14. IV. 23

Vector Bundles

Already seen: • $\begin{array}{c} TM \\ \downarrow \pi \\ M \end{array}$ tangent bundle with vector space T_pM over $p \in M$

• For $M \subseteq \mathbb{R}^n$, $\begin{array}{c} NM \\ \downarrow \\ M \end{array}$ normal bundle with vector space $N_pM := \underbrace{(T_pM)^\perp}$ over $p \in M$

orthogonal complement of $T_pM \subseteq T_p\mathbb{R}^n \cong \mathbb{R}^n$

In each case, get a family of vector spaces smoothly parametrized by points of M.

This is codified by the notion of a vector bundle over M:

A <u>vector bundle</u> (of rank k) over M is a space E along with surj cts map $\pi : E \longrightarrow M$ s.t.

(a) For $p \in M$, $E_p := \pi^{-1}\{p\}$ is endowed with the structure of a k-dim'l $\mathbb{R}$-vector space.

(b) $\forall p \in M$ $\exists$ nbhd U of p and a homeo $\Phi : \pi^{-1}U \longrightarrow U \times \mathbb{R}^k$ s.t.

- $\pi_U \circ \Phi = \pi$ (for $\pi_U : U \times \mathbb{R}^k \longrightarrow U$ proj'n)
- $\forall q \in U$, $\Phi|_{E_q} : E_q \to \{q\} \times \mathbb{R}^k \cong \mathbb{R}^k$ is a linear iso

Call Φ a <u>local trivialization</u>. If M, E are smooth mflds and π is smooth, call $\pi : E \longrightarrow M$ a <u>smooth vector bundle</u>.

If π admits a <u>global trivialization</u>

$$\begin{tikzcd} E \arrow[rr, "\cong"] \arrow[dr, "\pi"'] & & M \times \mathbb{R}^k \arrow[dl, "\pi_M"] \\ & M & \end{tikzcd}$$

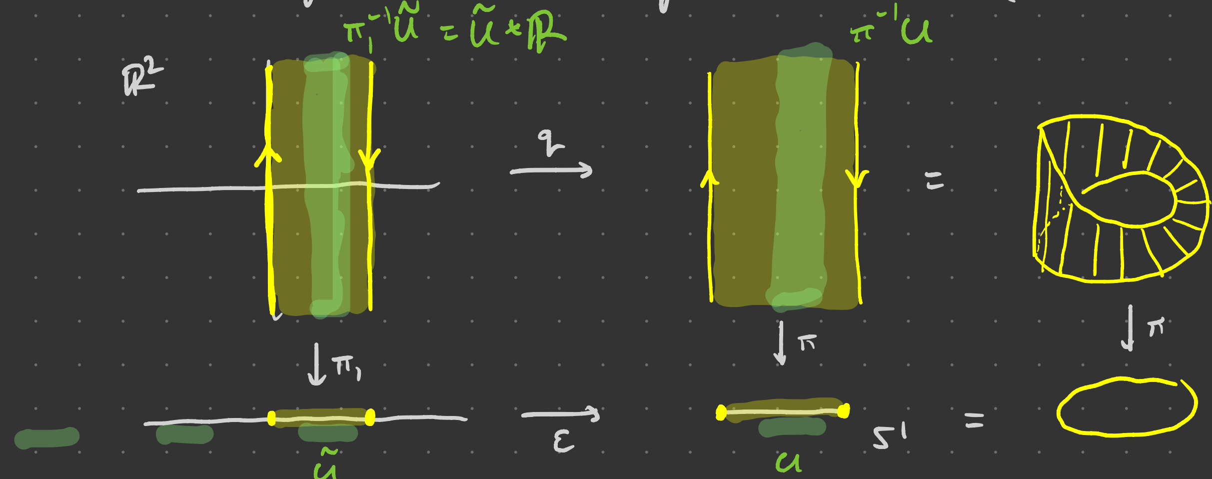
call π a (rank k) trivial bundle.

E.g. • Trivial/product bundles $M \times \mathbb{R}^k \to M$.

• Tangent bundle $TM \to M$.

• For $M \subseteq \mathbb{R}^n$, normal bundle $NM \to M$.

• Möbius bundle: Equiv reln on $\mathbb{R}^2$: $(x,y) \sim (x',y')$
$\iff (x',y') = (x+n, (-1)^n y)$ for some $n \in \mathbb{Z}$.



For any $U \subseteq S^1$ evenly covered by ε, $\tilde{U} \subseteq \mathbb{R}$ component of $\varepsilon^{-1}U$ evenly covered, $\tilde{U} \times \mathbb{R} \xrightarrow[\approx]{q} \pi^{-1}U$ gives local triv'n of π.

Transition Functions & Cocycles

Lemma $\begin{array}{c} E \\ \downarrow \pi \\ M \end{array}$ smooth vb of rk k, local trivializations

$\Phi : \pi^{-1}U \longrightarrow U \times \mathbb{R}^k$, $\Psi : \pi^{-1}V \longrightarrow V \times \mathbb{R}^k$ with $U \cap V \neq \emptyset$

There exists a smooth map $\tau : U \cap V \longrightarrow GL_k\mathbb{R}$ s.t.

$$\Phi \circ \Psi^{-1} : (U \cap V) \times \mathbb{R}^k \longrightarrow (U \cap V) \times \mathbb{R}^k$$

$$(p, v) \longmapsto (p, \underbrace{\tau(p)v}_{\text{matrix } \tau(p) \text{ acting on } v}).$$

Pf

$$\begin{tikzcd} (U\cap V)\times\mathbb{R}^k \arrow[dr, "\pi_1"'] & \pi^{-1}(U\cap V) \arrow[l, "\Psi"'] \arrow[r, "\Phi"] \arrow[d, "\pi"] & (U\cap V)\times\mathbb{R}^k \arrow[dl, "\pi_1"] \\ & U\cap V & \end{tikzcd}$$

commutes

so $\pi_1 \circ (\Phi\circ\Psi^{-1}) = \pi_1 \Longrightarrow \Phi\circ\Psi^{-1}(p,v) = (p, \sigma(p,v))$

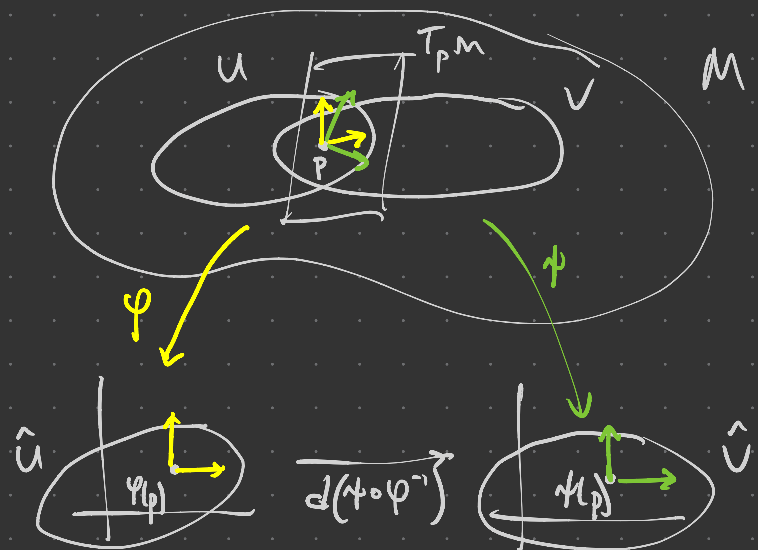
for some smooth $\sigma : (U\cap V)\times\mathbb{R}^k \longrightarrow \mathbb{R}^k$.

For $p \in U\cap V$ fixed, get $\mathbb{R}^k \longrightarrow \mathbb{R}^k$ invertible linear map

$\Longrightarrow \exists\, \tau(p) \in GL_k(\mathbb{R})$ s.t. $\sigma(p,v) = \tau(p)v$. $\square$

Call $\underset{\tau_{UV}}{\tau} : U\cap V \longrightarrow GL_k\mathbb{R}$ the transition function for U, V.

E.g. For the tangent bundle, $\tau : U\cap V \longrightarrow GL_n\mathbb{R}$ is the Jacobian of $\psi\circ\varphi^{-1}$ (smooth charts):



For $\begin{matrix} E \\ \downarrow\pi \\ M \end{matrix}$ vb, write τ_{UV} for the transition fn $U\cap V \longrightarrow GL_k\mathbb{R}$

Note $\tau_{VU} = \tau_{UV}^{-1}$ (pointwise matrix inverse).

What happens on triple intersections $U\cap V\cap W$?

$$\begin{tikzcd}
& \pi^{-1}(U\cap V\cap W) \arrow[dl, "\Phi_U"'] \arrow[dr, "\Phi_W"] \arrow[dd, "\Phi_V"] & \\
U\cap V\cap W\times\mathbb{R}^k \arrow[rr, "\tau_{UW}", "\wr"'] \arrow[dr, "\tau_{UV}"', "\wr"] & & U\cap V\cap W\times\mathbb{R}^k \\
& U\cap V\cap W\times\mathbb{R}^k \arrow[ur, "\tau_{VW}"', "\wr"] &
\end{tikzcd}$$

τ_{UW} is computed from $\Phi_W\circ\Phi_U^{-1} = \underbrace{\Phi_W\circ\Phi_V^{-1}}_{\tau_{VW}}\circ\underbrace{\Phi_V\circ\Phi_U^{-1}}_{\tau_{UV}}$

cocycle condition

so $\boxed{\tau_{UW}(p) = \tau_{VW}(p)\cdot\tau_{UV}(p)}$

Defn A $GL_k\mathbb{R}$-cocycle on M is an open cover $\{V_\alpha \mid \alpha\in A\}$ of M along with (cts or smooth) maps $\tau_{\alpha\beta}: V_\alpha\cap V_\beta \longrightarrow GL_k\mathbb{R}$ $\forall\alpha,\beta\in A$ s.t. $\tau_{\alpha\gamma} = \tau_{\beta\gamma}\,\tau_{\alpha\beta}$ $\forall\alpha,\beta,\gamma\in A$.

Get $GL_k\mathbb{R}$-cocycle from any rank k vb on M.

Converse: Set $\tilde{E} = \coprod_{\alpha \in A} V_\alpha \times \mathbb{R}^k$ and define equiv reln

$$\underset{(V_\alpha \cap V_\beta)\times\mathbb{R}^k}{(p,v)} \sim \underset{(V_\alpha \cap V_\beta)\times\mathbb{R}^k}{(p,w)} \quad \text{when } \tau_{\alpha\beta}(p)v = w$$

Let $E = \tilde{E}/\sim$ and produce

$$\begin{tikzcd} \tilde{E} \arrow[r] \arrow[d, "\coprod \pi_1"'] & E \arrow[d, "\pi"] \\ M \arrow[r, equal] & M \end{tikzcd}$$

TPS Why is π a vector bundle?

- Two $GL_k\mathbb{R}$-cocycles are equivalent if $\exists$ $GL_k\mathbb{R}$-cocycle in which both are contained.

- $\{\text{rk } k \text{ vb on } M\}/_{\cong} \quad \underset{\text{bij}}{\longleftrightarrow} \quad \{GL_k(\mathbb{R})\text{-cocycles on } M\}/_{\sim}$

  (we define later

- See ISM Lemma 10.6 for a less robust construction of vbs from local trivializing data.

Morphisms

$$\begin{tikzcd} E \arrow[rr, "F\ \text{cts}"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\ & M & \end{tikzcd}$$

F has a vb morphism inverse iff it's bij

Constructions

- Whitney sum: $\begin{matrix} E \\ \pi\downarrow \\ M \end{matrix}$, $\begin{matrix} E' \\ \pi'\downarrow \\ M \end{matrix}$ vbs over M

$\begin{matrix} E\oplus E' \\ \downarrow \\ M \end{matrix}$ has $(E\oplus E')_p = E_p \oplus E'_p$

$$\begin{pmatrix} \tau_{\alpha\beta} & 0 \\ 0 & \tau'_{\alpha\beta} \end{pmatrix}$$

- Tensor product: $\begin{array}{c} E \otimes E' \\ \downarrow \\ M \end{array}$ with $(E \otimes E')_p = E_p \otimes E'_p$.

- Pullback:

$$\begin{tikzcd} N \times_M E \arrow[r] \arrow[d, "F^*\pi"'] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d, "\pi"] \\ N \arrow[r, "F"'] & M \end{tikzcd}$$

with $N \times_M E = \{ (n, e) \mid F(n) = \pi(e) \}$

$$\begin{array}{cc} (n,e) & N \times_M E \\ \downarrow & \downarrow \\ n & N \end{array}$$

Note $(N \times_M E)_n \cong E_{F(n)}$.

If $F: N \subseteq M$, this is the restriction of E to N.

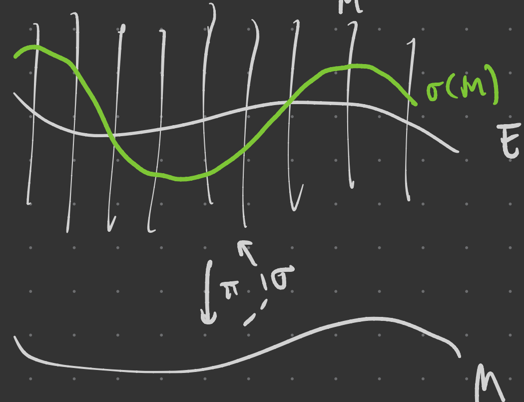
- Duals: $\begin{array}{c} E^* \\ \downarrow \\ M \end{array}$ with E^*_p = $\mathbb{R}$-linear dual of E_p.

$$\tau^*_{\alpha\beta} = \tau_{\alpha\beta}^T$$

Sections

A section of a vb $\begin{array}{c} E \\ \downarrow{\scriptstyle \pi} \\ M \end{array}$ is a [cts or smooth] map $\begin{array}{c} E \\ \uparrow{\scriptstyle \sigma} \\ M \end{array}$ s.t.

$\pi \circ \sigma = \mathrm{id}_M$. Write $E \underset{\sigma}{\overset{\pi}{\rightleftarrows}} M$



... { Sections of line bundles are "generalized functions"

E.g.

- Zero section $p \longmapsto 0_p \in E_p \quad \forall p \in M$
- sections of TM = vector fields

Write $\Gamma(E)$ for the $\mathbb{R}$-vs of global sections of E.

Note $\mathfrak{X}(M) = \Gamma(TM)$.

in fact, $C^\infty(M)$-module:

$(f\sigma)(p) = f(p)\sigma(p)$