MATH 546: MANIFOLDS
HOMEWORK DUE FRIDAY WEEK 8

Problems taken from *Introduction to Smooth Manifolds* are marked ISM x–y. Please review the syllabus for expectations and policies regarding homework.

Problem 1 (ISM 17–1). Let M be a smooth manifold with or without boundary, and let $\omega \in \Omega^p(M)$, $\eta \in \Omega^q(M)$ be closed forms. Show that the de Rham cohomology class of $\omega \wedge \eta$ depends only on the cohomology classes of ω and η, and thus there is a well-defined bilinear map

$$\begin{aligned} \smile\colon H^p_{\mathrm{dR}}(M) \times H^q_{\mathrm{dR}}(M) &\longrightarrow H^{p+q}_{\mathrm{dR}}(M) \\ ([\omega],[\eta]) &\longmapsto [\omega] \smile [\eta] := [\omega \wedge \eta] \end{aligned}$$

called the *cup product*.

Problem 2 (ISM 17–5). For each $n \geq 1$, compute the de Rham cohomology groups of $\mathbb{R}^n \smallsetminus \{e_1, -e_1\}$; and for each nonzero cohomology group, give specific differential forms whose cohomology classes form a basis.

Problem 3 (ISM 17–6). Let M be a connected smooth manifold of dimension $n \geq 3$. For any $x \in M$ and $0 \leq p \leq n-2$, prove that the map $H^p_{\mathrm{dR}} \to H^p_{\mathrm{dR}}(M \smallsetminus \{x\})$ induced by the inclusion $M \smallsetminus \{x\} \hookrightarrow M$ is an isomorphism. Prove that the same is true for $p = n-1$ if M is compact and orientable. [*Hint*: Use the Mayer–Vietoris theorem. The cases $p = 0$, $p = 1$, and $p = n-1$ require special handling.]

Problem 4 (ISM 17–10). Let $p \in \mathbb{C}[z]$ be a nonzero polynomial in the variable z with complex coefficients. Read Problems 2–8 and 2–9 and convince yourself that there is a unique well-defined smooth map $\tilde{p}\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ such that $\tilde{p}([z,1]) = [p(z),1]$. (You do not need to write up solutions to these progblems.) Prove that the degree of $\tilde{p}$ (as a smooth map between manifolds) is equal to the degree of the polynomial p in the usual sense.

Problem 5 (ISM 17–12). Suppose M and N are compact, connected, oriented smooth n-manifolds, and $F\colon M \to N$ is a smooth map. Prove that if $\int_M F^*\eta \neq 0$ for some $\eta \in \Omega^n(N)$, then F is surjective. Give an example to show that F can be surjective even if $\int_M F^*\eta = 0$ for every $\eta \in \Omega^n(N)$.