Tensors

24. IV. 23

Defn (physicists) A **tensor** is anything that transforms like a tensor.

Defn (mathematicians) A **tensor** is a representing object for bilinear transformations.

Suppose V, W are k-vector spaces, k a field.
(If you wish, pretend $k = \mathbb{R}$ throughout. Or pretend k is a commutative ring and V, W are k-modules!)

For U another k-vs, a map $V \times W \xrightarrow{B} U$ is **bilinear** when

- $B(\lambda v_1 + v_2, w) = \lambda B(v_1, w) + B(v_2, w)$
- $B(v, \lambda w_1 + w_2) = \lambda B(v, w_1) + B(v, w_2)$

i.e. B is linear in each variable.

The job of $V \otimes W = V \otimes_k W$ is to turn bilinear maps into linear maps:

(⊗)
$$\begin{tikzcd} V \times W \arrow[r, "B"] \arrow[d] & U \\ V \otimes W \arrow[ur, dashed, "\tilde{B}"'] & \end{tikzcd}$$

$\forall$ bilinear $B : V \times W \longrightarrow U$
$\exists!$ linear $\tilde{B} : V \otimes W \longrightarrow U$
making the diagram commute.

We now construct $V \otimes W$ and show it satisfies the universal property (⊗).

Free vector spaces

Given a set S, the free k-vector space on S is

$$k\cdot S = \left\{ f : S \to k \;\middle|\; \begin{array}{l} f \text{ a function}, \\ f(s) = 0 \text{ for all} \\ \text{but finitely many } s\in S \end{array} \right\}$$

$$(f+g)(s) = f(s) + g(s)$$

$$(\lambda f)(s) = \lambda f(s)$$

$$f \longmapsto \sum_{s\in S} f(s)\, s$$

$$\cong \left\{ \begin{array}{l} \text{formal } k\text{-linear combinations of} \\ \text{elts of } S,\ \sum_{s\in S} \lambda_s s \end{array} \;\middle|\; \begin{array}{l} \lambda_s \in k \text{ is } 0 \text{ for} \\ \text{all but finitely many} \\ s\in S \end{array} \right\}$$

For any function $F : S \longrightarrow V$ to a k-vs V, $\exists!\ \tilde{F} : k\cdot S \longrightarrow V$ linear such that

$$\begin{tikzcd} S \arrow[r, "F"] \arrow[d] & V \\ k\cdot S \arrow[ur, dashed, "\tilde{F}"'] & \end{tikzcd}$$

$$s \longmapsto t : \delta_s \longmapsto \begin{cases} 1 & t = s \\ 0 & t \neq s \end{cases}$$

$$f \longmapsto \sum_{s\in S} f(s)\, F(s)$$

$\square$

<u>Note</u>

- $k\cdot S$ has basis S

- $k\cdot S$ is functorial in S and is part of the "free-forgetful" adjunction $\quad F : \mathsf{Set} \rightleftarrows \mathsf{Vect}_k : U$

$$\begin{tikzcd}
S \arrow[r, mapsto] \arrow[d, "g"'] \arrow[dr, phantom, "\longmapsto"] & k\cdot S \arrow[d] & \sum \lambda_s s \arrow[d, mapsto] \\
T \arrow[r, mapsto] & k\cdot T & \sum \lambda_s g(s)
\end{tikzcd}$$

In particular, $\mathsf{Vect}_k(k\cdot S, V) \cong \mathsf{Set}(S, \underbrace{U(V)}_{\text{set underlying } V})$.

Tensor products

Let $R \subseteq k\cdot(V\times W)$ be the subspace spanned by

$$\left.\begin{array}{l} \bullet\ (\lambda v_1 + v_2, w) - \lambda(v_1, w) - (v_2, w) \\[1em] \bullet\ (v, \lambda w_1 + w_2) - \lambda(v, w_1) - (v, w_2) \end{array}\right\} \text{formal sums}$$

$\forall \lambda \in k$, $v, v_i \in V$, $w, w_i \in W$.

Look a lot like bilinearity conditions...

Then $V \otimes W := k \cdot (V \times W) / R$ is the **tensor product** of V, W.

Given, $v \in V$, $w \in W$, define the **simple tensor**

$$v \otimes w := (v, w) + R \in V \otimes W.$$

$V \otimes W$ is spanned by simple tensors; general elements are k-linear combinations of simple tensors.

Note

- $(\lambda v_1 + v_2) \otimes w = \lambda (v_1 \otimes w) + (v_2 \otimes w)$
- $v \otimes (\lambda w_1 + w_2) = \lambda (v \otimes w_1) + (v \otimes w_2)$ (*)

<u>Thm</u> $V \otimes W$ satisfies universal property $(\otimes)$.

<u>Pf</u> Suppose $B : V \times W \longrightarrow U$ is bilinear. By the universal property of free vector spaces, we get an extension

$$\begin{tikzcd} V \times W \arrow[r, "B"] \arrow[d] & U & \sum \lambda_{(v,w)} B(v,w) \\ k \cdot (V \times W) \arrow[ur, "\tilde{B}"'] & & \sum \lambda_{(v,w)} (v,w) \arrow[u, mapsto] \end{tikzcd}$$

By bilinearity of B, $R \subseteq \ker \tilde{B}$, so $\tilde{B}$ descends to the quotient $V \otimes W = k \cdot (V \times W) / R$:

$$\begin{tikzcd} V \times W \arrow[rr, "B"] \arrow[d] & & U \\ {} & {} \arrow[ur, "\tilde{B}"] & {} \arrow[u] \end{tikzcd}$$

$$\begin{tikzcd} k\cdot(V\times W)' \arrow[d] \\ V\otimes W \arrow[ur, "\tilde{\tilde{B}}"'] \end{tikzcd}$$

The diagram forces $\tilde{\tilde{B}}(v\otimes w) = B(v,w)$, so $\tilde{\tilde{B}}$ is the unique linear map making

$$\begin{tikzcd} V\times W \arrow[r, "B"] \arrow[d] & U \\ V\otimes W \arrow[ur, "\tilde{\tilde{B}}"'] & \end{tikzcd}$$

commute. $\square$

Exc If $V\times W \longrightarrow V\tilde{\otimes} W$ is some other map satisfying $(\otimes)$

then $\exists!$ iso

$$\begin{tikzcd} & V\times W \arrow[dl] \arrow[dr] & \\ V\otimes W \arrow[rr, dashed, "\exists!\cong"'] & & V\tilde{\otimes} W \end{tikzcd}$$

Exc Compute $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$

Prop If V has basis $v^1, \dots, v^m$ and W has basis $w^1, \dots, w^n$, then $V \otimes W$ has basis $\{v^i \otimes w^j \mid 1 \le i \le m,\ 1 \le j \le n\}$. In particular,

$$\dim V \otimes W = (\dim V)(\dim W).$$

(Only for k a field.)

Pf We have already noted that simple tensors span, so the rules (*) imply that the $v^i \otimes w^j$ span.

For linear independence, take $\{\varphi^i\}$ dual to $\{v^i\}$,
$\{\psi^j\}$ dual to $\{w^j\}$

basis of V^*, W^*, respectively.

Now define $\eta_{ij} : V \otimes W \longrightarrow k$ by univ property $(\otimes)$:

$$(v,w) \longmapsto \varphi^i(v) \cdot \psi^j(w) \quad \text{(Check this is bilinear.)}$$

$$\begin{tikzcd}
V \times W \arrow[r] \arrow[d] & k \\
V \otimes W \arrow[ur, "\eta_{ij} =: \varphi^i \otimes \psi^j"'] &
\end{tikzcd}$$

Then $\eta_{ij}(v^k \otimes w^l) = \begin{cases} 1 & \text{if } i=k, j=l \\ 0 & \text{o/w}. \end{cases}$

Applying η_{ij} to an expression $\sum \lambda_{kl} v^k \otimes v^l$ reveals that this is 0 iff $\lambda_{ij} = 0 \ \forall i,j$, so $\{v^i \otimes w^j\}$ is lin ind. $\square$

<u>Fact</u> $\otimes$ is naturally associative : $U \otimes (V \otimes W) \cong (U \otimes V) \otimes W$ since

$u \otimes (v \otimes w) \longmapsto (u \otimes v) \otimes w$

both objects represent trilinear maps out of $U \times V \times W$.

$k \in \mathbb{N}$ now.

Terminology A covariant k-tensor on V is an element of

$$\underbrace{V^* \otimes \cdots \otimes V^*}_{k \text{ times}} = (V^*)^{\otimes k}.$$

Equivalent data:
- multilinear $V^{\times k} \longrightarrow \mathbb{R}$
- element of $(\underbrace{V \otimes \cdots \otimes V}_{k \text{ times}})^*$

E.g. det as a function of n vectors

dim $V = n$

$\det : V^{\times n} \longrightarrow \mathbb{R}$

$\cap$

$(V^*)^{\otimes n}$

A contravariant k-tensor is an element of $V^{\otimes k}$

A mixed tensor of type (k, l) is an element of

$$T^{(k,l)}(V) := V^{\otimes k} \otimes (V^*)^{\otimes l}.$$

More flavors

\mathfrak{S}_k — symm gp

- Symmetric tensors: $V^{\otimes k} \circlearrowleft \mathfrak{S}_k$ via $(v_1 \otimes \cdots \otimes v_k)^\sigma = v_{\sigma 1} \otimes \cdots \otimes v_{\sigma k}$

Fixed points are symmetric tensors $\mathrm{Sym}^k(V) := (V^{\otimes k})^{\mathfrak{S}_k}$

$$= \{ x \in V^{\otimes k} \mid x^\sigma = x \ \forall \sigma \in \mathfrak{S}_k \}$$

This is a subspace of $V^{\otimes k}$ admitting a natural projection map

$$\mathrm{sym} : V^{\otimes k} \longrightarrow \mathrm{Sym}^k V$$
$$x \longmapsto \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} x^\sigma$$

$\mathrm{Sym}^k(\mathbb{R}\{x_1, \ldots, x_n\})$
$\cong$ homogeneous deg k polynomials in $x_1, \ldots, x_n$

- Alternating tensors

Can also ask that $x^\sigma = \mathrm{sgn}(\sigma)\, x \quad \forall \sigma \in \mathfrak{S}_k$ to form the <u>k-th</u> <u>alternating</u> <u>power</u> of V, $\Lambda^k V \subseteq V^{\otimes k}$.

Equivalently, $x \in \Lambda^k V \iff x^\sigma = -x \quad \forall$ transposition $\sigma \in \mathfrak{S}_k$.

Alternation map

$$\begin{aligned} \mathrm{alt}: V^{\otimes k} &\longrightarrow \Lambda^k V \\ x &\longmapsto \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \mathrm{sgn}(\sigma)\, x^\sigma . \end{aligned}$$

We will present the <u>algebra</u> of alternating tensors later.