19. IV. 23

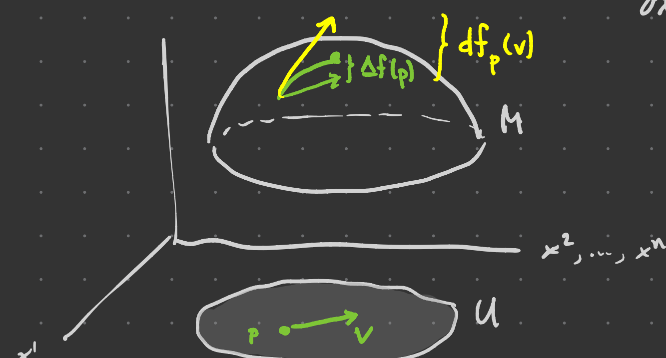
Recall $T^*M = (TM)^*$ is the cotangent bundle

$\mathfrak{X}^*(M) =$ covector fields $=$ sections of T^*M

$$d : C^\infty(M) \longrightarrow \mathfrak{X}^*(M)$$

$$f \longmapsto \begin{array}{c} T^*M \\ \downarrow \uparrow df \\ M \end{array} \qquad \begin{array}{c} df_p \\ \uparrow \\ p \end{array} : T_pM \longrightarrow \mathbb{R}$$
$$v \longmapsto vf$$

In local coords (x^i), $df = \sum \frac{\partial f}{\partial x^i} dx^i$.



Prop For $f \in C^\infty(M)$, $df = 0$ iff f is $\underbrace{\text{locally constant}}_{\text{i.e. constant on conn'd components}}$

Pf $\Leftarrow$ ✓

$\Rightarrow$ WLOG, assume M conn'd and $df = 0$. WTS f constant.

For $p \in M$, let $C = \{q \in M \mid f(q) = f(p)\}$. If $q \in C$, let U be a smooth coordinate ball centered at q. Then $\frac{\partial f}{\partial x^i} = 0$ on U $\forall i$.

By calculus, f constant on U. Thus C is open. By continuity

$C = \underbrace{f^{-1}\{f(p)\}}_{\text{closed}}$

of f, C is closed. $p \in C \neq \emptyset$, M conn'd $\Rightarrow C = M$. $\square$

Prop $\gamma : J \to M$ smooth curve, $f : M \to \mathbb{R}$ smooth then

$$(f \circ \gamma)'(t) = df_{\gamma(t)}(\gamma'(t)).$$

Pf Compute $df_{\gamma(t_0)}(\gamma'(t_0)) = \gamma'(t_0) f$

$$= d\gamma_{t_0}\left(\frac{d}{dt}\Big|_{t_0}\right) f$$

$$= \frac{d}{dt}\Big|_{t_0} (f \circ \gamma)$$

$$= (f \circ \gamma)'(t_0). \quad \square$$

Pullback of covector fields

$F : M \longrightarrow N$ smooth, $p \in M$

$dF_p : T_pM \longrightarrow T_{F(p)}N$ has dual $dF_p^* : T^*_{F(p)}N \longrightarrow T^*_pM$

the pullback by F at p / cotangent map of F.

We get the following commutative diagram:

$$\begin{tikzcd}
T^*M \arrow[d] & T^*N \arrow[l, "dF^*"'] \arrow[d] \\
M \arrow[r, "F"'] \arrow[u, bend left, dashed, "F^*\omega"] & N \arrow[u, bend right, dashed, "\omega"']
\end{tikzcd}$$

Given a covector field $\omega \in \mathfrak{X}^*(N)$ define $F^*\omega \in \mathfrak{X}^*(M)$ by

$$(F^*\omega)_p = dF_p^*(\omega_{F(p)}).$$

Thus $(F^*\omega)_p(v) = \omega_{F(p)}(dF_p(v))$.

$\mathfrak{X}^*$ is functorial $\mathrm{Diff}^{op} \longrightarrow \mathrm{Vect}_{\mathbb{R}}$

$$\begin{tikzcd}
M \arrow[d, "F"'] \arrow[r, mapsto] & \mathfrak{X}^*(M) \\
N \arrow[r, mapsto] \arrow[ur, phantom, "\longmapsto"] & \mathfrak{X}^*(N) \arrow[u, "F^*"']
\end{tikzcd}$$

Prop $F: M \longrightarrow N$ smooth, $u \in C^\infty(N)$, $\omega \in \mathfrak{X}^*(N)$ then

$$F^*(u\omega) = (u \circ F) F^*\omega \quad \text{and} \quad F^* du = d(u \circ F).$$

Pf Compute. $\square$

In particular, $F^*\omega = F^*\left(\sum \omega_j dy^j\right) = \sum (\omega_j \circ F) F^* dy^j$

$$= \sum (\omega_j \circ F)\, d(y^j \circ F)$$

gives a coordinate expression for the F pullback of ω.

E.g. $F: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$, $\omega = u\,dv + v\,du$. then

$(x,y,z) \longmapsto (u,v) = (x^2 y, y \sin z)$

$$F^*\omega = (u \circ F)\, d(v \circ F) + (v \circ F)\, d(u \circ F)$$

$$= (x^2 y)\, d(y \sin z) + (y \sin z)\, d(x^2 y)$$

$$= x^2 y\,(\sin z\, dy + y\cos z\, dz) + y\sin z\,(2xy\, dx + x^2 dy)$$

$$= 2xy^2\sin z\, dx + 2x^2 y\sin z\, dy + x^2y^2\cos z\, dz$$

Line Integrals

$[a,b] \subseteq \mathbb{R}$, $\omega \in \mathfrak{X}^*([a,b])$ so $\omega_t = f(t)\,dt$, $f:[a,b] \longrightarrow \mathbb{R}$ smooth

The integral of ω over $[a,b]$ is

$$\int_{[a,b]} \omega := \int_a^b f(t)\,dt.$$

$$\begin{tikzcd} T^*[a,b] \arrow[d] & \omega_t \\ {[a,b]} \arrow[u, dashed, bend right, "\omega"'] & t \arrow[u, mapsto] \end{tikzcd}$$

Prop $\omega \in \mathfrak{X}^*([a,b])$, $\varphi:[c,d] \longrightarrow [a,b]$ increasing diffeo, then

$$\int_{[c,d]} \varphi^*\omega = \int_{[a,b]} \omega.$$

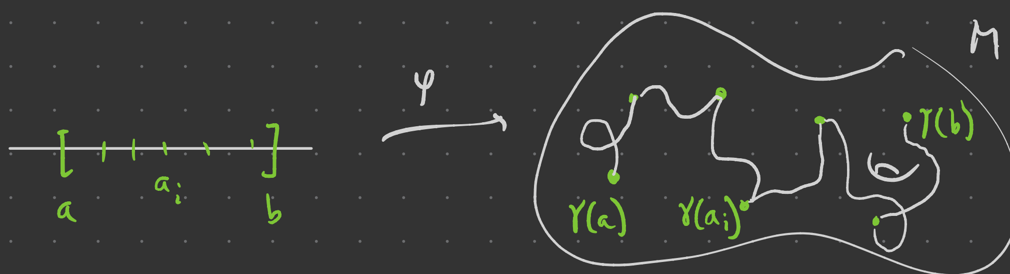
Pf Take s coord for $[c,d]$, t for $[a,b]$. Then $(\varphi^*\omega)_s = f(\varphi(s))\varphi'(s)\,ds.$

Thus $\displaystyle\int_{[c,d]} \varphi^*\omega = \int_c^d f(\varphi(s))\varphi'(s)\,ds$

$$= \int_a^b f(t)\,dt = \int_{[a,b]} \omega. \quad \square$$

Pullback = u-substitution

Dfn by picture: A **piecewise smooth curve segment** is



Note M conn'd smooth mfld then any 2 pts of M can be joined by a piecewise smooth curve segment.

For $\gamma : [a,b] \longrightarrow M$ a smooth curve segment, $\omega \in \mathfrak{X}^*(M)$, define

$$\int_\gamma \omega := \int_{[a,b]} \gamma^* \omega .$$

If γ is piecewise smooth,

$$\int_\gamma \omega := \sum_i \int_{[a_{i-1}, a_i]} \gamma^* \omega .$$

Prop

- $\int_\gamma : \mathfrak{X}^*(M) \longrightarrow \mathbb{R}$ is a linear transformation
- If γ is constant, then $\int_\gamma = 0$.
- If $\gamma_1 = \gamma|_{[a,c]}$ and $\gamma_2 = \gamma|_{[c,b]}$ for some $a<c<b$, then
$$\int_\gamma = \int_{\gamma_1} + \int_{\gamma_2} .$$

- If $F: M \to N$ is smooth, $\eta \in \mathfrak{X}^*(N)$, then

$$\int_\gamma F^*\eta = \int_{F\circ\gamma} \eta .$$

<u>E.g.</u> $M = \mathbb{R}^2 \setminus 0$, $\omega = \dfrac{x\,dy - y\,dx}{x^2+y^2} \in \mathfrak{X}^*(M)$, $\gamma : [0,2\pi] \to M$
$t \mapsto (\cos t, \sin t)$

Then $$\int_\gamma \omega = \int_{[0,2\pi]} \frac{\cos t\,(\cos t\,dt) - \sin t\,(-\sin t\,dt)}{\sin^2 t + \cos^2 t} = \int_0^{2\pi} dt = 2\pi .$$

<u>Prop</u> $\tilde{\gamma}$ a reparametrization of γ (i.e. $\tilde{\gamma} = \gamma\circ\varphi$ for $\underbrace{\varphi : [c,d] \xrightarrow{\approx} [a,b]}$)
$\omega \in \mathfrak{X}^*(M)$ then

"forward" if φ inc,
"backward" if φ dec

$$\int_{\tilde{\gamma}} \omega = \pm \int_\gamma \omega$$

forward vs backward reparam

$\square$

Note $\displaystyle\int_\gamma \omega = \int_a^b \underbrace{\omega_{\gamma(t)}(\gamma'(t))}_{=\gamma^*\omega}\, dt$

Thm (Fundamental Thm for Line Integrals) $f \in C^\infty(M)$, $\gamma : [a,b] \to M$ piecewise smooth curve segment. Then

$$\int_\gamma df = f(\gamma(b)) - f(\gamma(a)).$$

Pf $\displaystyle\int_\gamma df = \int_a^b df_{\gamma(t)}(\gamma'(t))\, dt = \int_a^b (f\circ\gamma)'(t)\, dt$

So reduces to FTC. $\square$