MATH 546: MANIFOLDS
HOMEWORK DUE FRIDAY WEEK 9

Problems taken from *Introduction to Smooth Manifolds* are marked ISM x–y. Please review the syllabus for expectations and policies regarding homework.

Problem 1 (ISM 18–7, THE POINCARÉ DUALITY THEOREM). Let M be an oriented smooth n-manifold. Define a map $\mathrm{PD}\colon \Omega^p(M) \to \Omega_c^{n-p}(M)^*$ by

$$\mathrm{PD}(\omega)(\eta) = \int_M \omega \wedge \eta.$$

(a) Show that PD descends to a linear map (still denoted by the same symbols) $\mathrm{PD}\colon H^p_{\mathrm{dR}}(M) \to H_c^{n-p}(M)^*$.
(b) Show that PD is an isomorphism for each p [*Hint*: Imitate the proof of the de Rham theorem. You will need Lemma 17.27 and you may assume that compactly supported de Rham cohomology has a Mayer–Vietoris sequence as in Problem 18–6 without proof.]

Problem 2 (ISM 18–8). Let M be a compact smooth n-manfiold.

(a) Show that all de Rham groups of M are finite-dimensional. [*Hint*: for the orientable case, use Poincaré duality to show that $H^p_{\mathrm{dR}}(M) \cong H^p_{\mathrm{dR}}(M)^{**}$. (You may use the fact that an infinite-dimensional vector space is not isomorphic to its double dual.) For the nonorientable case, use Lemma 17.33.]
(b) Show that if M is orientable, then $\dim H^p_{\mathrm{dR}}(M) = \dim H^{n-p}_{\mathrm{dR}}(M)$ for all p.

Problem 3 (ISM 18–9). Let M be a smooth n-manifold, all of whose de Rham groups are finite-dimensional. The *Euler characteristic* of M is the number

$$\chi(M) := \sum_{p=0}^{n} (-1)^p \dim H^p_{\mathrm{dR}}(M).$$

Show that $\chi(M)$ is a homotopy invariant of M, and $\chi(M) = 0$ when M is compact, orientable, and odd-dimensional.