

# Tensors

Defn (physicists) A tensor is anything that transforms like a tensor.

Defn (mathematicians) A tensor is a representing object for bilinear transformations.

Suppose  $V, W$  are  $k$ -vector spaces,  $k$  a field.

(If you wish, pretend  $k = \mathbb{R}$  throughout. Or pretend  $k$  is a commutative ring and  $V, W$  are  $k$ -modules!)

For  $U$  another  $k$ -vs, a map  $V \times W \xrightarrow{B} U$  is bilinear when

- $B(\lambda v_1 + v_2, w) = \lambda B(v_1, w) + B(v_2, w)$
- $B(v, \lambda w_1 + w_2) = \lambda B(v, w_1) + B(v, w_2)$

i.e.  $B$  is linear in each variable

The job of  $V \otimes W = V \otimes_k W$  is to turn bilinear maps into linear

maps:

$$\begin{array}{ccc} V \times W & \xrightarrow{B} & U \\ \downarrow & \nearrow \tilde{B} & \\ V \otimes W & & \end{array}$$

(\*)

$\forall$  bilinear  $B: V \times W \rightarrow U$

$\exists!$  linear  $\tilde{B}: V \otimes W \rightarrow U$

making the diagram commute.

We now construct  $V \otimes W$  and show it satisfies the universal property (\*).

Free vector spaces

Given a set  $S$ , the free  $k$ -vector space on  $S$  is

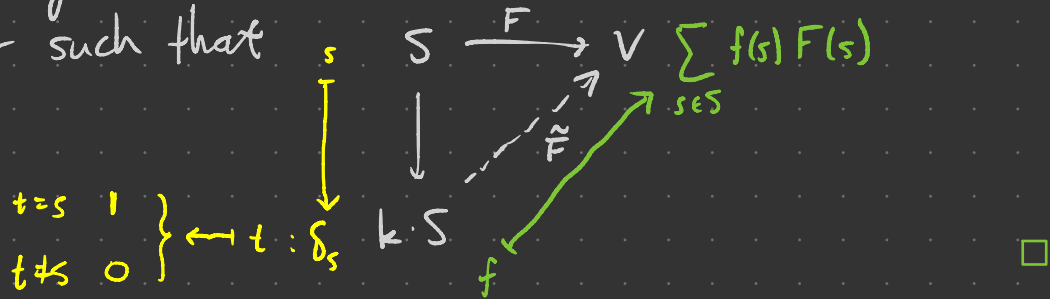
$f$   
 $\downarrow$   
 $\sum_{s \in S} f(s)s$

$k \cdot S = \left\{ f: S \rightarrow k \mid \begin{array}{l} f \text{ a function,} \\ f(s) = 0 \text{ for all} \\ \text{but finitely many } s \in S \end{array} \right\}$

$\approx \left\{ \begin{array}{l} \text{formal } k\text{-linear combinations of} \\ \text{elts of } S, \sum_{s \in S} \lambda_s s \end{array} \mid \begin{array}{l} \lambda_s \in k \text{ is } 0 \text{ for} \\ \text{all but finitely many} \\ s \in S \end{array} \right\}$

$(f+g)(s) = f(s) + g(s)$   
 $(\lambda f)(s) = \lambda f(s)$

For any function  $F: S \rightarrow V$  to a  $k$ -vs  $V$ ,  $\exists!$   $\tilde{F}: k \cdot S \rightarrow V$  linear such that



Note •  $k \cdot S$  has basis  $S$

- $k \cdot S$  is functorial in  $S$  and is part of the "free-forgetful" adjunction  $F: \text{Set} \rightleftarrows \text{Vect}_k: U$

$$\begin{array}{ccc}
 S & \xrightarrow{\quad} & k \cdot S & \sum \lambda_s s \\
 g \downarrow & \xrightarrow{\quad} & \downarrow & \downarrow \\
 T & \xrightarrow{\quad} & k \cdot T & \sum \lambda_s g(s)
 \end{array}$$

In particular,  $\text{Vect}_k(k \cdot S, V) \cong \text{Set}(S, \underbrace{U(V)}_{\text{set underlying } V})$ .

### Tensor products

Let  $R \subseteq k \cdot (V \times W)$  be the subspace spanned by

- $(\lambda v_1 + v_2, w) - \lambda(v_1, w) - (v_2, w)$
  - $(v, \lambda w_1 + w_2) - \lambda(v, w_1) - (v, w_2)$
- } formal sums

$$\forall \lambda \in k, v, v_i \in V, w, w_i \in W.$$

Look a lot like  
bilinearity conditions...

Then  $V \otimes W := k \cdot (V \times W) / \mathcal{R}$  is the tensor product of  $V, W$ .

Given,  $v \in V, w \in W$ , define the simple tensor

$$v \otimes w := (v, w) + \mathcal{R} \in V \otimes W.$$

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$V \otimes W$  is spanned by simple tensors; general elements are  $k$ -linear combinations of simple tensors.

Note

$$\bullet (\lambda v_1 + v_2) \otimes w = \lambda (v_1 \otimes w) + (v_2 \otimes w)$$

$$\bullet v \otimes (\lambda w_1 + w_2) = \lambda (v \otimes w_1) + (v \otimes w_2)$$

(\*)

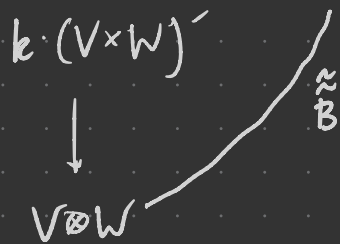
Thm  $V \otimes W$  satisfies universal property  $(*)$ .

Pf Suppose  $B: V \times W \rightarrow U$  is bilinear. By the universal property of free vector spaces, we get an extension

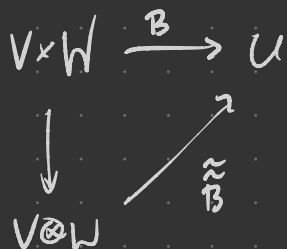
$$\begin{array}{ccc} V \times W & \xrightarrow{B} & U \\ \downarrow & \nearrow \tilde{B} & \\ k \cdot (V \times W) & & \end{array} \quad \begin{array}{l} \sum \lambda_{(v,w)} B(v,w) \\ \sum \lambda_{(v,w)} (v,w) \end{array}$$

By bilinearity of  $B$ ,  $R \in \ker \tilde{B}$ , so  $\tilde{B}$  descends to the quotient  $V \otimes W = k \cdot (V \times W) / R$ :

$$\begin{array}{ccc} V \times W & \xrightarrow{B} & U \\ \downarrow & \nearrow \tilde{B} & \end{array}$$

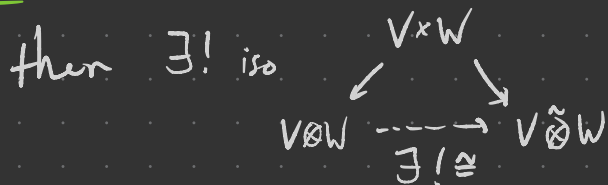


The diagram forces  $\tilde{B}(v \otimes w) = B(v, w)$ , so  $\tilde{B}$  is the unique linear map making



commute.  $\square$

Exc If  $V \times W \rightarrow V \tilde{\otimes} W$  is some other map satisfying  $(\otimes)$



Exc Compute  $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$

Prop If  $V$  has basis  $v^1, \dots, v^m$  and  $W$  has basis  $w^1, \dots, w^n$ ,

then  $V \otimes W$  has basis  $\{v^i \otimes w^j \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ . In particular,

↗  $\dim V \otimes W = (\dim V)(\dim W)$

(Only for  $k$  a field.)

Pf We have already noted that simple tensors span, so the rules (\*) imply that the  $v^i \otimes w^j$  span.

For linear independence, take  $\{\varphi^i\}$  dual to  $\{v^i\}$ ,  
 $\{\psi^j\}$  dual to  $\{w^j\}$

basis of  $V^*$ ,  $W^*$ , respectively.



Now define  $\eta_{ij} : V \otimes W \rightarrow k$  by univ property  $(\otimes)$ :

$$\begin{array}{ccc} (v, w) & \mapsto & \varphi^i(v) \cdot \psi^j(w) \\ V \times W & \longrightarrow & k \end{array} \quad \left. \vphantom{\begin{array}{ccc} (v, w) & \mapsto & \varphi^i(v) \cdot \psi^j(w) \\ V \times W & \longrightarrow & k \end{array}} \right\} \text{(check this is bilinear.)}$$

$$\begin{array}{ccc} & & \nearrow \eta_{ij} =: \varphi^i \otimes \psi^j \\ \downarrow & & \\ V \otimes W & & \end{array}$$

$$\text{Then } \eta_{ij}(v^k \otimes w^d) = \begin{cases} 1 & \text{if } i=k, j=d \\ 0 & \text{o/w.} \end{cases}$$

Applying  $\eta_{ij}$  to an expression  $\sum \lambda_{kd} v^k \otimes w^d$  reveals that this is 0 iff  $\lambda_{ij} = 0 \forall i, j$ , so  $\{v^i \otimes w^j\}$  is lin ind.  $\square$

Fact  $\otimes$  is naturally associative:  $U \otimes (V \otimes W) \cong (U \otimes V) \otimes W$  since  $u \otimes (v \otimes w) \mapsto (u \otimes v) \otimes w$

both objects represent trilinear maps out of  $U \times V \times W$ .

$k \in \mathbb{N}$  now.

Terminology A covariant k-tensor on  $V$  is an element of

$$\underbrace{V^* \otimes \dots \otimes V^*}_{k \text{ times}} = (V^*)^{\otimes k}$$

Equivalent data: • multilinear  $V^{\times k} \rightarrow \mathbb{R}$

• element of  $\underbrace{(V \otimes \dots \otimes V)^*}_{k \text{ times}}$

E.g. det as a function of  $n$  vectors.

$$\dim V = n$$

A contravariant k-tensor is an element of  $V^{\otimes k}$ .

A mixed tensor of type  $(k, d)$  is an element of

$$\det: V^{\times n} \rightarrow \mathbb{R}$$
$$\prod_{i=1}^n (V^*)^{\otimes n}$$

$$T^{(k,l)}(V) := V^{\otimes k} \otimes (V^*)^{\otimes l}$$

Mora flavors

$\mathfrak{S}_k$  — symm gp

• Symmetric tensors:  $V^{\otimes k} \supset \mathfrak{S}_k$  via  $(v_1 \otimes \dots \otimes v_k)^\sigma = v_{\sigma_1} \otimes \dots \otimes v_{\sigma_k}$

Fixed points are symmetric tensors  $\text{Sym}^k(V) := (V^{\otimes k})^{\mathfrak{S}_k}$

$$= \{x \in V^{\otimes k} \mid x^\sigma = x \forall \sigma \in \mathfrak{S}_k\}$$

This is a subspace of  $V^{\otimes k}$  admitting a natural projection map

$$\begin{aligned} \text{sym} : V^{\otimes k} &\longrightarrow \text{Sym}^k V \\ x &\longmapsto \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} x^\sigma \end{aligned}$$

$\text{Sym}^k(\mathbb{R}\{x_1, \dots, x_n\})$   
 $\cong$  homogeneous deg  $k$   
 polynomials in  
 $x_1, \dots, x_n$

- Alternating tensors

Can also ask that  $x^\sigma = \text{sgn}(\sigma)x \quad \forall \sigma \in \mathfrak{S}_k$  to form the  $k$ -th alternating power of  $V$ ,  $\Lambda^k V \subseteq V^{\otimes k}$

Equivalently,  $x \in \Lambda^k V \iff x^\sigma = -x \quad \forall$  transposition  $\sigma \in \mathfrak{S}_k$ .

$$\begin{aligned} \text{Alternation map } \text{alt}: V^{\otimes k} &\longrightarrow \Lambda^k V \\ x &\longmapsto \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \text{sgn}(\sigma) x^\sigma \end{aligned}$$

We will present the algebra of alternating tensors later.