MATH 546: MANIFOLDS
HOMEWORK DUE FRIDAY WEEK 3

Problems taken from *Introduction to Smooth Manifolds* are marked ISM x–y. Please review the syllabus for expectations and policies regarding homework.

Problem 1 (ISM 7–15). Let G be a Lie group and let G_0 be its identity component. Then G_0 is a normal subgroup of G, and is the only connected open subgroup. Every connected component of G is diffeomorphic to G_0.

Problem 2 (ISM 8–10). Let M be the open submanifold of $\mathbb{R}^2$ where both x and y are positive, and let $F\colon M \to M$ be the map $F(x,y) = (xy, y/x)$. Show that F is a diffeomorphism, and compute F_*X and F_*Y, where

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \qquad Y = y\frac{\partial}{\partial x}.$$

Also compute $[X, Y]$ and $[F_*X, F_*Y]$.

Problem 3 (ISM 8–25). Prove that if G is an Abelian Lie group, then $\mathfrak{g} = \mathrm{Lie}(G)$ is Abelian. [*Hint*: Show that the inversion map $i\colon G \to G$ is a group isomorphism. Prove any parts of ISM Problem 7–2 that might be relevant.]

Problem 4 (ISM 8–29). Theorem 8.46 implies that the Lie algebra of any Lie subgroup of $\mathrm{GL}_n\mathbb{R}$ is canonically isomorphic to a subalgebra of $\mathfrak{gl}_n\mathbb{R}$, with a similar statement for Lie subgroups of $\mathrm{GL}_n\mathbb{C}$. Under this isomorphism, show that

$$\mathrm{Lie}(\mathrm{SO}(n)) \cong \mathfrak{o}(n) = \{A \in \mathfrak{gl}_n\mathbb{R} \mid A^T + A = 0\}$$

and

$$\mathrm{Lie}(\mathrm{U}(n)) \cong \mathfrak{u}(n) = \{A \in \mathfrak{gl}_n\mathbb{C} \mid A^* + A = 0\}.$$

Problem 5 (ISM 9–3(b)). Compute the flow the vector field

$$W = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}$$

on $\mathbb{R}^2$.