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Lie Groups

G smooth manifold + group

$\mu : G \times G \longrightarrow G$, $\iota : G \longrightarrow G$ both smooth

mult'n inversion

E.g. $(\mathbb{R}^n, +)$

For $g \in G$, $L_g : G \longrightarrow G$, $R_g : G \longrightarrow G$
$h \longmapsto gh$ $\qquad$ $h \longmapsto hg$

left translation by g $\qquad$ right translation by g

Both are smooth with smooth inverses $L_{g^{-1}}$, $R_{g^{-1}}$, hence diffeomorphisms

$$\begin{tikzcd} G \arrow[r] \arrow[dr, "L_g"'] & G \times G \arrow[d, "\mu"] \\ & G \end{tikzcd} \qquad h \longmapsto (g,h)$$

E.g.

- $GL_n\mathbb{R}$ (dimension?) n^2 — open submfld $\mathbb{R}^{n\times n}$
- open subgroups of Lie groups
- $\mathbb{R}^n$, $\mathbb{C}^n$, $\mathbb{R}^\times$, $\mathbb{C}^\times$, $GL_n\mathbb{C}$
- S^1
- finite products of Lie groups
- tori $\mathbb{T}^n = \underbrace{S^1 \times \cdots \times S^1}_{n}$
- countable discrete groups

For G, H Lie groups, a Lie group homomorphism $G \to H$ is a homomorphism of groups which is also smooth.

E.g. Here is a diagram of Lie group homomorphisms:

$$\begin{tikzcd}
 & \mathbb{R} \arrow[r, "\exp"] \arrow[d, hook] & \mathbb{R}^\times \arrow[d, hook] & GL_n\mathbb{R} \arrow[l, "\det"'] \arrow[d, hook] \\
 & \mathbb{C} \arrow[r, "\exp"'] & \mathbb{C}^\times & GL_n\mathbb{C} \arrow[l, "\det"] \\
\mathbb{R} \arrow[r, "\cong"] \arrow[rr, bend right, "\varepsilon"'] & 2\pi i\mathbb{R} \arrow[r] \arrow[u, hook] & S^1 \arrow[u, hook] &
\end{tikzcd}$$

Thm Every Lie group homomorphism has constant rank.

Pf For $f: G \longrightarrow H$ a Lie gp hom and $g \in G$, the diagram

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "L_g"'] & H \arrow[d, "L_{f(g)}"] \\ G \arrow[r, "f"'] & H \end{tikzcd}$$

commutes

$$\left(\begin{tikzcd} g' \arrow[r, mapsto] \arrow[d, mapsto] & f(g') \arrow[d, mapsto] \\ gg' \arrow[r, mapsto] & f(gg') = f(g)f(g') \end{tikzcd} \right)$$

so taking diff'ls at identities gives

$$\begin{tikzcd} T_eG \arrow[r, "df_e"] \arrow[d, "d(L_g)_e"', "\cong"] & T_eH \arrow[d, "\cong"', "d(L_{f(g)})_e"] \\ T_gG \arrow[r, "df_g"'] & T_{f(g)}H \end{tikzcd}$$

so $\operatorname{rank}(df_g) = \operatorname{rank}(df_e)\ \forall g \in G$. $\square$

Cor A Lie gp hom is iso $\Leftrightarrow$ it is bijective. $\square$

// Reading: pg. 154-155 — universal covering groups of Lie gps exist and are unique. Their covering maps are Lie gp homs. //

E.g.:

- $\varepsilon^n : \mathbb{R}^n \longrightarrow \mathbb{T}^n$
- $\exp : \mathbb{C} \longrightarrow \mathbb{C}^\times$
- $\widetilde{SL_2\mathbb{R}} \longrightarrow SL_2\mathbb{R}$

  ↳ example of Lie group which is <u>not</u> a matrix group

A <u>Lie</u> sub<u>group</u> of a Lie gp G is $H \leq G$ endowed with a topology and smooth structure making it a Lie gp and submanifold.

Prop G Lie gp, $H \le G$ & H embedded submfld. potentially immersed!

Then H is a Lie subgp of G.

Pf Restrict mult + inv'n maps on domain and codomain. $\square$

E.g. Open subgps are embedded hence Lie subgps. But...

Lemma Every open $H \le G$ is also closed, hence a union of components of G.

Pf $G \setminus H = \bigcup_{g \in G \setminus H} gH = \bigcup_{g \in G \setminus H} \underbrace{L_g}_{\text{diffeo}} H$ is open, so H is closed. $\square$

Prop If $W \subseteq G$ is a conn'd nbhd of e, then $\underbrace{\langle W \rangle}_{\text{subgp gen'd by } W} = G_0$, the conn'd comp't of e in G. $\square$

In HW, you'll show $G_0 \trianglelefteq G$ and every conn'd comp't of G is $\approx G_0$.

Kernels of Lie gp homs give a rich class of Lie subgps (generally not open).

Prop If $f: G \longrightarrow H$ is a Lie gp hom, then $\ker(f)$ is a properly embedded Lie subgp of G with $\operatorname{codim} \ker(f) = \operatorname{rank}(f)$.

Pf Since f is constant rank, $\ker(f) = f^{-1}\{e\}$ is an embedded subgp. $\square$

Eg.

- $SL_n\mathbb{R} = \ker(\det : GL_n\mathbb{R} \longrightarrow \mathbb{R}^\times)$
- $SL_n\mathbb{C} = \ker(\det : GL_n\mathbb{C} \longrightarrow \mathbb{C}^\times)$

Here's another embedded subgp:

$$GL_n \mathbb{C} \hookrightarrow GL_{2n} \mathbb{R}$$

$$(a_{kl} + i b_{kl}) \longmapsto \text{matrix with } 2\times 2 \text{ blocks } \begin{pmatrix} a_{kl} & -b_{kl} \\ b_{kl} & a_{kl} \end{pmatrix}$$

E.g. $\mathbb{C}^\times \hookrightarrow GL_2 \mathbb{R}$

$a+ib \longmapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

$(a+ib)(c+id) = (ac-bd) + i(ad+bc)$

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} c & -d \\ d & c \end{pmatrix} = \begin{pmatrix} ac-bd & -(ad+bc) \\ ad+bc & ac-bd \end{pmatrix}$$

Thm (deep) If $H \le G$ is a Lie subgp, then

H is closed iff H is embedded.

(Pf pp. 159–161)

Thm (deeper — closed subgp thm) $\{H \le G \mid H \text{ closed}\} = \{H \le G \mid H \text{ emb Lie subgp}\}$

no mfld / str assumed

(Pf Ch. 20)

<u>Q</u> What happens when $f: G \to H$ Lie gp hom and $\mathrm{rank}(f) < \dim H$?

<u>A</u> Constant rank level set thm: any level set is properly embedded submfld.

<u>Note</u> If f is trivial: $g \mapsto e$ $\forall g \in G$
then $\ker(f) = G$ — not discrete!

$\mathrm{codim}(\ker(f)) = \mathrm{rank}(f)$
so $\ker(f)$ 0-dim'l iff $\mathrm{rank}(f) = \dim G$

Q Is image of Lie gp hom a Lie subgp?

A $f : G \longrightarrow H$ Lie gp hom

$f(G) \leq H$ ✓

If f inj, then $f(G)$ Lie subgp

$$\begin{array}{c} \mathbb{R} \\ \downarrow \\ S^1 \end{array}$$

$\exists$ non inj f w/ $f(G)$ still a Lie subgp.

$$1 \longrightarrow \ker(f) \longrightarrow G \xrightarrow{f} f(G)$$

$f(G) \cong G/\ker(f)$ — as gps

maybe not a mfld!

<u>Note</u> Lie groups = groups in the category Diff

A gp object in a Cartesian cat $\mathcal{C}$, w/ terminal obj e is $G \in \mathrm{ob}\,\mathcal{C}$

equipped with
$$\mu : G \times G \longrightarrow G$$
$$\iota : G \longrightarrow G$$
$$\eta : e \longrightarrow G$$

s.t.
$$\begin{tikzcd}
e \times G \arrow[r] \arrow[dr, "\cong"'] & G \times G \arrow[d, "\mu"] & G \times e \arrow[l] \arrow[dl, "\cong"] \\
& G &
\end{tikzcd}
\qquad
\begin{tikzcd}
G \times G \times G \arrow[r, "\mathrm{id} \times \mu"] \arrow[d, "\mu \times \mathrm{id}"'] & G \times G \arrow[d, "\mu"] \\
G \times G \arrow[r, "\mu"'] & G
\end{tikzcd}$$

$$g \longmapsto (g,g) \longmapsto (g,g^{-1})$$

$$\begin{tikzcd}
G \arrow[r, "\Delta"] \arrow[d] & G \times G \arrow[r, "id \times i"] & G \times G \arrow[d, "\mu"] \\
e \arrow[rr, "\eta"'] & & G
\end{tikzcd}$$

Note

$$\begin{tikzcd}
& G \arrow[d, dashed, "\exists!"] \arrow[ddl, bend right, "="'] \arrow[ddr, bend left] & \\
& G \times e \arrow[dl] \arrow[dr] & \\
G & & e
\end{tikzcd}
\qquad
\begin{tikzcd}
& H \arrow[d, "f"] \arrow[ddl, bend right, "f"'] \arrow[ddr, bend left] & \\
& G \arrow[dl, "="'] \arrow[dr] & \\
G & & e
\end{tikzcd}$$