MATH 545: MANIFOLDS
WEDNESDAY WEEK 10

Let $U \xrightarrow{f} Y \xleftarrow{g} V$ be smooth maps between smooth manifolds. We say f and g *intersect transversely* when the product map $f \times g\colon U \times V \to Y \times Y$ intersects the diagonal $\Delta_Y \subseteq Y \times Y$ transversely.

Problem 1. Prove that if f and g intersect transversely, then the fiber product

$$W = U \times_Y V := \{(p,q) \in U \times V \mid f(p) = g(q)\}$$

is a submanifold of $U \times V$, and for any $(p,q) \in W$,

$$T_{(p,q)}W = \{(u,v) \in T_pU \times T_qV \mid df_p(u) = dg_q(v)\}.$$

(*Hint*: $W = (f \times g)^{-1}\Delta_Y$.)

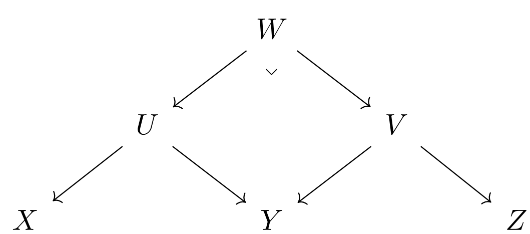
If $U \xrightarrow{f} Y$ is a smooth submersion, then f is transverse to every smooth map $Y \xleftarrow{g} V$ (proof presented in class), whence the pullback $W = U \times_Y V \subseteq U \times V$ is automatically a smooth submanifold.

Problem 2. Given smooth maps between smooth manifolds $U \xrightarrow{f} Y \xleftarrow{g} V$ where f is a submersion, prove that the induced map $U \times_Y V \to V$ is a submersion.

Given the above property, we say that submersions are *stable* under pullback.

If X, Y are smooth manifolds, then a diagram of smooth maps $X \leftarrow U \to Y$ is called a *smooth span*. (*Note*: By the universal property of products, this is the same data as a smooth map $U \to X \times Y$.) If the map $U \to Y$ is also a submersion, we will call the diagram $X \leftarrow U \to Y$ as *submersive span*.

Problem 3. Given submersive spans $X \leftarrow U \to Y$ and $Y \leftarrow V \to Z$, form the span $X \leftarrow W \to Z$ as the composition indicated by



where $W = U \times_Y V$ is the pullback of $U \to Y \leftarrow V$. Prove that $X \leftarrow W \to Z$ is a well-defined submersive span.

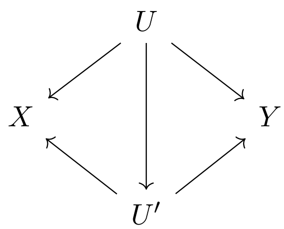
Problem 4. Under the above hypotheses, show that W is an embedded submanifold of $X \times \Delta_Y \times Z$ and compute its codimension.

We now attempt to construct a category $\mathsf{Span}^{\mathrm{sub}}$ whose objects are smooth manifolds and with $\mathsf{Span}^{\mathrm{sub}}(X,Y)$ the collection of submersive spans $X \leftarrow U \to Y$. The composition of submersive spans $X \leftarrow U \to Y$ and $Y \leftarrow V \to Z$ is given by the pullback submersive span $X \leftarrow U \times_Y Z \to Z$ of the previous problem.

Problem 5. What submersive span $X \leftarrow U \to X$ acts as the identity morphism on X?

Problem 6. Given composable spans $X \leftarrow U \rightarrow Y \leftarrow V \rightarrow Z \leftarrow W \rightarrow A$, observe that while there is a canonical natural isomorphism $(U \times_Y V) \times_Z W \cong U \times_Y (V \times_Z W)$, it is not the case that these manifolds are literally equal.

This means that $\mathsf{Span}^{\mathrm{sub}}$ is *not* a category! Instead, it's a bicategory, which is a certain flavor of weak 2-category. (I'll say some words about this during class.) Bicategories have 2-morphisms between morphisms, and this is no exception: a *2-morphism* between submersive spans $X \leftarrow U \rightarrow Y$ and $X \leftarrow U' \rightarrow Y$ is a smooth map $U \rightarrow U'$ making the diagram



commute. One way to extract a category from $\mathsf{Span}^{\mathrm{sub}}$ is to take $\mathsf{Span}^{\mathrm{sub}}(X, Y)$ to be the *isomorphism classes* of submersive spans. This is the perspective we'll take from here on.

We now turn to the relation between Diff, the category of smooth manifolds, and $\mathsf{Span}^{\mathrm{sub}}$. Each smooth map $Y \xrightarrow{f} X$ has an associated graph

$$\Gamma_f = \{(f(y), y) \mid y \in Y\} \subseteq X \times Y.$$

Problem 7. Check that $X \leftarrow \Gamma_f \rightarrow Y$ is a submersive span for all smooth maps $Y \xrightarrow{f} X$.

Problem 8. Prove that for composable smooth maps $Z \xrightarrow{g} Y \xrightarrow{f} X$, the submersive span $X \leftarrow \Gamma_f \times_Y \Gamma_g \rightarrow Z$ is isomorphic to the submersive span $X \leftarrow \Gamma_{f \circ g} \rightarrow Z$. Also show that identity maps are taken to identity spans, and thus $\Gamma \colon \mathsf{Diff}^{\mathrm{op}} \rightarrow \mathsf{Span}^{\mathrm{sub}}$ (taking a smooth manifold to itself and a smooth map to its graph span) is a functor.

Problem 9. In fact, Γ is a *faithful* functor which is *not full* (*i.e.*, it is injective but not surjective on morphism sets). Give an example of a submersive span which is *not* isomorphic to the graph of a smooth function.

Problem 10. In some applications, it is desirable to restrict submersive spans to those induced by submanifolds $U \subseteq X \times Y$ and restricting the projection maps onto each factor of $X \times Y$. In this case the composition (pullback) W of $X \leftarrow U \rightarrow Y$ and $Y \leftarrow V \rightarrow Z$ is naturally a submanifold of $X \times \Delta_Y \times Z$. Find conditions guaranteeing that the image of W under the projection $X \times \Delta_Y \times Z \rightarrow X \times Z$ is a submanifold. Use this to define an alternative category of submersive spans.