \oplus \underbrace{V}_{1} \oplus \underbrace{V^{\otimes 2}}_{2} \oplus \underbrace{V^{\otimes 3}}_{3} \oplus \cdots$$

$$= \bigoplus_{n \geq 0} V^{\otimes n}$$

with mult'n induced by $(v_1 \otimes \cdots \otimes v_s)(v_{s+1} \otimes \cdots \otimes v_t)$

$$:= v_1 \otimes \cdots \otimes v_t .$$

E.g. $T(F) \cong F[x]$

$T(F^n) =$ "noncommuting polynomials in n variables"

Note $T(V)/(v_1\otimes v_2 - v_2\otimes v_1) = \mathrm{Sym}(V)$ with $\mathrm{Sym}^k(V) \cong$ k-th graded piece of $\mathrm{Sym}(V)$; $\mathrm{Sym}(V) \cong F[x_1, \dots, x_{\dim V}]$.

cold

Defn Let $I = (v\otimes v \mid v \in V) \subseteq T(V)$ be the ideal of $T(V)$ gen'd by elts of the form $v\otimes v$. The *exterior algebra* of V is $\Lambda V := T(V)/I$. For $v_1, \dots, v_k \in V$, set $v_1 \wedge \dots \wedge v_k := v_1 \otimes \dots \otimes v_k + I \in \Lambda V$. Elts of ΛV are linear combos of such terms.

Observe $v \wedge v = 0$ $\xRightarrow{\text{so}}$ $v_1 \wedge \dots \wedge v_k = 0$ if any v_i is repeated or the $v_1, \dots, v_k$ are lin dep.

$$0 = (v_1 + v_2) \wedge (v_1 + v_2)$$

$$= \cancel{v_1 \wedge v_1} + v_1 \wedge v_2 + v_2 \wedge v_1 + \cancel{v_2 \wedge v_2}$$

$$= v_1 \wedge v_2 + v_2 \wedge v_1 \implies \boxed{v_1 \wedge v_2 = - v_2 \wedge v_1}$$

Note $\Lambda^k V$ = degree k terms of $\Lambda V \cong$ old $\Lambda^k V$.

(better defn!

Universal property

$$\begin{tikzcd}
V^{\times k} \arrow[r, "\text{alternating multilinear}"] \arrow[d] & U \\
\Lambda^k V \arrow[ur, dashed, "\exists!\text{ linear map}"'] &
\end{tikzcd}$$

Exc Show

$$A(\dots, v, \dots, w, \dots) = -A(\dots, w, \dots, v, \dots)$$

$\Longrightarrow A(\dots, v, \dots, v, \dots) = 0$

char $\neq 2$

Prop If $e_1, \dots, e_n$ is a basis for V, then

$$\{ e_{i_1} \wedge \cdots \wedge e_{i_k} \mid 1 \le i_1 < i_2 < \cdots < i_k \le n \} \text{ is a basis of } \Lambda^k V.$$

Thus $\dim \Lambda^k V = \binom{n}{k}$ (in particular, $\dim \Lambda^n V = 1$, and $\Lambda^k V = 0$ for $k > n$).

Note: $\Longrightarrow$ $\dim \Lambda V = \sum_{k=0}^{n} \binom{n}{k} = 2^n$

E.g. For $\mathbb{R}^3$,

$$\Lambda^0 \mathbb{R}^3 = \mathbb{R}\{1\}$$

$$\Lambda^1 \mathbb{R}^3 = \mathbb{R}\{e_1, e_2, e_3\}$$

$$\Lambda^2 \mathbb{R}^3 = \mathbb{R}\{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\}.$$

$$\Lambda^3 \mathbb{R}^3 = \mathbb{R}\{e_1 \wedge e_2 \wedge e_3\}$$

Pf of Prop Spanning follows from universal property: reordering wedge factors changes sign at worst.

For linear independence, we seek an alternating k-multilinear map $V^{\times k} \longrightarrow F$ taking the value 1 on a given $e_{i_1} \wedge \cdots \wedge e_{i_k}$ and 0 on all others.

Observe that $T^k(V)$ has basis $\{e_{j_1} \otimes \cdots \otimes e_{j_k} \mid 1 \leq j_1, \dots, j_k \leq n\}$ so we may define

$$\begin{tikzcd} V^{\times k} \arrow[r] \arrow[d] & F \\ T^k(V) \arrow[ur] & \end{tikzcd}$$

$$e_{j_1} \otimes \cdots \otimes e_{j_k} \longmapsto \begin{cases} \operatorname{sgn}(\sigma) & \text{if } \exists \sigma \in \mathfrak{S}_k \text{ with } (e_{j_1} \otimes \cdots \otimes e_{j_k})^\sigma = e_{i_1} \otimes \cdots \otimes e_{i_k} \\ 0 & \text{o/w} \end{cases}$$

Exc Check that the composite $V^{\times k} \longrightarrow F$ is alternating. $\square$

TPS For $\omega \in \Lambda^k V$, $\eta \in \Lambda^l V$, which sign appears in $\omega \wedge \eta = \pm \eta \wedge \omega$?

Hint Start with $k=1$ and think about $v \wedge w_1 \wedge \cdots \wedge w_l$ vs $w_1 \wedge \cdots \wedge w_l \wedge v$.

$$= (-1)^l \, v \wedge w_1 \wedge \cdots \wedge w_l$$

$$v_1 \wedge \cdots \wedge v_k \wedge w_1 \wedge \cdots \wedge w_l$$

$$\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$$ — graded commutative

<u>Functoriality</u> T^k, Sym^k, and Λ^k are functorial in linear transformations:

$$\varphi : V \longrightarrow W$$

$$\rightsquigarrow$$

$$T^k(\varphi) : v_1 \otimes \cdots \otimes v_k \longmapsto \varphi(v_1) \otimes \cdots \otimes \varphi(v_k)$$

$$\mathrm{Sym}^k(\varphi) \qquad \Lambda^k(\varphi)$$

These assemble to give graded F-algebra homomorphisms

$$T(\varphi) : TV \longrightarrow TW$$

$$\mathrm{Sym}(\varphi) : \mathrm{Sym}(V) \longrightarrow \mathrm{Sym}(W)$$

$$\Lambda\varphi : \Lambda V \longrightarrow \Lambda W$$

Suppose $\dim V = n$ and $\varphi : V \longrightarrow V$ is a linear endomorphism.

Then $\Lambda^n \varphi (e_1 \wedge \cdots \wedge e_n) = \varphi(e_1) \wedge \cdots \wedge \varphi(e_n) \in \Lambda^n V$

$$= D(\varphi)\, e_1 \wedge \cdots \wedge e_n$$

for some $D(\varphi) \in F$. Now $\mathrm{End}(V) \cong V^n$ so this induces

matrix rep'n of endos

$$D : V^n \longrightarrow F$$

Then D is alternating, multilinear, and $D(e_1, \dots, e_n) = D(\mathrm{id}) = 1$.

By univ property of determinant, $D = \det$!

<u>Prop</u> For φ as above, $\Lambda^n \varphi(w) = \det(\varphi) w \quad \forall w \in \Lambda^n V$. $\square$

<u>Interior</u> <u>mult'n</u>

For $v \in V$,
$$i_v : \Lambda^k(V^*) \longrightarrow \Lambda^{k-1}(V^*)$$
$$\varphi_1 \wedge \cdots \wedge \varphi_k \longmapsto \varphi_1(v)\, \varphi_2 \wedge \cdots \wedge \varphi_k$$
is called

<u>interior</u> <u>multiplication</u> by v. For $\omega \in \Lambda^k(V^*)$, write

$v \lrcorner \omega := i_v \omega$ — read "v into ω."

Lemma For V finite dim'l,

(a) $i_v \circ i_v = 0$

(b) $\omega \in \Lambda^k V^*$, $\eta \in \Lambda^l V^*$ then

$$i_v(\omega \wedge \eta) = (i_v \omega) \wedge \eta + (-1)^k \omega \wedge (i_v \eta).$$

Pf pp. 358-359 $\square$