3.IV.23

Vector Fields

Smooth mfld M

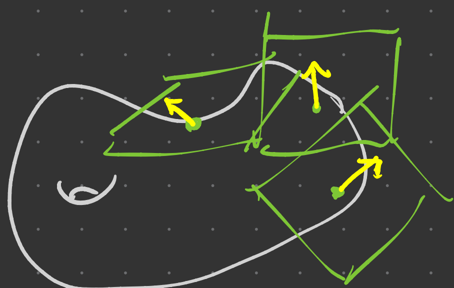
Tangent bundle TM

$$\begin{tikzcd} TM \arrow[d, "\pi"'] \\ M \arrow[u, dashed, bend right, "X"'] \end{tikzcd}$$

cts section : $\pi \circ X = \mathrm{id}_M$

Call X a vector field on M.

(Have smooth & "rough" variants as well.)



$\Big\{$ X = ctsly varying velocity (direction + magnitude) on M

Note $\pi \circ X = \mathrm{id}_M$ just means $X(p) = X_p \in T_pM \;\; \forall p \in M$

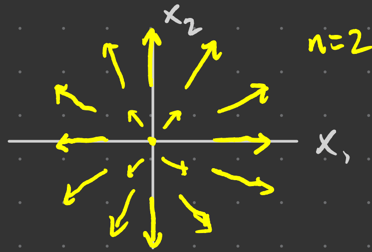
Coordinates $X : M \to TM$ (rough) vector field, $(U, (x^i))$ smooth coordinates on M, then

$$X_p = \sum_i X^i(p) \frac{\partial}{\partial x^i}\Big|_p$$

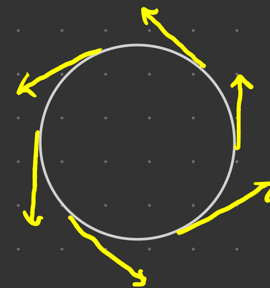
for $p \in U$; call X^i the component function of X in U.

Have $X|_U$ smooth iff $X^1, \dots, X^n$ smooth.

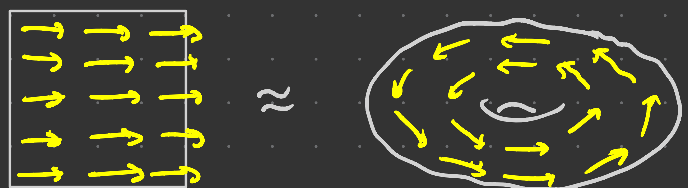
E.g. (Euler vector field) $V_x = x^1 \frac{\partial}{\partial x^1}\Big|_x + \dots + x^n \frac{\partial}{\partial x^n}\Big|_x$ for $x \in \mathbb{R}^n$.



E.g. $\frac{d}{d\theta}$ on S^1:



E.g. $\mathbb{T}^2 =$



$\frac{\partial}{\partial \theta^1}$

$\mathbb{T}^2 =$

$\frac{\partial}{\partial \theta^2}$

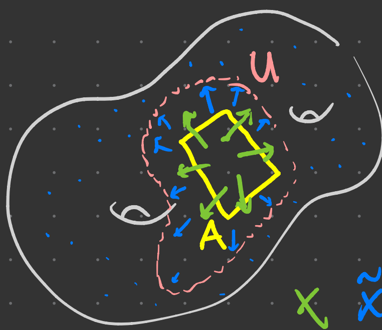
Lemma M a smooth mfld w/ or w/o ∂, $A \subseteq M$ closed.

Suppose X is a smooth vector field on A. If $A \subseteq \underset{\text{open}}{U} \subseteq M$

then $\exists$ smooth vector field $\tilde{X}$ on M s.t.

$$\tilde{X}\big|_A = X \quad \text{and} \quad \operatorname{supp} \tilde{X} \subseteq U$$

$$\operatorname{supp} \tilde{X} = \overline{\{p \in M \mid \tilde{X}_p \neq 0\}}$$



Pf POU $\square$

Cor For $p \in M$, $v \in T_pM$ $\exists$ smooth v.f. X on M

s.t. $X_p = v$.

Notation $\mathfrak{X}(M) := C^\infty(M)$-module of smooth vector fields on M.

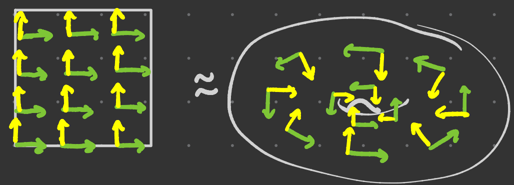
- $(X+Y)_p = X_p + Y_p$ for $X, Y \in \mathfrak{X}(M)$
- $(fX)_p = f(p) X_p$ for $f \in C^\infty(M)$, $X \in \mathfrak{X}(M)$

Frames $X_1, \dots, X_k \in \mathfrak{X}(A)$, $A \subseteq M$, are linearly independent when $(X_1)_p, \dots, (X_k)_p \in T_p M$ are lin ind $\forall p \in A$.

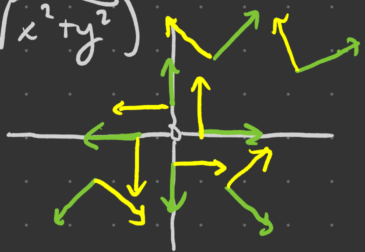
A local frame for an open $U \subseteq M$ is an ordered n-tuple $(E_1, \dots, E_n) \in \mathfrak{X}(U)^n$ s.t. $((E_1)_p, \dots, (E_n)_p)$ form a basis of T_pM $\forall p \in U$; it's a global frame if $U = M$.

E.g.

- If $(U,(x^i))$ is a smooth coord patch for M then $\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right)$ is a local frame on U.
- $\left(\frac{\partial}{\partial \theta^1}, \frac{\partial}{\partial \theta^2}\right)$ is a global frame on $\mathbb{T}^2$:



- $E_1 = \frac{x}{r}\frac{\partial}{\partial x} + \frac{y}{r}\frac{\partial}{\partial y}$, $E_2 = \frac{-y}{r}\frac{\partial}{\partial x} + \frac{x}{r}\frac{\partial}{\partial y}$ is a global frame on $\mathbb{R}^2 \setminus 0$ $\left(r = \sqrt{x^2+y^2}\right)$



The last two examples are orthonormal frames.

By Gram-Schmidt, any local frame may be orthonormalized.

Defn A smooth mfld is *parallelizable* when it admits a smooth global frame.

E.g. $\mathbb{R}^n$, $\mathbb{T}^n$ are parallelizable

Deep Thm S^0, S^1, S^3, S^7 are the only parallelizable spheres

Upcoming Thm Every Lie group is parallelizable.

Fact S^0, S^1, S^3 admit Lie group structures but S^7 does not.

Recall that $v \in T_pM$ is a derivation $C^\infty(M) \longrightarrow \mathbb{R}$: a linear trans'n s.t. $v(fg) = f(p)\,v(g) + v(f)\,g(p)$.

Given $X \in \mathfrak{X}(M)$ and $f \in C^\infty(U)$, $U \subseteq M$ open, this allows us to define

$$Xf : U \longrightarrow \mathbb{R}$$
$$p \longmapsto X_p f$$

$\cancel{f \cdot X}$

... Directional derivatives of f along directions provided by X

Have $Xf \in C^\infty(U)$; may view X as a derivation $C^\infty(M) \longrightarrow C^\infty(M)$: linear trans'n s.t. $X(fg) = f(Xg) + g\cdot(Xf)$.

Prop $X : M \longrightarrow TM$ rough vector field. TFAE

(a) X is smooth

(b) $\forall f \in C^\infty(M)$, Xf is smooth

(c) $\forall$ open $U \subseteq M$, $f \in C^\infty(U)$, Xf is smooth   pf pp. 180-181

Prop A map $D: C^\infty(M) \longrightarrow C^\infty(M)$ is a derivation iff $\exists X \in \mathfrak{X}(M)$ s.t. $Df = Xf$ $\forall f \in C^\infty(M)$.

Pf ($\Leftarrow$) Previous prop.

($\Rightarrow$) Define X_p so that $X_p f = Df(p)$. Then $X_p : C^\infty(M) \longrightarrow \mathbb{R}$ is a derivation at p, i.e. $X_p \in T_pM$. Since $Xf = Df \in C^\infty(M)$, previous prop implies X smooth. $\square$

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$\mathfrak{X}$ is not a functor!

$F : M \longrightarrow N$ smooth, $X \in \mathfrak{X}(M)$

Might want $F_* X \in \mathfrak{X}(N)$ with

$dF_p X_p = (F_* X)_{F(p)}$ — what might go wrong?

$$\begin{tikzcd} TM \arrow[r, "dF"] \arrow[d] & TN \arrow[d] \\ M \arrow[r, "F"'] \arrow[u, bend left, dashed, "X"] & N \arrow[u, bend right, dashed, "F_*X\,?"'] \end{tikzcd}$$

(1) No rule for $(F_* X)_q$, $q \in N \setminus FM$

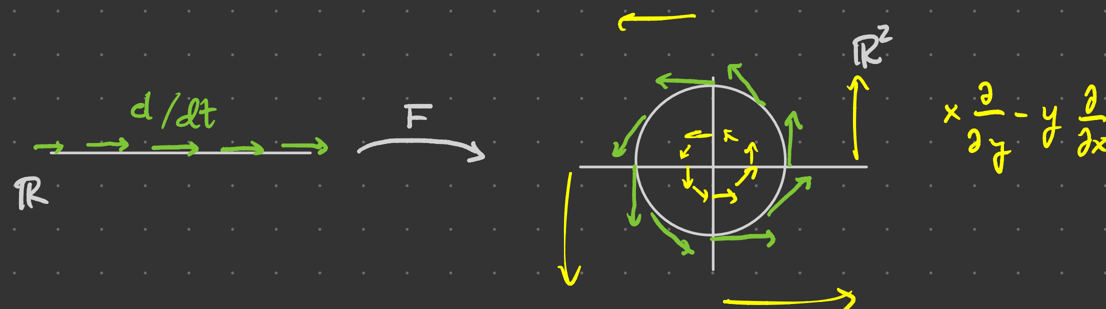
(2) Still a section if well-defined...

(3)

Call $X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(N)$ F-related ($F: M \to N$ smooth)

when $dF_p X_p = Y_{F(p)}$ $\forall p \in M$.

E.g. $F: \mathbb{R} \to \mathbb{R}^2$
$t \mapsto (\cos t, \sin t)$ $\quad \frac{d}{dt}$ is F-related to $x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$:



Prop $F: M \to N$ smooth, $X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(N)$ are F-related iff
$\forall U \subseteq N$ open and $f \in C^\infty(U)$,
$$X(f \circ F) = (Yf) \circ F.$$

Pf $\quad X(f\circ F)(p) = X_p(f\circ F) = dF_p(X_p)f$

and $\quad (Yf)\circ F(p) = Yf(F(p)) = Y_{F(p)}f$. $\square$

E.g. (ct'd) $\quad F: \mathbb{R} \to \mathbb{R}^2$, $t \mapsto (\cos t, \sin t)$, $\quad X = \frac{d}{dt}$, $\quad Y = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$

For $U \subseteq \mathbb{R}^2$ open, $f \in C^\infty(U)$, $p \in U$

$$X(f\circ F)(p) = \left(-\sin(p)\frac{\partial}{\partial x}\Big|_{(\cos p,\, \sin p)} + \cos(p)\frac{\partial}{\partial y}\Big|_{(\cos p,\, \sin p)}\right)f$$

$$(Yf)\circ F(p) = \left(\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\Big|_{(\cos p,\, \sin p)}\right)f$$