<u>Good</u> <u>news</u> : $\mathfrak{X}$ is functorial in diffeomorphisms!

I.e. $F : M \xrightarrow[\approx]{} N$ and $X \in \mathfrak{X}(M)$ then $\exists!\ F_* X \in \mathfrak{X}(N)$

s.t. $(F_* X)_q = dF_{F^{-1}q}\, X_{F^{-1}(q)}$; moreover, $id_* X = X$ and

if $G : N \xrightarrow[\approx]{} P$ then $(G \circ F)_* X = G_*(F_* X)$.

$$
\begin{tikzcd}
TM \arrow[r, "dF", "\approx"'] \arrow[d, "\pi_M"] & TN \arrow[d, "\pi_N"'] \\
M \arrow[r, "F", "\approx"'] \arrow[u, bend left, "X"] & N \arrow[u, bend right, "F_* X"']
\end{tikzcd}
\qquad F_* X = dF \circ X \circ F^{-1}
$$

Call $F_* X$ the pushforward of X along F.

Since X and $F_* X$ are F-related,

$$X(f \circ F) = ((F_* X) f) \circ F.$$

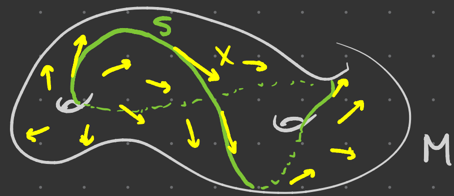
See pp. 183-184 for detailed comp'n of $F_* X$.

Vector fields & submanifolds

$S \subseteq M$ immersed or embedded submfld

For $p \in S$, $X \in \mathfrak{X}(M)$ call X tangent to S at p when

$$X_p \in T_p S \subseteq T_p M.$$



Note $S \subseteq M$ imm submfld, $X \in \mathfrak{X}(M)$ tangent to S, then $\exists!\, X|_S \in \mathfrak{X}(S)$ that is ι-related to X $(\iota : S \hookrightarrow M)$.

p. 185

Prop $S \subseteq M$ emb submfld. Then $X \in \mathfrak{X}(M)$ is tangent to S iff

$$Xf|_S = 0 \quad \forall f \in C^\infty(M) \text{ s.t. } f|_S = 0.$$

$\square$

$\therefore$ $\left\{ \begin{array}{l} \text{Directional derivs in directions} \\ \text{tangent to } S \text{ are } 0 \text{ on} \\ \text{fns constant on } S. \end{array} \right.$

Lie brackets

Recall $\mathfrak{X}(M) \cong \{\text{derivations } C^\infty(M) \to C^\infty(M)\}$

$$X \longmapsto (f \mapsto Xf).$$

Given $X, Y \in \mathfrak{X}(M)$, we may form XY & YX as composites:

$$XY : f \mapsto X(Yf)$$

$$YX : f \mapsto Y(Xf)$$

These are fns $XY, YX : C^\infty(M) \to C^\infty(M)$ but not derivations.

But the Lie bracket

$$[X,Y] := XY - YX : f \longmapsto X(Yf) - Y(Xf)$$

is a derivation $C^\infty(M) \to C^\infty(M)$ so $[X,Y] \in \mathfrak{X}(M)$!

Pf $[X,Y](fg) = (XY)(fg) - (YX)(fg)$

$$= X\big(f(Yg) + (Yf)g\big) - Y\big(f(Xg) + (Xf)g\big)$$

$$= f\cdot XYg + \cancel{Xf\cdot Yg} + \cancel{Yf\cdot Xg} + XYf\cdot g$$

$$- \big(f\cdot YXg + \cancel{Yf\, Xg} + \cancel{Xf\cdot Yg} + YXf\cdot g\big)$$

$$= f\cdot[X,Y]g + [X,Y]f\cdot g$$

so $[X,Y]$ is a derivation. $\square$

Later $[X,Y]$ is the "Lie derivative" — directional derivs of Y along X.

Prop $X, Y \in \mathfrak{X}(M)$, $X = \sum_i X^i \frac{\partial}{\partial x^i}$, $Y = \sum_j Y^j \frac{\partial}{\partial x^j}$ for some smooth local coords (x^i) for M. Then

$$[X, Y] = \sum_{i,j} \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}.$$

$$= \sum_j \left(X Y^j - Y X^j \right) \frac{\partial}{\partial x^j}$$

Pf $\frac{\partial^2}{\partial x^i \partial x^j} = \frac{\partial^2}{\partial x^j \partial x^i}$ on smooth fns. $\square$

E.g.

- $\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right] = 0 \quad \forall i, j$
- $X = x \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + x(y+1) \frac{\partial}{\partial z}$, $Y = \frac{\partial}{\partial x} + y \frac{\partial}{\partial z} \in \mathfrak{X}(\mathbb{R}^3)$

Then $[X, Y] = (X1 - Yx) \frac{\partial}{\partial x} + (X0 - Y1) \frac{\partial}{\partial y} + (Xy - Y[x(y+1)]) \frac{\partial}{\partial z}$

$$= (0-1)\frac{\partial}{\partial x} + 0\frac{\partial}{\partial y} + (1-(y+1))\frac{\partial}{\partial z}$$

$$= -\frac{\partial}{\partial x} - y\frac{\partial}{\partial z}.$$

Prop Lie brackets are

- **bilinear** $[aX+Y, Z] = a[X,Z] + [Y,Z]$
  $[X, aY+Z] = a[X,Y] + [X,Z]$ $\quad \forall a \in \mathbb{R},\ X,Y,Z \in \mathfrak{X}(M)$

- **antisymmetric** $[X,Y] = -[Y,X]$ $\ \forall X,Y \in \mathfrak{X}(M)$.

They also satisfy the

- **Jacobi identity** $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \quad \forall X,Y,Z$

and for $f,g \in C^\infty(M)$, $\quad [fX, gY] = fg[X,Y] + (fXg)Y - (gYf)X$

The proof consists of comp'ns following from defns. $\square$

A **Lie algebra** is an $\mathbb{R}$-vector space $\mathfrak{g}$ together with a bilinear transformation $[\,,\,]: \mathfrak{g}\times\mathfrak{g} \longrightarrow \mathfrak{g}$ which is antisymmetric and satisfies the Jacobi identity.

Thus $\mathfrak{X}(M)$ is a Lie algebra!

The Lie algebra of a Lie group

G a Lie group. $X \in \mathfrak{X}(G)$ is **left invariant** when $(L_g)_* X = X \;\; \forall g \in G$.

Write $\mathrm{Lie}(G) = \mathfrak{g} \subseteq \mathfrak{X}(G)$ for the $\mathbb{R}$-vector subspace of left-invariant vector fields on G.

Prop If $X, Y \in \mathfrak{g}$, then $[X,Y] \in \mathfrak{g}$ so $\mathfrak{g}$ is a Lie algebra.

Pf $\quad (L_g)_*[X,Y] = [(L_g)_*X, (L_g)_*Y] = [X,Y]$ $\square$

Naturality of $[\,,\,]$: If $F: M \to N$ smooth, $X_1, X_2 \in \mathfrak{X}(M)$, $Y_1, Y_2 \in \mathfrak{X}(N)$ s.t. X_i is F-related to Y_i, $i=1,2$, then $[X_1, X_2]$ is F-related to $[Y_1, Y_2]$. (Apply to $F = L_g$.)

Pf $\quad X_1 X_2 (f \circ F) = X_1((Y_2 f) \circ F) = (Y_1 Y_2 f) \circ F$

$\quad X_2 X_1 (f \circ F) = X_2((Y_1 f) \circ F) = (Y_2 Y_1 f) \circ F$

$$\Rightarrow [X_1, X_2](f \circ F) = X_1 X_2 (f \circ F) - X_2 X_1 (f \circ F)$$
$$= (Y_1 Y_2 f) \circ F - (Y_2 Y_1 f) \circ F$$
$$= ([Y_1, Y_2] f) \circ F. \quad \square$$

A **Lie algebra homomorphism** is a linear map $A : \mathfrak{g} \to \mathfrak{h}$ s.t.

$$A[X,Y] = [AX, AY] \quad \forall X, Y \in \mathfrak{g}.$$

E.g.

- $\mathfrak{gl}_n \mathbb{R} := \mathbb{R}^{n \times n}$ with $[A,B] := AB - BA$
- $\mathfrak{gl}_n \mathbb{C} := \mathbb{C}^{n \times n}$ as $2n^2$-dim'l $\mathbb{R}$-vs with $[A,B] := AB - BA$

(Later we'll see these are $\cong \operatorname{Lie}(GL_n F)$ for $F = \mathbb{R}, \mathbb{C}$ resp.)

Thm Let G be a Lie group. The evaluation map

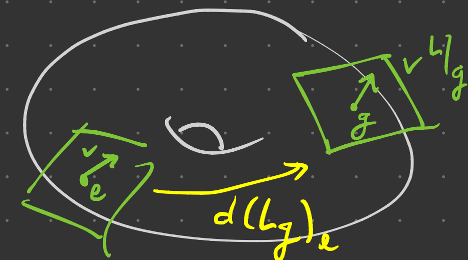
$$\begin{aligned} \varepsilon : \operatorname{Lie}(G) &\longrightarrow T_e G \\ X &\longmapsto X_e \end{aligned}$$

is a vector space isomorphism. Thus $\dim_{\mathbb{R}} \operatorname{Lie}(G) = \dim G$.

Pf ε is linear and if $\varepsilon(X) = X_e = 0$ then $X = 0$ b/c (by left-invariance) $X_g = d(L_g)_e(X_e) = 0 \ \forall g$.

For surjectivity, let $v \in T_eG$ be arbitrary and define

$$v^L|_g := d(L_g)_e(v).$$



v^L is smooth: Suffices to show $v^L f$ smooth $\forall f \in C^\infty(G)$. Take $\gamma : (-\delta, \delta) \to G$ smooth with $\gamma(0) = e$, $\gamma'(0) = v$. Then for $g \in G$,

$$(v^L f)(g) = v^L|_g f = d(L_g)_e(v) f = v(f \circ L_g)$$

$$= \gamma'(0)(f \circ L_g) = \frac{d}{dt}\Big|_{t=0} (f \circ L_g \circ \gamma)(t)$$

Define $\varphi : (-\delta, \delta) \times G \longrightarrow \mathbb{R}$
$$(t, g) \longmapsto f \circ L_g \circ \gamma(t) = f(g\gamma(t)).$$

Then $(v^L f)(g) = \frac{\partial \varphi}{\partial t}(0, g)$. Since φ is smooth, $v^L f$ is smooth. ✓

v^L is left-invariant: WTS $d(L_h)_g (v^L|_g) = v^L|_{hg}$ $\forall g, h \in G$.

By defn, $d(L_h)_g (v^L|_g) = d(L_h)_g \left(d(L_g)_e v \right)$
$$= d(L_h \circ L_g)_e \, v$$
$$= d(L_{hg})_e \, v$$
$$= v^L|_{hg}.$$

Finally, $\varepsilon(v^L) = v^L|_e = v$ so ε is surjective. $\square$