8. III. 23

Submersive spans

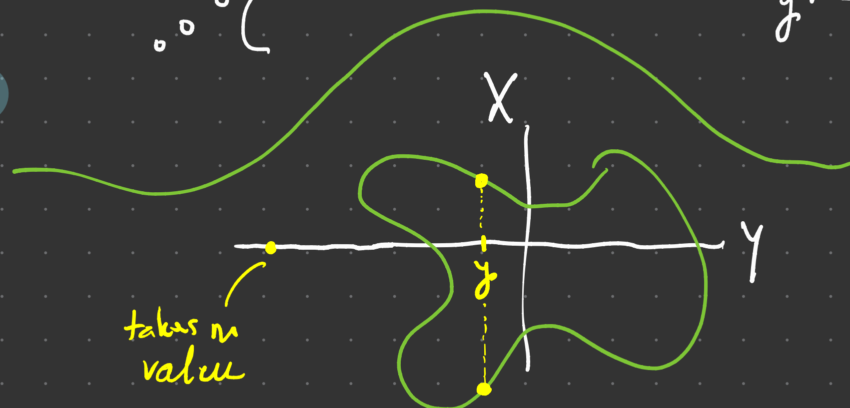
$V \subseteq X \times Y$ submanifold

$$\begin{tikzcd} & X\times Y \arrow[ddl, bend right, "\pi_1"'] \arrow[ddr, bend left, "\pi_2"] & \\ & V \arrow[u, phantom, "\cup"] \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

span

$\circ \circ \circ$ { V is a partially defined, multivalued function $Y \longrightarrow X$

$y \longmapsto \{x \mid (x,y) \in V\}$ }



Given spans
$$\begin{tikzcd}[column sep=small] & U \arrow[dl] \arrow[dr] & & V \arrow[dl] \arrow[dr] & \\ X & & Y & & Z \end{tikzcd}$$
we want to compose

via pullback
$$\begin{tikzcd}[column sep=small] & & W \arrow[dl] \arrow[dr] & & \\ & U \arrow[dl] \arrow[dr, "f"'] \arrow[rr, phantom, "\vee"] & & V \arrow[dl, "g"] \arrow[dr] & \\ X & & Y & & Z \end{tikzcd}$$

$W := U \times_Y V = \{(u,v) \in U \times V \mid f(u) = g(v)\}$

$\subseteq U \times \Delta_Y \times V$

In order for W to be a smooth mfld, need f, g to be transverse: $f \times g : U \times V \longrightarrow Y \times Y$ tranrverse to $\Delta_Y = \{(y,y) \mid y \in Y\}$.

(Cf. Problem 1 on handout / final practice #5.)

Prop If $f: U \to Y$ is a submersion then it is transverse to every smooth $g: V \to Y$.

Pf Need to show $f \times g: U \times V \to Y \times Y$ transverse to $\Delta_Y \subseteq Y \times Y$,

i.e. $\forall (u,v) \in (f \times g)^{-1} \Delta_Y = U \times_Y V$,

$$T_{(f(u), g(v))} Y \times Y = T_{(f(u), g(v))} \Delta_Y + d(f \times g)_{(u,v)} T_{(u,v)} U \times V$$

$$\overset{y = f(u) = g(v)}{\iff} \quad T_y Y \times T_y Y = \Delta_{T_y Y} + \underbrace{df_u T_u U}_{T_y Y \text{ since } f \text{ submersive}} \times dg_v T_v V$$

Know $T_y Y \times T_y Y \supseteq \Delta_{T_y Y} + T_y Y \times dg_v Y \supseteq \Delta_{T_y Y} + T_y Y \times 0$.

For $(v,w) \in T_y Y \times T_y Y$, $(w,w) + (v-w, 0) = (v,w) \in RHS$ ✓ $\square$

Have $W \subseteq X \times \Delta_Y \times Z$, not $X \times Z$.

$\exists$ conditions s.t. the proj'n of W is a submfld of $X \times Z$ (cf. Prob 10) but its easier to generalize our notion of span to all diagrams

$$\begin{array}{ccc} & U & \\ \swarrow & & \searrow \\ X & & Y \end{array}$$

of smooth maps s.t. $U \to Y$ is a submersion.

$$X \xleftarrow{f} Y \quad \rightsquigarrow \quad \Gamma_f = \{(f(y), y) \mid y \in Y\} \subseteq X \times Y$$

$$\begin{array}{ccc} & X \times Y & \\ \swarrow & & \searrow \\ \underline{X} & & \underline{Y} \end{array}$$

Symplectic geom: Lagrangian correspondence

Weinstein