Note $\mathfrak{X}(M) = \Gamma(TM)$. — in fact, $C^\infty(M)$-module:

$(f\sigma)(p) = f(p)\,\sigma(p)$

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Reduction of Structure

Recall that a vector bundle $\begin{matrix} E \\ \downarrow\pi \\ X \end{matrix}$ may be specified by cocycle data: open cover $\{V_\alpha \mid \alpha \in A\}$ of X + transition fns

$$\tau_{\alpha\beta} : V_\alpha \cap V_\beta \longrightarrow GL_k\mathbb{R}$$

satisfying the cocycle condition: $\forall \alpha, \beta, \gamma \in A$

$$\tau_{\alpha\gamma} = \tau_{\beta\gamma}\,\tau_{\alpha\beta}.$$

Defn Let $G \leq GL_k\mathbb{R}$, $\pi : E \longrightarrow M$ vb of rank k on M. A reduction of the structure group of E to G is a

$GL_k\mathbb{R}$-cocycle representing the iso class of E, all of whose transition maps are valued in G.

E.g. • $G = GL_k^+\mathbb{R} = \det^{-1}(\mathbb{R}_{>0}) \leq GL_k\mathbb{R}$

If E admits a $GL_k^+\mathbb{R}$-cocycle call it orientable.

If TM is orientable, call M orientable.

• $G = O(k) \leq GL_k\mathbb{R}$

If E admits an $O(k)$-cocycle, then we can endow each of its fibers with an inner product and transition data will respect this.

If TM has structure gp reduced to $O(n)$, call

M a Riemannian manifold and say M has been given a Riemannian structure.

Fact Every smooth mfld admits a Riemannian str.

Twisting Fix a representation (i.e. homomorphism)
$\rho : GL_k(\mathbb{R}) \longrightarrow GL_m(\mathbb{R})$. Composing transition data for $\begin{array}{c} E \\ \downarrow \pi \\ M \end{array}$
with ρ gives new cocycles

$$\begin{tikzcd} \rho\circ\tau_{\alpha\beta} : V_\alpha \cap V_\beta \arrow[rr] \arrow[dr, "\tau_{\alpha\beta}"'] & & GL_m\mathbb{R} \\ & GL_k\mathbb{R} \arrow[ur, "\rho"'] & \end{tikzcd}$$

The associated vb on M is the ρ-twisting of E.

$$\underbrace{E}_{\text{rk } k \text{ vb on } M} \longmapsto \underbrace{E^\rho}_{\text{rk } m \text{ vb on } M}$$

E.g.

- $\det : GL_k\mathbb{R} \longrightarrow GL_1\mathbb{R} = \mathbb{R}^\times \rightsquigarrow \underset{\displaystyle \Lambda^k E}{\det E \,\shortparallel}$ line bundle on M

- $\begin{array}{ccc} GL_n\mathbb{R} & \longrightarrow & GL_n\mathbb{R} \\ A & \longmapsto & (A^{-1})^T \end{array} \rightsquigarrow E^*$ dual of E

Local & Global Frames

$U \subseteq M$ open, $\begin{array}{c} E \\ \pi\big\downarrow \\ M \end{array}$ vb of rank k

$(\sigma_1, \dots, \sigma_k) \in \underbrace{\Gamma(E, U)}_{\text{local sections of } E \text{ over } U}{}^k$ is a **local frame** if $\sigma_1(p), \dots, \sigma_k(p)$ is a basis of E_p $\forall p \in U$;

$$\begin{tikzcd} & E \arrow[d, "\pi"] \\ U \subseteq M \arrow[ur, dashed, "\sigma_i"] & \end{tikzcd}$$

global frame when $U = M$.

Note Frame for M = frame for TM.

E.g. • Trivial bundles $\begin{array}{c} M \times \mathbb{R}^k \\ \downarrow \\ M \end{array}$ admit the global frame

$(\tilde{e}_1, \dots, \tilde{e}_k)$ where $\tilde{e}_i(p) = (p, e_i)$.

• Local trivializations induce local frames:

$$\begin{tikzcd} \pi^{-1}U \arrow[rr, "\Phi", "\approx"'] \arrow[dr] & & U \times \mathbb{R}^k \arrow[dl] \\ & U \arrow[ul, dashed, bend left, "\sigma_i"] \arrow[ur, dashed, bend right, "\tilde{e}_i"'] & \end{tikzcd}$$

so define $\sigma_i = \Phi^{-1} \circ \tilde{e}_i$.

Prop Every smooth local/global frame for a smooth vb is associated with a smooth local/global trivialization.

Pf p. 259 $\square$

Cor A smooth manifold is parallelizable iff TM is trivial. $\square$

defined as TM admitting a global frame

The Cotangent Bundle

Duals V an $\mathbb{R}$-vector space.

$V^* := \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R}) = \{ \underbrace{\omega : V \longrightarrow \mathbb{R}}_{\text{a \underline{covector}}} \mid \omega \text{ linear} \}$ is the dual space of V.

Given basis $v_1, \ldots, v_n$ of V let $v_1^*, \ldots, v_n^* \in V^*$ be the covectors defined by

$$v_i^*(v_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \qquad \text{(then extended linearly).}$$

Then $v_1^*, \ldots, v_n^*$ is the dual basis (to (v_i)) of V^*.

Given $A : V \longrightarrow W$ linear, the dual of A is

$$A^* : W^* \longrightarrow V^*$$

$$\begin{array}{c} W \\ \omega \downarrow \\ \mathbb{R} \end{array} \longmapsto \begin{array}{c} V \\ A^*\omega \downarrow \\ \mathbb{R} \end{array}$$

$$\begin{tikzcd} V \arrow[r, "A"] \arrow[d, "A^*\omega"'] & W \arrow[dl, "\omega"] \\ \mathbb{R} & \end{tikzcd}$$

i.e. $(A^*\omega)(v) = \omega(Av)$ for $v \in V$, $\omega \in W^*$.

Have $(A \circ B)^* = B^* \circ A^*$ and $\mathrm{id}_V^* = \mathrm{id}_{V^*}$ so duality is a functor $\mathrm{Vect}_{\mathbb{R}}^{op} \longrightarrow \mathrm{Vect}_{\mathbb{R}}$.

Double dual $V^{**} = (V^*)^*$. Have

$$\xi = \xi_V : V \longrightarrow V^{**}$$

$$\begin{tikzcd} & V^* \arrow[d, "\xi(v)"'] & \omega : V \to \mathbb{R} \arrow[d, mapsto] \\ v \arrow[ru, mapsto] & \mathbb{R} & \omega(v) \end{tikzcd}$$

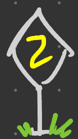
Then $\xi : \mathrm{id}_{\mathrm{Vect}_{\mathbb{R}}} \Longrightarrow (\ )^{**}$ is a natural transformation:

for $A : V \longrightarrow W$ linear,

$$\begin{tikzcd} V \arrow[r, "A"] \arrow[d, "\xi_V"'] & W \arrow[d, "\xi_W"] \\ V^{**} \arrow[r, "A^{**}"'] & W^{**} \end{tikzcd}$$

commutes.

Prop If $\dim V < \infty$, then $\xi : V \xrightarrow{\cong} V^{**}$ is a natural isomorphism.



- $V \not\cong V^{**}$ for infinite dim'l vector spaces
- $V \cong V^{*}$ for V finite dim'l, but the iso is not natural.

Tangent covectors on manifolds

$p \in M$ smooth mfld, the cotangent space at p is

$$T_p^{*}M := (T_pM)^{*}.$$

The transition fns for T^*M are inverse-transpose of those for TM.

Write $\mathfrak{X}^*(M) = \Gamma(T^*M)$ for smooth covector fields on M.

$\phantom{\mathfrak{X}^*(M)} = \Gamma(T^*M, M)$

The differential

$$\begin{tikzcd} T^*M \arrow[d] \\ M \arrow[u, bend right, dashed, "\omega"'] \end{tikzcd}$$

$\omega_p : T_pM \to \mathbb{R}$ linear

For $f \in C^\infty(U)$, write $df \in \mathfrak{X}^*(U)$ defined by

U open in M

$$df : U \longrightarrow T^*M$$

$$p \longmapsto \begin{array}{ccc} T_pM & & v \\ \big\downarrow df_p & & \big\downarrow \\ \mathbb{R} & & df_p(v) = \underbrace{vf} \end{array}$$

Using $T_pM = \mathrm{Der}_p(C^\infty(U))$

Call df the *differential* covector field of f.

Take (x^i) smooth coords on $U \subseteq M$ open.

Get a frame $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$ of $TM|_U$ with dual frame

$\lambda^1, \dots, \lambda^n$ of $T^*M|_U$. $\left(\lambda_i|_p = \left(\frac{\partial}{\partial x^i}\Big|_p\right)^* \right)$

Have $df_p = \sum A_i(p)\, \lambda^i|_p$ for some fns $A_i : U \longrightarrow \mathbb{R}$.

By dfn of df,

$$A_i(p) = df_p\left(\frac{\partial}{\partial x^i}\Big|_p\right) = \frac{\partial}{\partial x^i}\Big|_p f = \frac{\partial f}{\partial x^i}\Big|_p .$$

Thus $df_p = \sum \frac{\partial f}{\partial x^i}(p)\, \lambda^i\Big|_p$. $\circ\circ\circ\left\{ \nabla f_p \right.$ in x^i coord system

In particular, applied to $x^j : U \longrightarrow \mathbb{R}$ coord fn, get

$$dx^j\big|_p = \sum_i \frac{\partial x^j}{\partial x^i}(p)\, \lambda^i\big|_p = \lambda^j\big|_p$$

I.e., $dx^1, \dots, dx^n$ is the dual frame of $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$.

Thus $$df_p = \sum \frac{\partial f}{\partial x^i}(p)\, dx^i\big|_p \in T_p^*M$$

$$\int f(x)\, \underline{dx}$$

and $$df = \sum \frac{\partial f}{\partial x^i}\, dx^i \in \mathfrak{X}^*(U).$$

Why covectors? They give us coordinate-free versions of gradients!

$$\begin{tikzcd} TM \arrow[r, "dF"] \arrow[d] & TN \arrow[d] \\ M \arrow[r, "F"'] & N \end{tikzcd}$$

$$\begin{tikzcd} TM \arrow[r, "df"] \arrow[d] & T\mathbb{R} = \mathbb{R} \times \mathbb{R} \arrow[d] \\ M \arrow[r, "f"'] & \mathbb{R} \end{tikzcd}$$

$\underline{\text{vs}}$ $\quad df \in \mathfrak{X}^*(M)$

$$\begin{tikzcd} T_pM \arrow[r] & \{p\} \times \mathbb{R} \\ v \arrow[r, mapsto] & (p, vf) \end{tikzcd}$$

$$\begin{tikzcd} T^*M \arrow[d] \\ M \arrow[u, bend right, dashed, "df"'] \end{tikzcd}$$

E.g. $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$

$(x,y) \longmapsto x^2 y \cos x$

$$df = \frac{\partial(x^2 y \cos x)}{\partial x}\,dx + \frac{\partial(x^2 y \cos x)}{\partial y}\,dy$$

$$= (2xy\cos x - x^2 y \sin x)\,dx + (x^2 \cos x)\,dy$$

while $\nabla f = (\quad , \quad)$.

Prop $f, g \in C^\infty(M)$, $a, b \in \mathbb{R}$:

(a) $d(af + bg) = a\,df + b\,dg$

(b) $d(fg) = f\,dg + g\,df$

(c) $d(f/g) = \frac{1}{g^2}(g\,df - f\,dg)$ on $g \neq 0$

(d) $\operatorname{im} f \subseteq J \subseteq \mathbb{R}$ (interval), $h: J \to M$
$\Rightarrow d(h \circ f) = (h' \circ f)\,df$

(e) f const $\Rightarrow df = 0$. $\square$