10. IV. 23

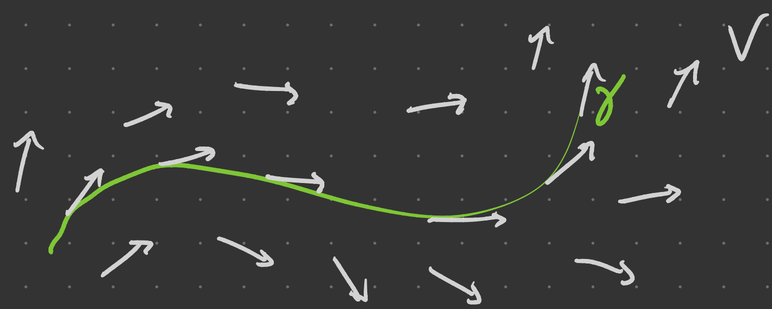
Integral curves & flows

$\gamma : J \longrightarrow M$ a smooth curve has velocity $\gamma'(t) \in T_{\gamma(t)} M$ for each $t \in J$.

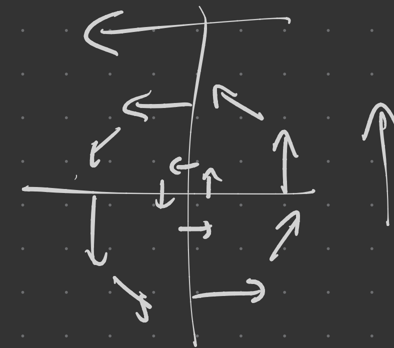
interval in $\mathbb{R}$

smooth mfld

If $V \in \mathfrak{X}(M)$, an integral curve of V is a smooth curve $\gamma : J \to M$ s.t. $\gamma'(t) = V_{\gamma(t)}$ $\forall t \in J$.



E.g. $W = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \in \mathfrak{X}(\mathbb{R}^2)$



$\gamma : \mathbb{R} \longrightarrow \mathbb{R}^2$ smooth

$t \longmapsto (x(t), y(t))$

Integral when

$$\underbrace{x'(t)\frac{\partial}{\partial x}\Big|_{\gamma(t)} + y'(t)\frac{\partial}{\partial y}\Big|_{\gamma(t)}}_{\gamma'(t)} = \underbrace{x(t)\frac{\partial}{\partial y}\Big|_{\gamma(t)} - y(t)\frac{\partial}{\partial x}\Big|_{\gamma(t)}}_{W_{\gamma(t)}}$$

I.e. $\begin{aligned} x'(t) &= -y(t) \\ y'(t) &= x(t) \end{aligned} \implies \begin{aligned} x(t) &= a\cos(t) - b\sin(t) \\ y(t) &= a\sin(t) + b\cos(t) \end{aligned}$

i.e. $\gamma(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$ with $\gamma(0) = (a, b)$

So circles = circulating integral flows.

More generally, if $V \in \mathfrak{X}(M)$, $\gamma: J \to M$ smooth, $U \subseteq M$ smooth coord patch, then $\gamma = (\gamma^1, \dots, \gamma^n)$ on U

$V = (V^1, \dots, V^n)$ on U

and $\gamma'(t) = V_{\gamma(t)}$ becomes on U

$$\sum_i \dot{\gamma}^i(t) \frac{\partial}{\partial x^i}\bigg|_{\gamma(t)} = \sum_i V^i(\gamma(t)) \frac{\partial}{\partial x^i}\bigg|_{\gamma(t)}$$

Thus

$$\left.\begin{aligned} \dot{\gamma}^1(t) &= V^1(\gamma^1(t), \dots, \gamma^n(t)) \\ &\vdots \\ \dot{\gamma}^n(t) &= V^n(\gamma^1(t), \dots, \gamma^n(t)) \end{aligned}\right\} \text{autonomous system of ordinary diff'l eqns (ODEs)}$$

Prop $\forall V \in \mathfrak{X}(M)$, $p \in M$ $\exists \varepsilon > 0$, smooth $\gamma: (-\varepsilon, \varepsilon) \to M$

that is an integral curve of V starting at p. $(\gamma(0) = p)$

Pf Appendix D.1. $\square$

A global flow on M (or one-parameter group action) is a cts left $\mathbb{R}$-action on M, $\theta : \mathbb{R} \times M \longrightarrow M$.

- For $t \in \mathbb{R}$, get $\theta_t : \begin{aligned} M &\xrightarrow{\cong} M \\ p &\longmapsto \theta(t,p) \end{aligned}$ with $\theta_t \circ \theta_s = \theta_{t+s}$, $\theta_0 = \mathrm{id}_M$.

- For $p \in M$, get $\theta^{(p)} : \begin{aligned} \mathbb{R} &\longrightarrow M \\ t &\longmapsto \theta(t,p) \end{aligned}$ with $\operatorname{im} \theta^{(p)} = \mathbb{R} \cdot p$

If θ is smooth, define $V = V(\theta) \in \mathfrak{X}(M)$ by $V_p = \theta^{(p)\prime}(0) \in T_pM$;
then V is the infinitessimal generator of θ.

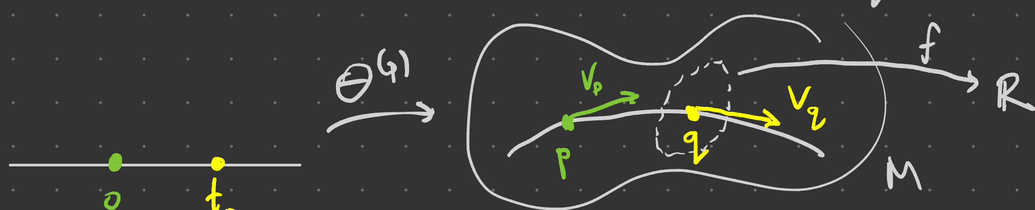
Prop V is really a smooth vector field, and $\Theta^{(p)}$ is an integral curve of V $\forall p \in M$.

Pf For smoothness of V, suffices to check Vf smooth $\forall f \in C^\infty(U)$, $U \subseteq M$ open.

Now $Vf(p) = V_p f = \Theta^{(p)\prime}(0) f = \frac{d}{dt}\Big|_{t=0} f(\Theta^{(p)}(t)) = \frac{\partial}{\partial t}\Big|_{t=0} f(\Theta(t,p))$.

Since $(t,p) \longmapsto f(\Theta(t,p))$ is smooth, so is its t-partial as a fn of p; thus $V \in \mathfrak{X}(M)$.

Now show $\Theta^{(p)\prime}(t) = V_{\Theta^{(p)}(t)}$ $\forall p \in M, t \in \mathbb{R}$. Fix $t_0 \in \mathbb{R}$ and set $q = \Theta^{(p)}(t_0) = \Theta_{t_0}(p)$. WTS $\Theta^{(p)\prime}(t_0) = V_q$



Now $\theta^{(q)}(t) = \theta_t(q) = \theta_t(\theta_{t_0}(p)) = \theta_{t+t_0}(p) = \theta^{(p)}(t+t_0)$.

Thus for $f : U \to \mathbb{R}$ smooth on $q \in U \subseteq M$ open,

$$V_q f = \theta^{(q)\prime}(0) f = \frac{d}{dt}\bigg|_{t=0} f(\theta^{(q)}(t))$$

$$= \frac{d}{dt}\bigg|_{t=0} f(\theta^{(p)}(t+t_0))$$

$$= \theta^{(p)\prime}(t_0) f$$

as desired. $\square$

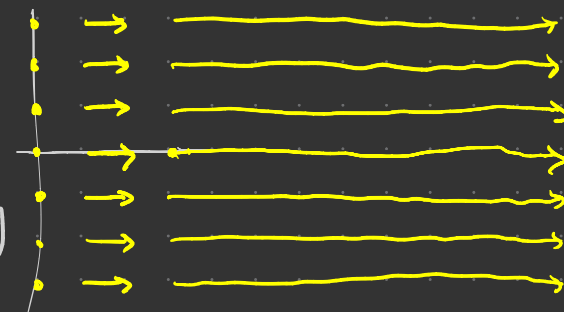
E.g. For $W = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ has global flow

$$\theta_t(x,y) = (x\cos t - y\sin t,\ x\sin t + y\cos t)$$

$$= \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Not all smooth vector fields admit global flows!

E.g. $W = x^2 \dfrac{\partial}{\partial x} \in \mathfrak{X}(\mathbb{R}^2)$

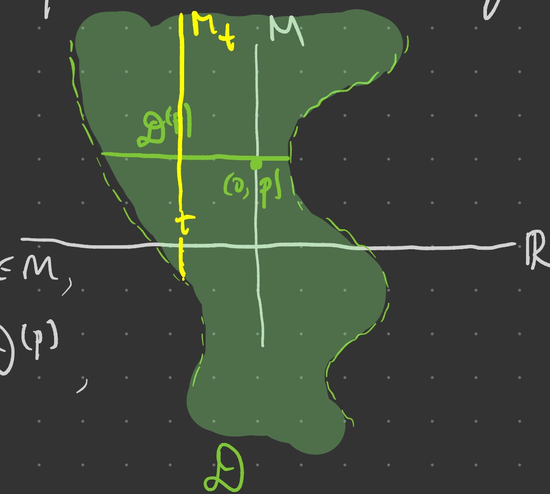


Integral curve starting at $(1,0)$ has eqn $\gamma(t) = (x(t), y(t))$ with $x(0) = 1$, $y(0) = 0$

$x'(t) = x(t)^2$

$y'(0) = 0 \implies \gamma(t) = \left(\dfrac{1}{1-t}, 0\right)$ which cannot be extended past $t = 1$.

A **flow domain** for M is an open set $\mathcal{D} \subseteq \mathbb{R} \times M$ s.t. for each $p \in M$, $\mathcal{D}^{(p)} = \{t \in \mathbb{R} \mid (t,p) \in \mathcal{D}\}$ is an open interval containing 0.



A **flow** on M is a cts map

$\theta : \mathcal{D} \longrightarrow M$ where $\mathcal{D} \subseteq \mathbb{R} \times M$

is a flow domain and s.t. $\theta(0,p) = p \ \forall p \in M$,

and for all $s \in \mathcal{D}^{(p)}$, $t \in \mathcal{D}^{(\theta(s,p))}$ s.t. $s+t \in \mathcal{D}^{(p)}$,

$$\theta(t, \theta(s,p)) = \theta(t+s, p).$$

For θ a flow, define $\theta_t(p) = \theta^{(p)}(t) = \theta(t,p)$ for $(t,p) \in \mathcal{D}$.

Also $M_t := \{p \in M \mid (t,p) \in \mathcal{D}\}$ so that

$$p \in M_t \iff t \in \mathcal{D}^{(p)} \iff (t,p) \in \mathcal{D}.$$

If θ is smooth, the infinitesimal generator of θ is

$$V_p := \theta^{(p)\prime}(0).$$

Prop If $\theta : \mathcal{D} \longrightarrow M$ is a smooth flow, then the infinitesimal generator V of θ is a smooth vector field, and each curve $\theta^{(p)}$ is an integral curve of V. $\square$

A maximal integral curve is one that can't be extended to a longer open interval, and a maximal flow is a flow that admits no extension to a flow on a larger flow domain.

Thm (Fundamental Theorem on Flows) Let $V \in \mathfrak{X}(M)$.

$\exists!$ smooth maximal flow $\theta : \mathcal{D} \longrightarrow M$ whose infinitesimal generator is V.

This flow has the following properties:

(a) $\forall p \in M$, $\theta^{(p)} : \mathcal{D}^{(p)} \longrightarrow M$ is the unique maximal integral curve of V starting at p.

(b) If $s \in \mathcal{D}^{(p)}$, then $\mathcal{D}^{(\theta(s,p))}$ is the interval $\mathcal{D}^{(p)} - s = \{t-s \mid t \in \mathcal{D}^{(p)}\}$.

(c) $\forall t \in \mathbb{R}$, M_t is open in M, and $\theta_t : M_t \longrightarrow M_{-t}$ is a diffeomorphism with inverse θ_{-t}.

Pf First show that $\gamma, \tilde{\gamma} : J \to M$ integral curves for V with $J \subseteq \mathbb{R}$ open interval, $\gamma(t_0) = \tilde{\gamma}(t_0)$ for some $t_0 \in J$, then $\gamma = \tilde{\gamma}$. (Use an old trick: $S = \{t \in J \mid \gamma(t) = \tilde{\gamma}(t)\} \subseteq J$ is nonempty, open, and closed.)

Now for $p \in M$ define $\mathcal{D}^{(p)} = \bigcup_{\substack{\exists \gamma : J \to M \\ \text{int for } V \text{ w/ } \gamma(0) = p}} J$. Define $\theta^{(p)}(t) = \gamma(t)$ for some (any) such γ.

Set $\mathcal{D} = \{(t,p) \in \mathbb{R} \times M \mid t \in \mathcal{D}^{(p)}\}$, $\theta : \mathcal{D} \longrightarrow M$

$(t,p) \longmapsto \theta^{(p)}(t)$

(a) ✓

Group action: $p \in M$, $s \in \mathcal{D}^{(p)}$, $q = \theta(s,p) = \theta^{(p)}(s)$. Then

$\gamma : \mathcal{D}^{(p)} - s \to M$, $t \mapsto \theta^{(p)}(t+s)$ is an integral curve of V starting at q

By uniqueness of ODE solns, $\gamma = \theta^{(q)}$ on their common domain

Thus $\theta(t, \theta(s,p)) = \theta(t+s, p)$ (and $\theta(0,p) = p$ is clear).

Also $\mathcal{D}^{(p)} - s \subseteq \mathcal{D}^{(q)}$ by maximality. $0 \in \mathcal{D}^{(p)} \Rightarrow -s \in \mathcal{D}^{(q)}$

and $\theta^{(q)}(-s) = p$. Same argument w/ $(-s, q)$ for (s,p) implies

$\mathcal{D}^{(q)} + s \subseteq \mathcal{D}^{(p)} \Longrightarrow \mathcal{D}^{(q)} \subseteq \mathcal{D}^{(p)} - s$, so (b) ✓.

$\mathcal{D}$ open, (c) : pp. 213 – 214. $\square$

Call $V \in \mathfrak{X}(M)$ complete if it generates a global flow.

Uniform time lemma $V \in \mathfrak{X}(M)$ with flow $\theta : \mathcal{D} \to M$. If $\exists\, \varepsilon > 0$ s.t. $(-\varepsilon, \varepsilon) \times M \subseteq \mathcal{D}$, then V is complete.

Pf Suppose for ~~c~~ $\exists\, p \in M$ with $\mathcal{D}^{(p)}$ bdd above. Let $b = \sup \mathcal{D}^{(p)}$, $b - \varepsilon < t_0 < b$, $q = \theta^{(p)}(t_0)$. Know domain of $\theta^{(q)}$ contains $(-\varepsilon, \varepsilon)$.

Define $\gamma : (-\varepsilon, t_0 + \varepsilon) \longrightarrow M$

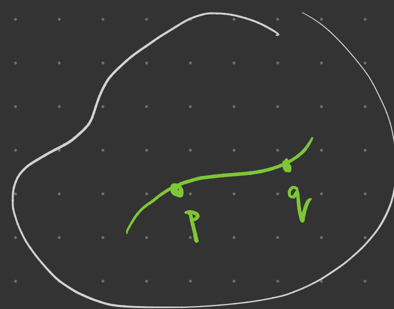
$$t \longmapsto \begin{cases} \theta^{(p)}(t) & -\varepsilon < t < b \\ \theta^{(q)}(t - t_0) & t_0 - \varepsilon < t < t_0 + \varepsilon \end{cases}.$$

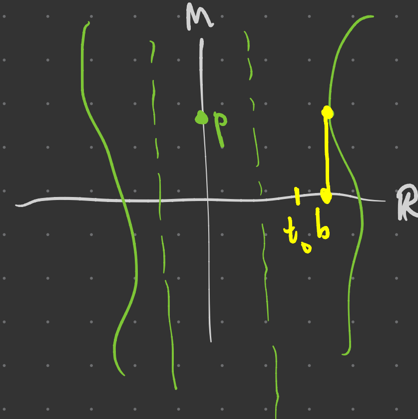
These agree on overlap b/c $\theta^{(q)}(t - t_0) = \theta_{t - t_0}(q) = \theta_{t - t_0}(\theta_{t_0}(p))$

$$= \theta_t(p) = \theta^{(p)}(t).$$

By translation lemma, γ is an integral curve for V starting at q.

Since $t_0+\varepsilon > b$, ※. $\square$





Thm Every compactly supported smooth vector field is complete.

Pf $K = \operatorname{supp} V$. For $p \in K$ have $\varepsilon_p > 0$ with flow on $(-\varepsilon_p, \varepsilon_p) \times U_p$.

Use compactness to find a valid uniform time. $\square$

Cor On a compact smooth mfld, every smooth vector field is complete. $\square$

Thm For G a Lie group, every $X \in \mathrm{Lie}(G)$ is complete.

Pf Choose ε that works at e. Push the integral curves at e around w/ the group action to see that ε works everywhere. $\square$