MATH 545: TOPOLOGY
WEDNESDAY WEEK 1

Given a group presentation

$$G = \langle x_1, x_2, \ldots, x_m \mid r_1, r_2, \ldots, r_n \rangle$$

we know how to build a 2-dimensional cell complex X_G (called the *presentation complex*) with

- one 0-cell,
- one 1-cell (necessarily a loop) for each generator x_i,
- one 2-cell for each relation r_j, glued onto the 1-skeleton according to its word.

For instance, if $G = \langle x, y \mid xyx^{-1}y^{-1} \rangle$ is the free Abelian group on two generators, then $(X_G)_1 = S^1 \vee S^1$ and X_G has a unique 2-cell with boundary traversing the figure-eight once in one orientation then once in the opposite orientation.

Recall that the presentation complex has a nice property:

$$\pi_1 X_G \cong G.$$

Indeed, we've set X_G up so that $\pi_1 X_G$ is generated by the loops in the 1-skeleton, $x_1, \ldots, x_m$, and the 2-cells witness the relations $r_1, \ldots, r_n$.

We will now construct a new 2-dimensional cell complex $\tilde{X}_G$ called the *Cayley complex* of G. It comes equipped with a map $\tilde{X}_G \to X_G$ whose properties will be of significant interest.

We begin by connecting some dots to form the *Cayley graph* Γ_G of G; this will ultimately be the 1-skeleton of the Cayley complex, $(\tilde{X}_G)_1 = \Gamma_G$.[1] The vertices of Γ_G are the elements of G, and there is a labeled directed edge $g \xrightarrow{x_i} h$ if and only if $h = gx_i$ where $x_i \in S$, the set of generators. In other words, the edges are of the form $g \xrightarrow{x_i} gx_i$.

Problem 1. Draw the Cayley graphs for the cyclic groups of order 2, 3, and ∞,[2]

$$C_2 = \langle x \mid x^2 \rangle, \qquad C_3 = \langle y \mid y^3 \rangle, \qquad C_\infty = \langle z \rangle.$$

To construct $\tilde{X}_G$, we need to glue some 2-cells to Γ_G. Here is the rule:

- For each $g \in G$ and relation r_j, attach a 2-cell based at g via the loop given by the word r_j.

Problem 2. Construct the Cayley complexes $\tilde{X}_{C_2}$, $\tilde{X}_{C_3}$, and $\tilde{X}_{C_\infty}$.

Problem 3. Observe that $\tilde{X}_G$ is simply connected (*i.e.* path connected with trivial fundamental group).

The Cayley complex $\tilde{X}_G$ admits a natural left action by G. For $h \in G$, we make the following definitions:

- on 0-cells $g \in G$, $h \cdot g = hg$,
- on 1-cells, $h \cdot (g \xrightarrow{x_i} gx_i) = hg \xrightarrow{x_i} hgx_i$,

---

[1] *Warning*: This construction depends on the chosen generators of G, so should perhaps be written $\Gamma_{G,S}$ for S a set of generators of G. If G is specified by a presentation, then we will always take S to be the given generators.

[2] Of course, these are isomorphic to $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$, and $\mathbb{Z}$, respectively, but this notation will help us remember the multiplicative notation.

· on 2-cells, if $g \in G$ and $r_j = xyz\cdots$, then the 2-cell attached along the loop

$$g \xrightarrow{x} gx \xrightarrow{y} gxy \xrightarrow{z} gxyz \to \cdots$$

is taken homeomorphically to the 2-cell attached along

$$hg \xrightarrow{x} hgx \xrightarrow{y} hgxy \xrightarrow{z} hgxyz \to \cdots .$$

Problem 4. Convince yourself that the above rules define a continuous left action of G on $\tilde{X}_G$.

Problem 5. Argue that the orbit space $\tilde{X}_G/G$ is homeomorphic to X_G, *i.e.*, the quotient of the Cayley complex by its G-action is the presentation complex.

We now have a natural quotient map

$$\tilde{X}_G \longrightarrow X_G.$$

Problem 6. Illustrate the quotient map $\tilde{X}_G \to X_G$ for $G = C_2, C_3, C_\infty$.

Problem 7. What general properties does $\tilde{X}_G \to X_G$ have? How is it similar to and different from $\mathbb{R} \to S^1$, $t \mapsto \exp(2\pi i t)$?

Problem 8. Illustrate the Cayley and presentation complexes for $C_2 * C_2 = \langle a, b \mid a^2, b^2 \rangle$ and the free group on two generators.