MATH 546: MANIFOLDS
HOMEWORK DUE FRIDAY WEEK 5

Problems taken from *Introduction to Smooth Manifolds* are marked ISM x–y. Please review the syllabus for expectations and policies regarding homework.

Problem 1 (ISM 11–5). For any smooth manifold M, show that T^*M is a trivial vector bundle if and only if TM is trivial.

Problem 2 (ISM 11–13). The *length* of a smooth curve segment $\gamma\colon [a,b] \to \mathbb{R}^n$ is defined to be the value of the (ordinary) integral

$$L(\gamma) = \int_a^b |\gamma'(t)|| \, dt.$$

Show that there is no smooth covector field $\omega \in \mathfrak{X}^*(\mathbb{R}^n)$ with the property that $\int_\gamma \omega = L(\gamma)$ for every smooth curve γ.

Problem 3 (ISM 11–14). Consider the following two covector fields on $\mathbb{R}^3$:

$$\omega = -\frac{4z\,dx}{(x^2+1)^2} + \frac{2y\,dy}{y^2+1} + \frac{2x\,dz}{x^2+1},$$
$$\eta = -\frac{4xz\,dx}{(x^2+1)^2} + \frac{2y\,dy}{y^2+1} + \frac{2\,dz}{x^2+1}.$$

(a) Set up and evaluate the line integral of each covector field along the straight line segment from $(0,0,0)$ to $(1,1,1)$.
(b) Determine whether either of these covector fields is exact.
(c) For each one that is exact, find a potential function and use it to recompute the line integral.

Problem 4 (ISM 11–16). Let M be a compact smooth manifold of positive dimension. Show that every exact covector field on M vanishes at least at two points in each component of M.

Problem 5 (ISM 12–7). Let (e^1, e^2, e^3) be the standard dual basis for $(\mathbb{R}^3)^*$. Show that $e^1 \otimes e^2 \otimes e^3$ is not equal to a sum of an alternating tensor and a symmetric tensor.