7. IV. 23

Cor Every left-invt vector field on G is smooth.

Pf If X is left-invt, then $X = \underbrace{(X_e)^L}_{\text{smooth}}$. □

Cor Every Lie group admits a left-invariant smooth global frame.

Pf If $e_1, \dots, e_n$ is a basis of T_eG, then $e_1^L, \dots, e_n^L$ is such a frame.

□

Examples of Lie algebras of Lie groups

(0) $\mathrm{Lie}(\mathbb{R}^n) \cong \mathbb{R}^n$ w/ trivial bracket:

For $b \in \mathbb{R}^n$, $L_b(x) = x + b$ and $d(L_b) = \mathrm{id} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

Thus $[X^i \frac{\partial}{\partial x^i} \in \mathrm{Lie}(\mathbb{R}^n)$ iff all X^i are constant.

Since $[\partial/\partial x^i, \partial/\partial x^j] = 0$ and $[\,,]$ is bilinear, $[\,,] = 0$.

Note When $[\,,\,]=0$, we call the Lie algebra **Abelian**.

(1) $\mathrm{Lie}(\mathbb{T}^n)=\mathbb{R}^n$ is Abelian.

(2) $\mathrm{Lie}(GL_n\mathbb{R}) \cong \mathfrak{gl}_n\mathbb{R} := \mathbb{R}^{n\times n}$ w/ $[A,B]=AB-BA$:

Recall $GL_n\mathbb{R} \subseteq \mathbb{R}^{n\times n}$ is an open submfld so admits global coords X^{ij} of matrix entries. Then

$$T_e GL_n\mathbb{R} \xrightarrow{\cong} \mathfrak{gl}_n\mathbb{R}$$

$$\sum A^{ij}\frac{\partial}{\partial X^{ij}} \longmapsto (A^{ij})$$

Given $A=(A^{ij})\in\mathfrak{gl}_n\mathbb{R}$, have $A^L\in \mathrm{Lie}\, GL_n\mathbb{R}$ given by

$$A^L|_X = d(L_X)_e(A) = d(L_X)_e\left(\sum A^{ij}\frac{\partial}{\partial x^{ij}}\right).$$

Since L_X is the restriction of the linear map $A \longmapsto XA$ on $\mathfrak{gl}_n\mathbb{R}$

dL_X is rep'd in coords by same map:

$$(A^L|_X)^{ik} = \sum_j X^{ij} A^{jk} \frac{\partial}{\partial X^{ik}}\bigg|_X$$

In Einstein notation, write this as

$$A^L|_X = X^{ij}A^{jk}\frac{\partial}{\partial x^{ik}}\bigg|_X .$$

Thus $$[A^L, B^L]_X = \left[X^{ij}A^{jk}\frac{\partial}{\partial x^{ik}}\bigg|_X ,\ X^{pq}B^{qr}\frac{\partial}{\partial x^{pr}}\bigg|_X \right]$$

$$= X^{ij}A^{jk}\frac{\partial}{\partial x^{ik}}(X^{pq}B^{qr})\frac{\partial}{\partial x^{pr}}$$

$$- X^{pq}B^{qr}\frac{\partial}{\partial x^{pr}}(X^{ij}A^{jk})\frac{\partial}{\partial x^{ik}}$$

$$= X^{ij} A^{jk} B^{kr} \frac{\partial}{\partial X^{ir}} - X^{pq} B^{qr} A^{rk} \frac{\partial}{\partial X^{pk}}$$

$$= \left(X^{ij} A^{jk} B^{kr} - X^{ij} B^{jk} A^{kr} \right) \frac{\partial}{\partial X^{ir}}$$

and $\quad [A^L, B^L]_e = \left(A^{jk} B^{kr} - B^{jk} A^{kr} \right) \frac{\partial}{\partial X^{ir}} \Big|_e$

$$= [A, B]^L \, .$$

□

Functoriality of Lie

Thm $\quad \mathrm{Lie} : \mathrm{LieGp} \longrightarrow \mathrm{LieAlg}$ is a functor, i.e.

$\forall$ Lie gp hom $F : G \longrightarrow H$ and $X \in \mathfrak{g}$, $\exists!\, F_* X \in \mathfrak{h}$ F-related to X, $F_* : \mathfrak{g} \longrightarrow \mathfrak{h}$ is a Lie alg hom, $id_* = id$, and $(G\circ F)_* = G_*\circ F_*$.

Pf Given $X \in \mathfrak{g}$ define $Y = F_* X = (dF_e(X_e))^L \in \mathfrak{h}$.

Then $dF(X_g) = dF(dL_g(X_e))$ (WTS $= Y_{F(g)}$)

Since $F(gg') = F(g)F(g')$, get $F\circ L_g = L_{F(g)}\circ F$

$$\Rightarrow dF\circ dL_g = dL_{F(g)}\circ dF.$$

$$= dL_{F(g)}(dF(X_e))$$

$$= dL_{F(g)}(Y_e)$$

$= Y_{F(g)}$ so X, Y F-related ✓

By naturality of Lie brackets,

$$F_*[X,Y] = [F_*X, F_*Y]$$

So $F_* : \mathfrak{g} \longrightarrow \mathfrak{h}$ is a Lie alg hom.

Moral exc: Check the rest (formal). $\square$

Thm $\iota : H \hookrightarrow G$ inclusion of a Lie subgp. Then

$$\mathfrak{h} = \iota_* \mathrm{Lie}(H) = \{ X \in \mathfrak{g} \mid X_e \in T_eH \} \qquad \square$$

w/ restricted bracket from $\mathfrak{g}$

Super helpful for describing Lie algs of Lie subgps of $GL_n\mathbb{R}$!

Examples of Lie Algs of Lie Gps (ct'd)

(13) $\mathfrak{o}(n) = \mathrm{Lie}[O(n)] = \{B \in \mathfrak{gl}_n\mathbb{R} = \mathbb{R}^{n\times n} \mid \underbrace{B^T + B = 0}_{\text{skew symmetric}}\}$

e.g. $\begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix} \in \mathfrak{o}(2)$

$$\Phi : GL_n\mathbb{R} \longrightarrow \mathbb{R}^{n\times n} \qquad \text{then } \Phi^{-1}\{I_n\} = O(n).$$
$$A \longmapsto A^TA$$

Thus $T_e O(n) = \ker d\Phi_e$; furthermore,

$$d\Phi_e : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{n\times n}$$
$$B \longmapsto B^T + B$$

✓

(4) $\mathfrak{gl}_n\mathbb{C} = \mathrm{Lie}(GL_n\mathbb{C}] = \mathbb{C}^{n\times n}$ w/ $[A,B] = AB-BA$.

(Use above techniques and $GL_n\mathbb{C} \longrightarrow GL_{2n}\mathbb{R}$. See p. 198.)

(5) Let's figure out the structure of $\mathfrak{sl}_n\mathbb{R}$ together!

$$SL_n\mathbb{R} = \ker\left(\det : GL_n\mathbb{R} \longrightarrow \underbrace{\mathbb{R}^{\times}\Big)}_{= GL_1\mathbb{R}}\right.$$

$$\mathfrak{gl}_1\mathbb{R} = \mathbb{R}^{1\times 1}$$
$$[a,b] = ab-ba$$
$$=0$$

so $$T_e SL_n\mathbb{R} = \ker\left(d\det_e : \underbrace{\mathfrak{gl}_n\mathbb{R} \longrightarrow \mathbb{R}}\right)$$

<u>TPS</u> Show this map is the trace function.

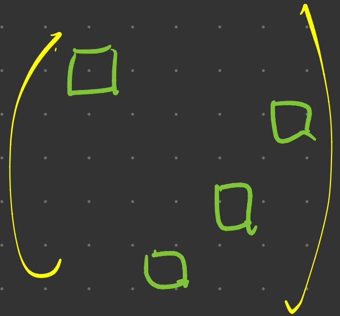
Thus $\mathfrak{sl}_n\mathbb{R} = \{B \in \mathfrak{gl}_n\mathbb{R} \mid \operatorname{tr} B = 0\}$.

Hints

- $$\mathrm{ddet}_e(A) = \frac{d}{dt}\Big|_{t=0} \underbrace{\det(I_n + tA)}_{\text{polynomial in } t \text{ — what is its } \underline{\underline{\text{linear term}}}\text{?}}$$

- $$\det\begin{pmatrix} 1+ta & tb \\ tc & 1+td \end{pmatrix} = 1 + ta + td + t^2(ad - bc)$$

- $$\det(b_{ij}) = \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\mathrm{sgn}(\sigma)} \prod_{i=1}^{n} b_{i\sigma(i)}$$



$$\frac{d}{dt}\Big|_{t=0} \det(\underbrace{I_n + tA})$$

$$\det t\cdot\left(\frac{1}{t}I_n + A\right) = t^n \cdot \det\left(\frac{1}{t}I_n + A\right)$$

$$= t^n \chi_{-A}\left(\frac{1}{t}\right)$$

$$\vdots$$

Vieta fmlae .....

$$= \operatorname{tr}(A)$$