MATH 546: MANIFOLDS
HOMEWORK DUE FRIDAY WEEK 4

Problems taken from *Introduction to Smooth Manifolds* are marked ISM x–y. Please review the syllabus for expectations and policies regarding homework.

Problem 1 (ISM 9–12). Suppose M_1 and M_2 are connected smooth n-manifolds and $M_1 \# M_2$ is their smooth connected sum (see Example 9.31). Show that the smooth structure on $M_1 \# M_2$ can be chosen in such a way that there are open subsets $\tilde{M}_1, \tilde{M}_2 \subseteq M_1 \# M_2$ that are diffeomorphic to $M_1 \smallsetminus \{p_1\}$ and $M_2 \smallsetminus \{p_2\}$, respectively, such that $\tilde{M}_1 \cup \tilde{M}_2 = M_1 \# M_2$ and $\tilde{M}_1 \cap \tilde{M}_2$ is diffeomorphic to $(-1,1) \times S^{n-1}$.

Problem 2 (ISM 9–16). Give an example of smooth vector fields V, $\tilde{V}$, and W on $\mathbb{R}^2$ such that $V = \tilde{V} = \partial/\partial x$ along the x-axis but $\mathscr{L}_V W \neq \mathscr{L}_{\tilde{V}} W$ at the origin.

Problem 3 (ISM 10–1). Let E be the total space of the Möbius bundle constructed in Example 10.3.

(a) Show that E has a unique smooth structure such that the quotient map $q\colon \mathbb{R}^2 \to E$ is a smooth covering map.
(b) Show that $\pi\colon E \to S^1$ is a smooth rank-1 vector bundle.
(c) Show that it is not the trivial bundle.

Problem 4 (ISM 10–11). Prove Proposition 10.26: A bijective bundle homomorphism is a bundle isomorphism.

Problem 5 (ISM 10–18). Suppose S is a properly embedded codimension k submanifold of $\mathbb{R}^n$. Show that the following are equivalent:

(a) There exists a smooth defining function for S on some neighborhood U of S in $\mathbb{R}^n$; *i.e.*, there is a smooth function $\Phi\colon U \to \mathbb{R}^k$ such that S is a regular level set of Φ.
(b) The normal bundle NS is a trivial vector bundle.