MATH 546: MANIFOLDS
HOMEWORK DUE FRIDAY WEEK 7

Problems taken from *Introduction to Smooth Manifolds* are marked ISM x–y. Please review the syllabus for expectations and policies regarding homework.

Problem 1 (14–4). Let V be a finite-dimensional vector space.

(a) Show that an ordered k-tuple $(v_1, \dots, v_k)$ of elements of V is linearly dependent if and only if $v_1 \wedge \cdots v_k = 0$.

(b) Show that two linearly independent ordered k-tuples $(v_1, \dots, v_k)$ and $(w_1, \dots, w_k)$ of elements of V have the same span if and only if

$$v_1 \wedge \cdots \wedge v_k = c w_1 \wedge \cdots \wedge w_k$$

for some nonzero scalar c.

Problem 2 (14–5, CARTAN'S LEMMA). Let M be a smooth n-manifold with or without boundary, and let $(\omega^1, \dots, \omega^k)$ be an ordered k-tuple of smooth 1-forms on an open subset $U \subseteq M$ such that $(\omega^1|_p, \dots, \omega^k|_p)$ is linearly independent for each $p \in U$. Given smooth 1-forms $\alpha^1, \dots, \alpha^k$ on U such that

$$\sum_{i=1}^{k} \alpha^i \wedge \omega^i = 0$$

show that each α^i can be written as a linear combination of $\omega^1, \dots, \omega^k$ with smooth coefficients.

Problem 3 (ISM 15–1). Suppose M is a smooth manifold that is the union of two orientable open submanifolds with connected intersection. Show that M is orientable. Use this to give another proof that S^n is orientable.

Problem 4 (ISM 16–3(a)). Suppose E and M are oriented smooth n-manifolds with or without boundary, and $\pi\colon E \to M$ is a smooth orientation-preserving k-sheeted covering map or generalized covering map. Show that

$$\int_E \pi^* \omega = k \int_M \omega$$

for any compactly supported n-form ω on M.

Problem 5. Let M be an oriented smooth n manifold, $f \in C^\infty(M)$, and $\omega \in \Omega_c^{n-1}(M)$. Use Stokes' theorem to prove the following generalization of integration by parts:

$$\int_M f \, \mathrm{d}\omega = \int_{\partial M} f\omega - \int_M \mathrm{d}f \wedge \omega.$$