10. III. 23

Review

Normal bundles / tubular nbhds

$M \subseteq \mathbb{R}^n$ embedded submfld

$T_pM \subseteq T_p\mathbb{R}^3$

$N_pM := (T_pM)^\perp$

$T_p\mathbb{R}^n \cong T_pM \oplus N_pM$

$$\begin{array}{ccccccc} & & C_\bullet(U\cap V) & & C_\bullet(U)\oplus C_\bullet(V) & & C_\bullet(X) \\ & & \parallel & & \parallel & & \parallel \\ 0 & \to & C_\bullet & \to & D_\bullet & \to & E_\bullet & \to & 0 \end{array}$$

$\Rightarrow \exists$ natural LES

$$\cdots \to H_n C_\bullet \to H_n D_\bullet \to H_n E_\bullet \to H_{n-1} C_\bullet \to H_{n-1} D_\bullet \to H_{n-1} E_\bullet \to \cdots$$

$C_\bullet$ ch cpx:

$$\cdots \to C_{n+1} \xrightarrow{d} C_n \xrightarrow{d} C_{n-1} \xrightarrow{d} \cdots \xrightarrow{d} C_0$$

$d \circ d = 0$

$$H_n C_\bullet := \frac{\ker(d : C_n \to C_{n-1})}{\operatorname{im}(d : C_{n+1} \to C_n)}$$

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & C_{n+1} & \to & D_{n+1} & \to & E_{n+1} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & C_n & \to & D_n & \to & E_n & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \end{array}$$

$[e] \in H_{n+1} E.$ WTD $\partial[e] \in H_n C.$ Take $e \in \ker(d: E_{n+1} \to E_n)$ rep'ing $[e]$

$= \ker(d: E_{n+1} \to E_n) \,/\, \operatorname{im}(d: E_{n+2} \to E_{n+1})$

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1} \arrow[r] \arrow[d] & D_{n+1} \arrow[r] \arrow[d, "d"'] & E_{n+1} \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & C_n \arrow[r] & D_n \arrow[r] & E_n \arrow[r] & 0
\end{tikzcd}
$$

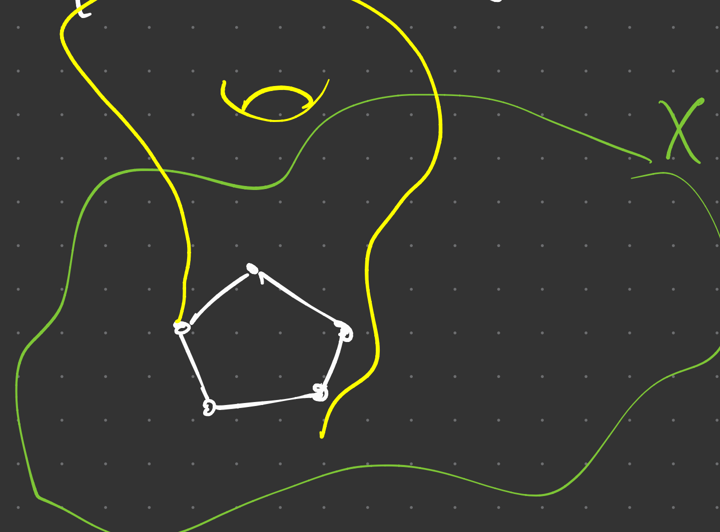
$$
\begin{tikzcd}
 & \delta \arrow[r, mapsto] \arrow[d] & e \arrow[d] \\
c \arrow[r, mapsto] & d(\delta) \arrow[r, mapsto] & 0
\end{tikzcd}
$$

Define $\partial[e] = [c]$

$$X \rightsquigarrow C_\bullet X \rightsquigarrow H_n X := H_n(C_\bullet X)$$

singular ch cpx of X

$$\begin{tikzcd} C_n X = \mathbb{Z}\{\sigma : \Delta^n \to X \mid \sigma \text{ cts}\} \arrow[d, "d"'] \\ C_{n-1} X \end{tikzcd}$$



ker d = "cycles"

im d = "boundaries"

$$X \xrightarrow{f} Y \rightsquigarrow C_\bullet(X) \xrightarrow{f_\#} C_\bullet(Y) \rightsquigarrow H_n X \xrightarrow[f_*]{} H_n Y$$

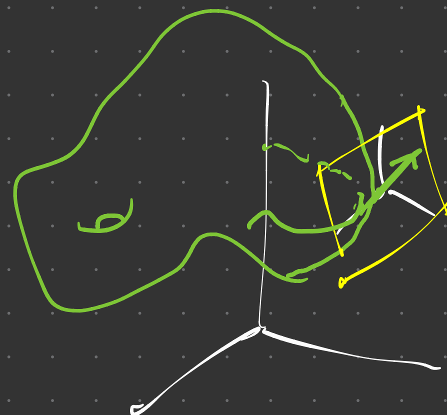
$$\begin{tikzcd} \sigma : \Delta^n \arrow[r] \arrow[dr, "f_\# \sigma"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

Hurewicz

$$\begin{tikzcd} \pi_1(X, x) \arrow[r] \arrow[dr] & H_1(X) = \pi_1 X^{ab} \arrow[d, dashed, "\exists!"] \\ & A \end{tikzcd}$$

$M \subseteq \mathbb{R}^n$

$T_pM \subseteq T_p\mathbb{R}^n = \mathbb{R}^n$



$S \subseteq \overset{\overset{\mathbb{R}^n}{\shortparallel}}{M}$ emb submfld

$T_pS \subseteq T_pM$

$T_pS = \{ v \in \underbrace{T_pM}_{\text{span}\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\}} \mid vf = 0 \;\forall f \in C^\infty(M) \text{ s.t. } f|_S = 0 \}$

$\mathbb{R}^n$

$$T_pS \subseteq T_p\mathbb{R}^n = \{\gamma'(0) \mid \gamma : J \to \mathbb{R}^n, \gamma(0) = p\}$$

$$\parallel$$

$$\{\gamma'(0) \mid \gamma : J \to S, \gamma(0) = p\}$$