MATH 546: MANIFOLDS
HOMEWORK DUE FRIDAY WEEK 2

Problems taken from *Introduction to Smooth Manifolds* are marked ISM x–y. Please review the syllabus for expectations and policies regarding homework.

Problem 1 (ISM 7–11). Considering S^{2n+1} as the unit sphere in $\mathbb{C}^{n+1}$, define an action of S^1 on S^{2n+1}, called the *Hopf action*, by

$$z \cdot (w^1, \dots, w^{n+1}) = (zw^1, \dots, zw^{n+1}).$$

Show that this action is smooth and its orbits are disjoint unit circles in $\mathbb{C}^{n+1}$ whose union is S^{2n+1}.

Problem 2 (ISM 7–22). Let $\mathbb{H} = \mathbb{C} \times \mathbb{C}$ and consider it as a real vector space. Define a bilinear product

$$\mathbb{H} \times \mathbb{H} \longrightarrow \mathbb{H}$$

$$((a,b),(c,d)) \longmapsto (ac - d\bar{b}, \bar{a}d + cb)$$

for $a, b, c, d \in \mathbb{C}$. This makes $\mathbb{H}$ a 4-dimensional $\mathbb{R}$-algebra called the *quaternions*. For each $p = (a,b) \in \mathbb{H}$, define $p^* = (\bar{a}, -b)$. It is useful to work with the basis $(1, i, j, k) = ((1,0),(i,0),(0,1),(0,-i))$ for $\mathbb{H}$. One may verify that this basis satisfies

$$\begin{aligned} &i^2 = j^2 = k^2 = -1, \qquad 1q = q1 = q \text{ for all } q \in \mathbb{H}, \\ &ij = -ji = k, \qquad jk = -kj = i, \qquad ki = -ik = j, \\ &1^* = 1, \qquad i^* = -i, \qquad j^* = -j, \qquad k^* = -k. \end{aligned}$$

A quaternion p is said to be *real* when $p^* = p$, and *imaginary* when $p^* = -p$. Real quaternions can be identified with real numbers via the correspondence $\lambda \leftrightarrow x1$.

(a) Show that quaternionic multiplication is associative but not commutative.
(b) Show that $(pq)^* = q^*p^*$ for all $p, q \in \mathbb{H}$.
(c) Show that $\langle p, q \rangle = \frac{1}{2}(p^*q + q^*p)$ is an inner product on $\mathbb{H}$ with associated norm satisfying $|pq| = |p||q|$.
(d) Show that every nonzero quaternion has a two-sided multiplicative inverse given by $p^{-1} = |p|^{-2}p^*$.
(e) Show that the set $\mathbb{H}^\times$ of nonzero quaternions is a Lie group under quaternionic multiplication.

Problem 3 (ISM 7–23). Let $\mathbb{H}^\times$ be the Lie group of nonzero quaternions, and let $S^3 \subseteq \mathbb{H}^\times$ be the set of norm 1 quaternions. Show that S^3 is a properly embedded Lie subgroup of $\mathbb{H}^\times$, isomorphic to $\mathrm{SU}(2)$.

Problem 4 (ISM 8–4). Let M be a smooth manifold with boundary. Show that there exists a global smooth vector field on M whose restriction to ∂M is everywhere inward-pointing, and one whose restriction to ∂M is everywhere outward-pointing.