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Group actions & equivariant maps

Call a left (resp. right) action of a Lie group G on a smooth mfld M a Lie group action when the action map $G \times M \xrightarrow{\theta} M$, $(g,x) \longmapsto g\cdot x$ (resp. $M \times G \longrightarrow M$, $(x,g) \longmapsto x\cdot g$) is smooth.

Prop $\theta_g = g\cdot : M \longrightarrow M$, $x \longmapsto g\cdot x$ is a diffeo $\forall g \in G$

Pf Smooth inverse $g^{-1}\cdot : M \longrightarrow M$. $\square$

May identify Lie action with hom
$$\begin{aligned} G &\longrightarrow \underbrace{\mathrm{Aut}(M)}_{\text{diffeos } M \to M \text{ under } \circ} \\ g &\longmapsto \theta_g \end{aligned}$$

Notation • $G\cdot x = \{g\cdot x \mid g\in G\} \subseteq M$ orbit of x

• $G_x = \{g\in G \mid g\cdot x = x\}$ the isotropy gp / stabilizer of x

• $G \circlearrowright M$ or $M \circlearrowleft G$ • $\Theta_g = g\cdot : M \longrightarrow M$

E.g. • $G \circlearrowright G$ by left translation; also by conjugation

$g * h := ghg^{-1}$

• trivial action $g\cdot x = x \ \forall g\in G, x\in M$

• $GL_n\mathbb{R} \circlearrowright \mathbb{R}^n$ by matrix mult'n

TPS What are the orbits of $GL_n\mathbb{R} \circlearrowright \mathbb{R}^n$?

of $SL_n\mathbb{R} \circlearrowright \mathbb{R}^n$?

of $SO(n) \circlearrowright \mathbb{R}^n$?

Try to solve $A\cdot x = y$ for $A \in \dots$

Here $SO(n) = \{A\in SL_n\mathbb{R} \mid AA^T = I_n\}$.

- $Gl_n \mathbb{R}$: $\mathbb{R}^n \setminus 0$, $\{0\}$

  $SL_1 \mathbb{R} = \{1\}$: singletons in $\mathbb{R}$

  $SL_2 \mathbb{R}$: $A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

  take $A = \begin{pmatrix} x & b \\ y & d \end{pmatrix}$ s.t. $xd - by = 1$

- $Sl_n \mathbb{R}$: $\mathbb{R}^n \setminus 0$, $\{0\}$

- $SO(n)$: λS^{n-1} for $\lambda \geq 0$

Suppose $G \curvearrowright M, N$ smoothly. Call $F: M \longrightarrow N$ **G-equivariant** when $F(g \cdot x) = g \cdot F(x)$ $\forall g \in G$, $x \in M$, i.e.

$$\begin{tikzcd} M \arrow[r, "F"] \arrow[d, "\theta_g"'] & N \arrow[d, "\varphi_g"] \\ M \arrow[r, "F"'] & N \end{tikzcd}$$

commutes $\forall g \in G$.

E.g. $\mathbb{R}^n \xrightarrow{\lambda \cdot} \mathbb{R}^n$, $x \longmapsto \lambda x$ is $GL_n\mathbb{R}$-equivariant

Equivariant rank thm $G \curvearrowright M, N$ smoothly, $F: M \longrightarrow N$ smooth & equivariant. If $G \curvearrowright M$ transitive, then F has constant rank.

Pf Same as "Lie gp homs have constant rank" but with

$$\begin{array}{ccc} M & \xrightarrow{F} & N \\ \theta_g \downarrow & & \downarrow \varphi_g \\ M & \xrightarrow[F]{} & N \end{array}$$ $\Longrightarrow$ rank F is constant on orbits

If $G \circlearrowright M$ transitive, then orbit is M.

$\square$

Application $x \in M$. Define $\theta^{(x)} : G \longrightarrow M$, $g \longmapsto g \cdot x$, the orbit map.

Note $\theta^{(x)}(G) = G \cdot x$, $(\theta^{(x)})^{-1}\{x\} = \{g \in G \mid g \cdot x = x\} = G_x$.

Prop $\theta^{(x)}$ is smooth of constant rank. Thus G_x is a properly embedded Lie subgp. If $G_x = e$, then $\theta^{(x)}$ is an injective smooth immersion, so $G \cdot x$ is an immersed submfld of M. (Ch. 21: $G \cdot x \subseteq M$ immersed always.)

$$gx = hx \Longrightarrow \underbrace{g^{-1}h}_{\in G_x} x = x$$

Pf Since

$$\begin{tikzcd} G \cong G \times \{x\} \arrow[r, hook] \arrow[dr, "\theta^{(x)}"'] & G \times M \arrow[d, "\theta"] \\ & M \end{tikzcd}$$

commutes, $\theta^{(x)}$ is smooth.

Further, $\theta^{(x)}$ is equivariant (wrt left trans'n on G, θ on M):

$$\theta^{(x)}(g'g) = (g'g) \cdot x = g' \cdot (g \cdot x) = g' \cdot \theta^{(x)}(g).$$

Since $G \circlearrowright G$ transitively, $\theta^{(x)}$ has constant rank. $\square$

The orthogonal group $O(n)$

$$O(n) := \{ A \in \mathbb{R}^{n \times n} \mid A^T A = I_n \}$$

- Let's show $O(n)$ is a Lie gp

Define $\Phi: GL_n\mathbb{R} \longrightarrow \mathbb{R}^{n\times n}$ so that $O(n) = \Phi^{-1}\{I_n\}$.

$$A \longmapsto A^T A$$

Let $GL_n\mathbb{R} \underset{\text{mult'n}}{\circlearrowleft} GL_n\mathbb{R}$, $\mathbb{R}^{n\times n} \circlearrowleft GL_n\mathbb{R}$ by $M\cdot B = B^T M B$.

These are smooth actions wrt which Φ is equivariant:

$$\Phi(AB) = (AB)^T(AB) = B^T A^T A B = B^T \Phi(A) B = \Phi(A)\cdot B$$

Thus Φ has constant rank, so $O(n)$ is a properly embedded Lie subgp of $GL_n\mathbb{R}$.

- Note $O(n)$ is compact as it is closed + bounded in $\mathbb{R}^{n\times n}$.
- $\dim O(n) = \frac{n(n-1)}{2}$: Compute rank of Φ at I_n.

For $B \in T_{I_n} GL_n\mathbb{R} = \mathbb{R}^{n\times n}$, define $\gamma : (-\varepsilon, \varepsilon) \longrightarrow GL_n\mathbb{R}$
$t \longmapsto I_n + tB$

Then $d\Phi_{I_n}(B) = \frac{d}{dt}\Big|_{t=0} \Phi\circ\gamma(t) = \frac{d}{dt}\Big|_{t=0} (I_n + tB)^T (I_n + tB)$

$= B^T + B$.

This is a symmetric matrix $(M^T = M)$ and for any B symm,
$d\Phi_{I_n}\left(\frac{1}{2}B\right) = B$, so $\operatorname{im} d\Phi_{I_n} = \underbrace{\text{symm matrices in } \mathbb{R}^{n\times n}}_{\dim\ n+(n-1)+\cdots+1 = \frac{n(n+1)}{2}}$.

Finally, $n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$.

- If $A^TA = I_n$, then $1 = \det(A^TA) = \det(A^T)\det A = (\det A)^2$

so $\det A = \pm 1 \quad \forall A \in O(n)$.

The special orthogonal group $SO(n)$

$SO(n) := O(n) \cap SL_n\mathbb{R}$ is the open subgp of $O(n)$ with $\det A = 1$.

$SO(2) \approx S^1 \qquad \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$

$SO(3) \approx \mathbb{R}P^3$

The unitary group $U(n)$

$z \cdot w = \sum z_i \bar{w}_i$ preserved

$U(n) = \{A \in \mathbb{C}^{n\times n} \mid A^* A = I_n\}$ is a properly emb

adjoint $A^* = \bar{A}^T$; satisfies $(AB)^* = B^* A^*$

subgp of $GL_n\mathbb{C}$ of dim n^2.

- $U(1) \cong S^1$

The special unitary group $SU(n)$

$SU(n) = U(n) \cap SL_n\mathbb{C}$ properly emb (n^2-1)-dim'l subgp of $U(n)$

$SU(2) \approx S^3$.

$\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$

$i^2 = j^2 = k^2 = -1$

$i \to j \to k \to i$

$\mathbb{H}^\times = \mathbb{H} \setminus 0 \supseteq S^3$

$\parallel$ 2

$SU(2)$ iso as Lie groups

$SU(2) \to SO(3)$

The Euclidean group $E(n)$

$O(n) \circlearrowright \mathbb{R}^n$ so we may form $E(n) = \mathbb{R}^n \rtimes O(n)$

with mult'n $(x, A)(y, B) = (x + Ay, AB)$

This is the $\underbrace{\text{isometry}}_{\text{distance preserving}}$ group of $\mathbb{R}^n$

$$\rho : E(n) \hookrightarrow GL_{n+1}\mathbb{R}$$
$$(x, A) \longmapsto \left(\begin{array}{c|c} A & x \\ \hline 0 & 1 \end{array}\right)$$

$$\mathrm{Im}(\mathbb{H}) := \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$$

$$p \in S^3 \subseteq \mathbb{H}^\times$$

$$S^3 \circlearrowright \mathrm{Im}(\mathbb{H})$$

$$p \cdot q = p q p^*$$

$$p \cdot : \mathrm{Im}(\mathbb{H}) \to \mathrm{Im}(\mathbb{H})$$
$$\in SO(\mathrm{Im}(\mathbb{H}))$$