21. IV. 23

Conservative Covector Fields

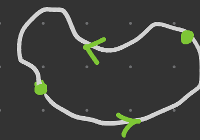
Defn • $\omega \in \mathfrak{X}^*(M)$ is *exact* when $\exists f \in C^\infty(M)$ s.t. $df = \omega$. In this case, call f a *potential* for ω (unique up to adding a locally constant function).

• Call ω *conservative* when $\forall$ pw smooth closed curve γ,

$$\int_\gamma \omega = 0.$$

By FTLI, exact $\Rightarrow$ conservative.

Prop ω is conservative iff its line integrals are path-independent

Pf  vs  $\square$

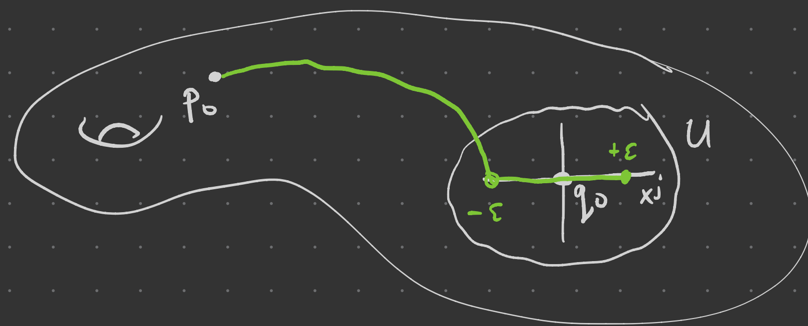
Thm $\omega \in \mathfrak{X}^*(M)$ is exact iff it's conservative.

Pf $\Rightarrow$ ✓ (FTLI)

$\Leftarrow$ Assume M conn'd at first so $\exists$ pw smooth path $p \rightsquigarrow q$ $\forall p, q \in M$. Write $\int_p^q \omega$ for $\int_\gamma \omega$, γ any pw smooth path $p \rightsquigarrow q$. This is well-defined b/c ω is conservative. Now fix $p_0 \in M$ and define

$$\begin{aligned} f : M &\longrightarrow \mathbb{R} \\ q &\longmapsto \int_{p_0}^{q} \omega . \end{aligned}$$

WTS: f smooth and $df = \omega$.



Take $q_0 \in M$, $(U, (x^i))$ smooth chart at q_0. Need to show

$$\frac{\partial f}{\partial x^j}(q_0) = \omega_j(q_0), \quad j = 1, \dots, n$$

(for $\omega = \sum \omega_i dx_i$ in local coords)

to conclude $df_{q_0} = \omega_{q_0}$.

Take $\gamma : [-\varepsilon, \varepsilon] \longrightarrow U$, set $p_1 = \gamma(-\varepsilon)$. Now define

$$t \longmapsto t e_j$$

$$\tilde{f} : M \longrightarrow \mathbb{R}$$
$$q \longmapsto \int_{p_1}^{q} \omega$$

and note $f(q) - \tilde{f}(q) = \int_{p_0}^{q} \omega - \int_{p_1}^{q} \omega$

$$= \int_{p_0}^{q} \omega + \int_{q}^{p_1} \omega$$

$$= \int_{p_0}^{p_1} \omega \quad \text{is constant.}$$

Thus it suffices to show

$$\frac{\partial \tilde{f}}{\partial x^j}(q_0) = \omega_j(q_0).$$

Have $\gamma'(t) = \left.\frac{\partial}{\partial x^j}\right|_{\gamma(t)}$ by construction

$$\Longrightarrow \quad \omega_{\gamma(t)}(\gamma'(t)) = \sum_i \omega_i(\gamma(t))\, dx_i \left(\left.\frac{\partial}{\partial x^j}\right|_{\gamma(t)} \right)$$

$$= \omega_j(\gamma(t))$$

Further, $\tilde{f} \circ \gamma(t) = \int_{p_1}^{\gamma(t)} \omega = \int_{-\varepsilon}^{t} \omega_{\gamma(s)}(\gamma'(s))\, ds = \int_{-\varepsilon}^{t} \omega_j(\gamma(s))\, ds$

Thus $\frac{\partial \tilde{f}}{\partial x^j}(q_0) = \gamma'(0)\tilde{f} = \left.\frac{d}{dt}\right|_{t=0} \tilde{f} \circ \gamma(t)$

$$= \left.\frac{d}{dt}\right|_{t=0} \int_{-\varepsilon}^{t} \omega_j(\gamma(s))\, ds = \omega_j(\gamma(0)) = \omega_j(q_0).$$

To do:

- boundary points (p. 294)
- $|\pi_0 M| > 1$

$\square$

Not every covector field is exact!

E.g. $\omega = \dfrac{x\,dy - y\,dx}{x^2+y^2} \in \mathfrak{X}^*(\mathbb{R}^2 \setminus 0)$ has

$$\int_{S^1} \omega = 2\pi \neq 0.$$

A simple obstruction to exactness:

If $\omega = df$ then in local coords (x^i), $\omega_i = \dfrac{\partial f}{\partial x^i}$

$$\Rightarrow \frac{\partial \omega_i}{\partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i} \underset{f \text{ smooth}}{=} \frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial \omega_j}{\partial x^i}.$$

Defn Call $\omega \in \mathfrak{X}^*(M)$ **closed** when $\forall$ smooth local coords (x^i),

$$\circledast \quad \frac{\partial \omega_i}{\partial x^j} = \frac{\partial \omega_j}{\partial x^i}.$$

Prop Exact $\Rightarrow$ closed. $\square$

Checking $\frac{\partial \omega_j}{\partial x^i} = \frac{\partial \omega_i}{\partial x^j}$ on all local coords sounds hard. But:

Prop TFAE:

(a) ω is closed

(b) ω satisfies $\circledast$ in some smooth chart around each point

(c) $\forall$ open $U \subseteq M$, $V, W \in \mathfrak{X}(U)$,

$$X(\omega(Y)) - Y(\omega(X)) = \omega([X, Y]).$$

Pf (a) $\Rightarrow$ (b) ✓

(b) $\Rightarrow$ (c) In local coords, $\omega = \sum \omega_i dx^i$, $X = \sum X_i \frac{\partial}{\partial x^i}$, $Y = \sum Y_i \frac{\partial}{\partial x^i}$

so $X(\omega(Y)) = X\left(\sum \omega^i Y_i\right) = \sum Y_i X\omega_i + \omega_i XY_i$

$$= \sum_i \left(Y_i \sum_j \left(X_j \frac{\partial \omega_i}{\partial x^j}\right) + \omega_i XY_i \right)$$

$$Y(\omega(X)) = \sum_i \left(X_i \sum_j \left[Y_j \frac{\partial \omega_i}{\partial x^j}\right] + \omega_i YX_i \right)$$

whence $X(\omega(Y)) - Y(\omega(X)) = \sum \omega_i (XY_i - YX_i)$

$$= \omega([X, Y]).$$

(c) $\Rightarrow$ (a) $X = \frac{\partial}{\partial x^i}$, $Y = \frac{\partial}{\partial x^j}$ + $\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0$

gives ⊛. $\square$

Closed $\not\Rightarrow$ exact in general. $\qquad \frac{\partial \omega_x}{\partial y} = \frac{\partial \omega_y}{\partial x}$

E.g. $\omega = \frac{x\,dy - y\,dx}{x^2+y^2} \in \mathfrak{X}^*(\mathbb{R}^2 \setminus 0)$ is closed but not exact.

The failure of closed $\Rightarrow$ exact is related to the "hole" in $\mathbb{R}^2 \setminus 0$.

Call $U \subseteq \mathbb{R}^n$ star-shaped when $\exists p \in U$ s.t. line segment $p + q$ is a subset of U $\forall q \in U$.

Poincaré Lemma for Covector Fields on Star-shaped domains:

If $U \subseteq \mathbb{R}^n$ or $\mathbb{H}^n$ is open, star-shaped, then every closed vector field on U is exact.

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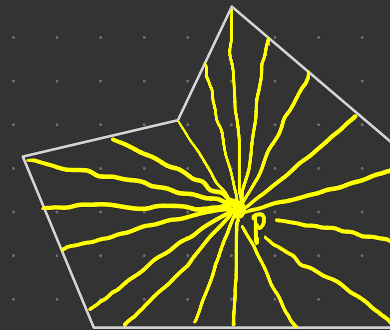

With de Rham cohomology, we'll build far more powerful answers to the closed $\overset{?}{\Rightarrow}$ exact question.