12.IV.23

Flowouts (informally; see pp. 217 – 227 for details/proofs)

Thm M a smooth mfld, $S \subseteq M$ emb k-dim'l submfld, $V \in \mathfrak{X}(M)$ nowhere tangent to S. Let $\theta : \mathcal{D} \longrightarrow M$ be the flow of V, $\mathcal{O} = (\mathbb{R} \times S) \cap \mathcal{D}$, $\Phi = \theta|_{\mathcal{O}}$.

(a) $\Phi : \mathcal{O} \longrightarrow M$ is an immersion

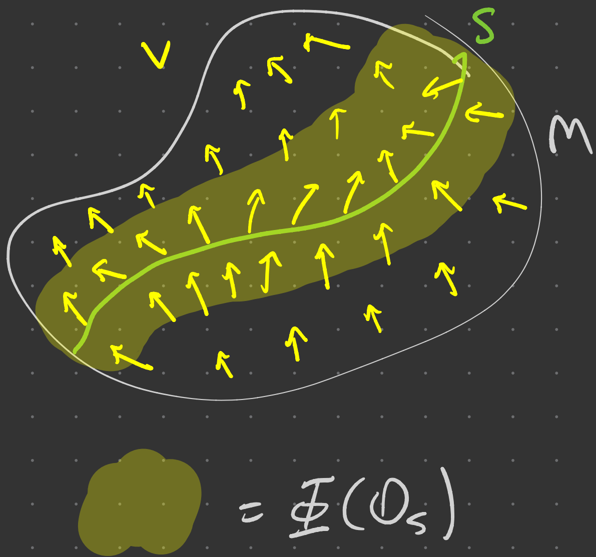
(b) $\frac{\partial}{\partial t} \in \mathfrak{X}(\mathcal{O})$ is Φ-related to V

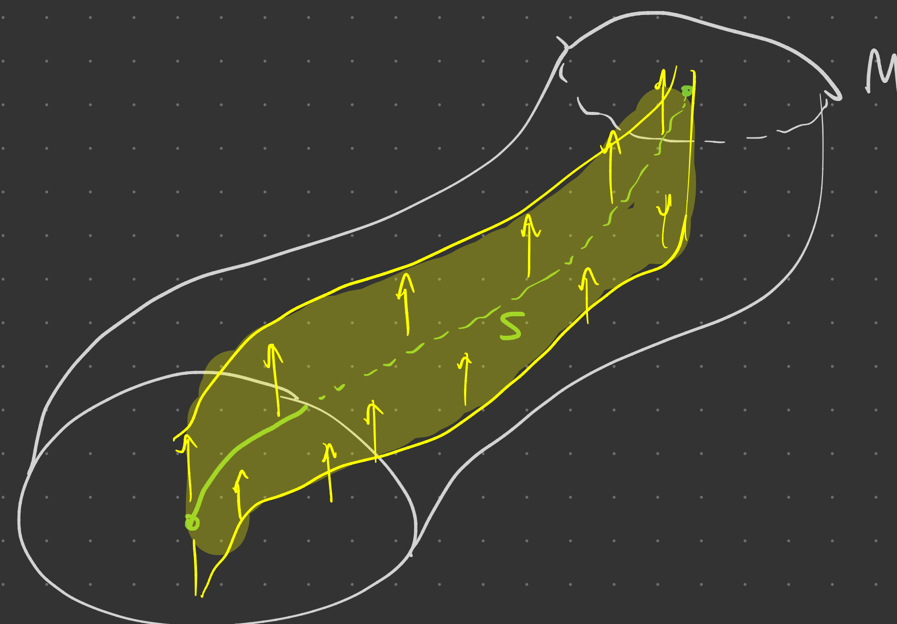
(c) $\exists\, \delta : S \longrightarrow \mathbb{R}_{>0}$ smooth s.t. $\Phi|_{\mathcal{O}_\delta}$ is injective, where

$$\mathcal{O}_\delta = \{(t,p) \in \mathcal{O} \mid |t| < \delta(p)\}.$$

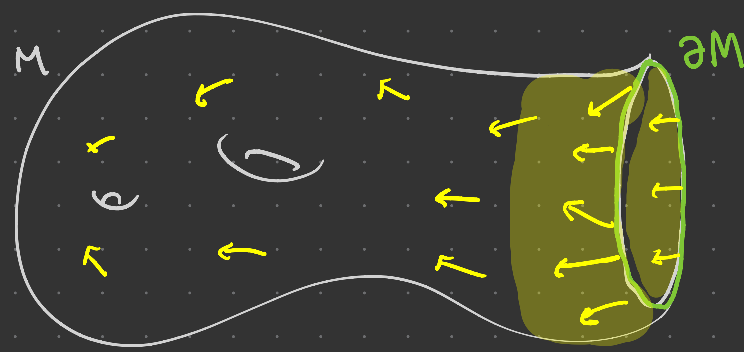
Thus $\Phi(\mathcal{O}_\delta)$ is an immersed submfld of M containing S, and V is tangent to $\Phi(\mathcal{O}_\delta)$.

(d) If $\operatorname{codim} S = 1$, then $\Phi|_{\mathcal{O}_S}$ is a diffeo onto an open submfld of M.





Boundary Flowout Thm M smooth mfld with $\partial M \neq \emptyset$, $N \in \mathfrak{X}(M)$ inward pointing on ∂M. $\exists \delta : \partial M \to \mathbb{R}_{>0}$ smooth and smooth emb $\Phi : \mathcal{P}_\delta \to M$ where $\mathcal{P}_\delta = \{(t,p) \mid p \in \partial M, 0 \le t < \delta(p)\} \subseteq \mathbb{R} \times \partial M$ s.t. $\Phi(\mathcal{P}_\delta)$ is a nbhd of ∂M, and $\forall p \in \partial M$, $t \mapsto \Phi(t,p)$ is an integral curve of N starting at p.



A nbhd of ∂M is called a **collar nbhd** if it is the image of a smooth emb $[0,1)\times\partial M \longrightarrow M$ s.t. $(0,p) \longmapsto p$ $\forall p \in \partial M$.

Collar Nbhd Thm If M is a smooth mfld w/ $\partial M \neq \emptyset$, then ∂M has a collar nbhd.

Pf By HW, $\exists\, N \in \mathfrak{X}(M)$ inward pointing on ∂M.

Take δ, Φ as in previous thm, define
$$\psi : [0,1)\times\partial M \xrightarrow{\approx} \mathcal{P}_\delta$$
$$(t,p) \longmapsto (t\delta(p), p)$$

$\Phi\circ\psi$ does the job. $\square$

Applications (1) Every smooth mfld is htpic to its interior.

(2) Whitney approximation for mflds w/ ∂:

cts maps b/w mflds w/ ∂ are htpic to smooth maps.

(3) Homotopic smooth maps are smoothly htpic.

(4) If $h : \partial N \xrightarrow{\approx} \partial M$, then the top'l mfld $M \cup_h N$ has a smooth structure with naturally emb submflds M, N intersecting in $\partial M = \partial N$

(5) Smooth connect sum & doubles of mflds.

Regular points, singular points, & canonical form

$V \in \mathfrak{X}(M)$

$p \in M$ is a singular point of V when $V_p = 0$
and a regular point of V when $V_p \neq 0$

Prop $V \in \mathfrak{X}(M)$, $\theta : \mathcal{D} \longrightarrow M$ flow gen'd by V.
If $p \in M$ is a singular point of V, then $\mathcal{D}^{(p)} = \mathbb{R}$ and
$\theta^{(p)}$ is the constant curve $\theta^{(p)}(t) = p$. If p is a regular point,
then $\theta^{(p)} : \mathcal{D}^{(p)} \longrightarrow M$ is a smooth immersion.

Pf Sing pts ✓
Suppose $\theta^{(p)}$ is not a smooth immersion. We show that p is

singular in this case (whence $\theta^{(p)}$ is in fact constant at p).

Know $\theta^{(p)\prime}(s)=0$ for some $s\in\mathcal{D}^{(p)}$. Let $q=\theta^{(p)}(s)$.

Then $\mathcal{D}^{(q)}=\mathbb{R}$ and $\theta^{(q)}(t)=q$ $\forall t\in\mathbb{R}$. But then $\mathcal{D}^{(p)}=\mathbb{R}$ as well and

$$\theta^{(p)}(t)=\theta_t(p)=\theta_{t-s}(\theta_s(p))=\theta_{t-s}(q)=q,$$

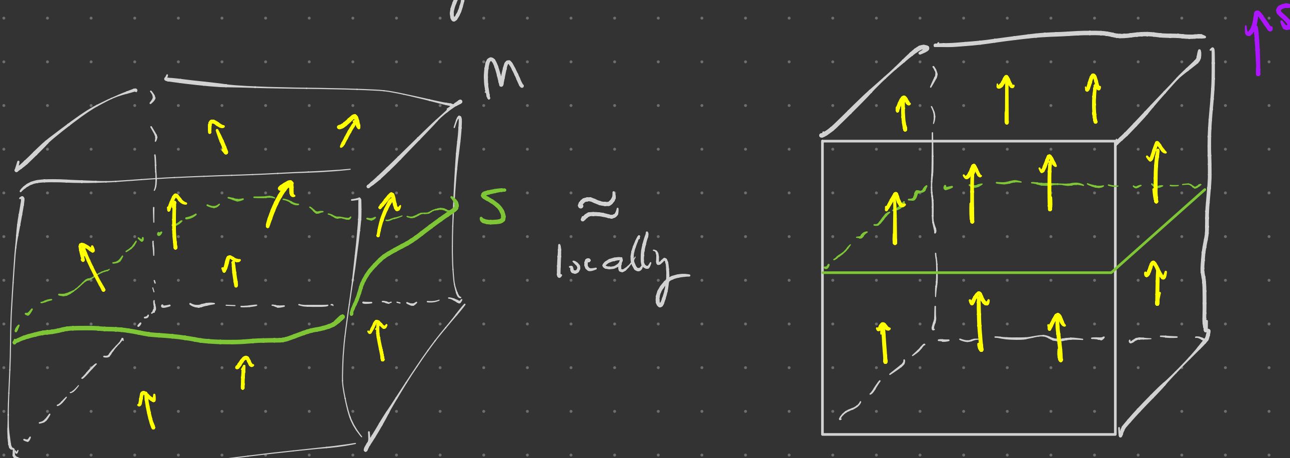
For $t=0$, get $p=q$. ✓ $\square$

If $\theta:\mathcal{D}\longrightarrow M$ is a flow, a point $p\in M$ is an equilibrium point of θ is $\theta(t,p)=p$ $\forall t\in\mathcal{D}^{(p)}$. In this case, p is a singular point of the infinitesimal generator of θ.

Thm (Canonical Form Near a Regular Point)

$V\in\mathfrak{X}(M)$, $p\in M$ regular pt of V. $\exists$ smooth coords

$(s^1, \dots, s^n)$ on a nbhd of p in which V has coordinate rep'n $\frac{\partial}{\partial s^1}$. If $S \subseteq M$ is an embedded hypersurface with $p \in S$ and $V_p \notin T_pS$, then the coords can also be chosen so that s^1 is a local defining function for S.



Pf Idea • If no S given, choose any smooth local coords $(U, (x^i))$ and let $S \subseteq U$ be given by $x^j = 0$ where $V^j(p) \neq 0$ (exists since p is regular).

• Now flow out from S to get open $W \subseteq M$ containing S and product nbhd $(-\varepsilon, \varepsilon) \times W_0$ of $(0, p)$ in $\mathcal{D}_\theta$.

• Choose smooth local param $X : \underset{\substack{\text{open} \cap \\ \mathbb{R}^{n-1}}}{\Omega} \longrightarrow S$ with image in W_0

$s^2, \dots, s^n$ coords

Then $\Psi : (-\varepsilon, \varepsilon) \times \Omega \xrightarrow{\approx} M$

$$(t, s^2, \dots, s^n) \longmapsto \theta(t, X(s^2, \dots, s^n))$$

with $\Psi_*\left(\frac{\partial}{\partial t}\right) = V = \Psi_*\left(\frac{\partial}{\partial t}\right)$. $\square$

Lie derivatives

In Euclidean space, we have directional derivatives:

$v \in T_p\mathbb{R}^n \cong \mathbb{R}^n$, $W \in \mathfrak{X}(\mathbb{R}^n)$,

$$D_v W(p) = \frac{d}{dt}\Big|_{t=0} W_{p+tv} = \lim_{t \to 0} \frac{W_{p+tv} - W_p}{t}$$

$$= \sum D_v W^i(p) \frac{\partial}{\partial x^i}\Big|_p$$

$p+tv$ doesn't make sense on a gen'l manfld.

Take $V, W \in \mathfrak{X}(M)$, $p \in M$, θ the flow of V.

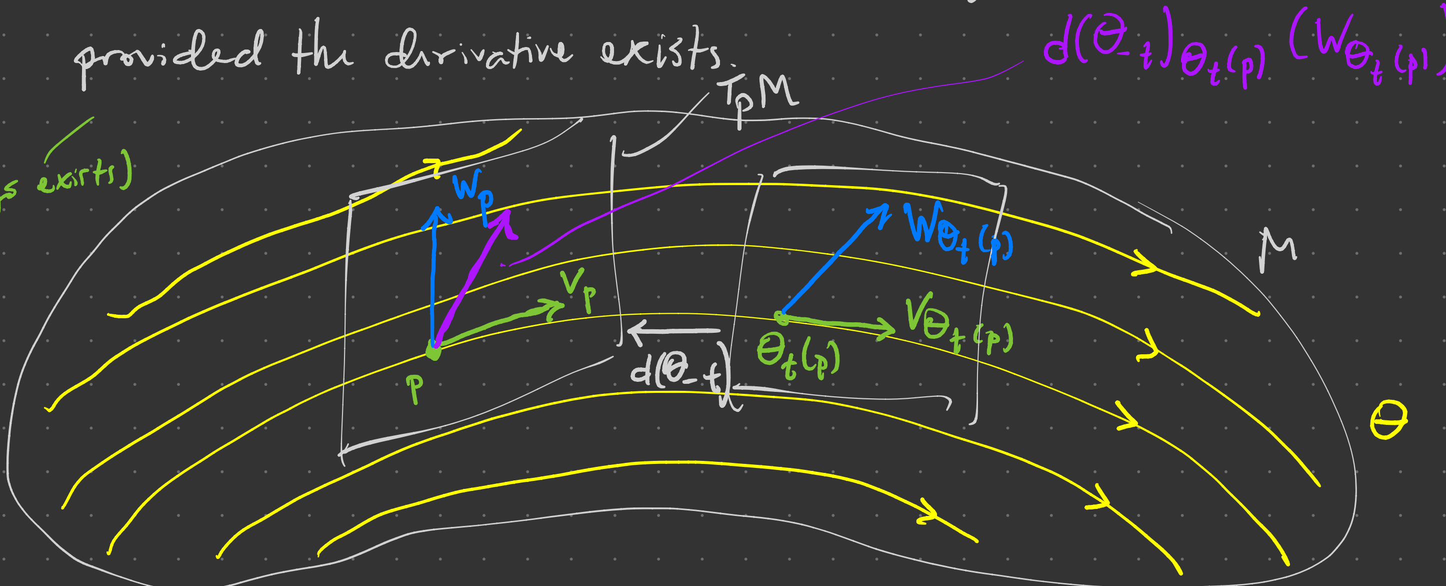
The Lie derivative of W with respect to V is

$$(\mathcal{L}_V W)_p := \frac{d}{dt}\Big|_{t=0} d(\theta_{-t})_{\theta_t(p)}\left(W_{\theta_t(p)}\right)$$

$$= \lim_{t\to 0} \frac{d(\theta_{-t})_{\theta_t(p)}\left(W_{\theta_t(p)}\right) - W_p}{t}$$

provided the derivative exists.

(it always exists)



Lemma $V, W \in \mathfrak{X}(M)$, V tangent to ∂M if $\partial M \neq \emptyset$, then $(\mathcal{L}_V W)_p$ exists for every $p \in M$ and $\mathcal{L}_V W \in \mathfrak{X}(M)$.

Pf idea Use coords to express $d(\theta_{-t})_{\theta_t(x)}(W_{\theta_t(x)})$ as a smooth function of (t, x). $\square$

Thm $V, W \in \mathfrak{X}(M)$ then $\mathcal{L}_V W = [V, W]$.

Pf Let $\mathcal{R}(V) \underset{\text{open}}{\subseteq} M$ be the regular pts of V.

Case 1 $p \in \mathcal{R}(V)$. Choose coords (u^i) with $V = \frac{\partial}{\partial u^1}$. In these coords, $\theta_t(u) = (u^1 + t, u^2, \dots, u^n)$. Get

$$d(\theta_{-t})_{\theta_t(u)}\left(W_{\theta_t(u)}\right) = d(\theta_{-t})_{\theta_t(u)}\left(\sum_j W^j(u^1+t, u^2, \dots, u^n)\frac{\partial}{\partial u^j}\bigg|_{\theta_t(u)}\right)$$

$$= \sum_j W^j(u^1+t, u^2, \dots, u^n)\frac{\partial}{\partial u^j}\bigg|_u .$$

By defn of Lie derivative,

$$(\mathcal{L}_V W)_u = \frac{d}{dt}\bigg|_{t=0} \sum W^j(u^1+t, u^2, \dots, u^n)\frac{\partial}{\partial u^j}\bigg|_u$$

$$= \sum \frac{\partial W^j}{\partial u^1}(u^1, \dots, u^n)\frac{\partial}{\partial u^j}\bigg|_u$$

$$= [V, W]_u$$ (by coord fmla for Lie bracket). ✓

Case 2 $p \in \operatorname{supp} V$. Since $\operatorname{supp} V = \overline{\mathcal{R}(V)}$, this case follows by continuity of both sides.

Case 3 $p \in M \setminus \operatorname{supp} V$. In this case, $V = 0$ on a nbhd of p.

so $d(\theta_{-t})_{\theta_t(p)}(W_{\theta_t(p)}) = W_p$ for t small

$$\Rightarrow (\mathcal{L}_V W)_p = 0 = [V, W]_p. \qquad \square$$