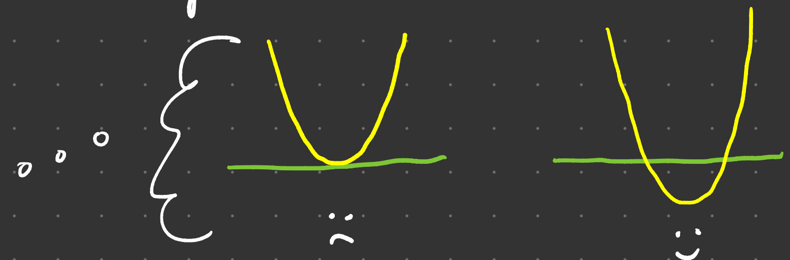
{2.III.23}

Transversality



Defn • Smooth mfld M, emb submflds $S, S' \subseteq M$ *intersect transversely* when $\forall p \in S \cap S'$, $T_pM = T_pS + T_pS'$.

• $F : N \to M$ smooth, emb submfld $S \subseteq M$ then F is *transverse* to S when $\forall x \in F^{-1}S$, $T_{F(x)}M = T_{F(x)}S + dF_x T_x N$.

Note
- Submersions are transverse to everything
- $S, S' \subseteq M$ transverse iff ι_S transverse to S'.
- $F: N \longrightarrow M$ transverse to $p \in M$ (viewed as a submfld) when dF_x surj $\forall x \in F^{-1}p$?

Thm N, M sm mflds, $S \subseteq M$ emb submfld

(a) If $F: N \longrightarrow M$ is smooth & transverse to S, then $F^{-1}S \subseteq N$ is an emb submfld with $\operatorname{codim}_N F^{-1}S = \operatorname{codim}_M S$.

(b) If $S' \subseteq M$ is an emb submfld intersecting S transversely then $S \cap S'$ is an emb submfld of M of codim $= \operatorname{codim}_M S + \operatorname{codim}_M S'$.

Pf (a) $\Rightarrow$ (b) by taking $F = \iota_{S'}$.

(a) Let $m = \dim M$, $k = \operatorname{codim}_M S$. For $x \in F^{-1}S$, take U nbhd of $F(x)$ and local defining fn $\varphi : U \to \mathbb{R}^k$ for S with $U \cap S = \varphi^{-1} 0$.

WTS 0 is a regular value of $\varphi \circ F$ (whence $F^{-1}S \cap F^{-1}U = (\varphi \circ F|_{F^{-1}U})^{-1} 0$ and done by reg level set thm).

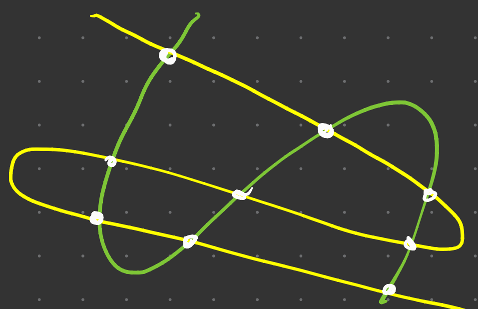
For $z \in T_0 \mathbb{R}^k$ & $p \in (\varphi \circ F)^{-1} 0$, φ regular $\Rightarrow \exists\, y \in T_{F(p)} M$ s.t. $d\varphi_{F(p)}(y) = z$. Since F transverse to S, we can write $y = y_0 + dF_p(v)$ for some $y_0 \in T_{F(p)} S$, $v \in T_p N$.

B/c φ const on $S \cap U$, $d\varphi_{F(p)}(y_0) = 0$. By chain rule

$$
\begin{aligned}
d(\varphi \circ F)_p(v) &= d\varphi_{F(p)}\big(dF_p(v)\big) \\
&= d\varphi_{F(p)}\big(y_0 + dF_p(v)\big) \\
&= d\varphi_{F(p)}(y) \\
&= z. \quad \checkmark
\end{aligned}
$$

$\square$

E.g.



Q Can two curves in $\mathbb{R}^3$ intersect transversely?

A smooth family of maps $F_s : N \to M$, $s \in S$ is a smooth map $F : N \times S \longrightarrow M$ with $F(-,s) = F_s$.

smooth mfld

Note If S is conn'd, then $F_s \simeq F_t$ $\forall s,t \in S$. Indeed, may restrict F to a path from s to t to get a htpy.

Thm (Parametric Transversality) N, M smooth mflds, $X \subseteq M$ emb submfld, $\{F_s : N \to M \mid s \in S\}$ smooth family of maps. If $F : N \times S \to M$ is transverse to X, then almost every F_s is transverse to X.

Pf $W = F^{-1}X \subseteq N\times S$ emb submfld. Let $\pi = \pi_2 : N\times S \to S$.

Claim $\{s\in S \mid s \text{ is a regular value of } \pi|_W\}$

$\subseteq \{s\in S \mid F_s \text{ transverse to } X\}$

Then LHS $\underbrace{\text{"full measure"}}_{\text{complement measure } 0}$ $\Rightarrow$ RHS full measure.

— true by Sard

Suppose $s\in S$ is a reg value of $\pi|_W$. For $p\in F_s^{-1}X$ arbitrary, set $q = F_s(p) \in X$. WTS $T_qM = T_qX + d(F_s)_p T_pN$.

- Know $T_qM \overset{1}{=} T_qX + dF_{(p,s)} T_{(p,s)}(N\times S)$ b/c F transverse to X.

- Also $T_sS \overset{2}{=} d\pi_{(p,s)} T_{(p,s)}(N\times S)$ b/c s reg value of $\pi|_W$.

- By HW (ISM 6-10), $T_{(p,s)}W = dF_{(p,s)}^{-1} T_q X$

$$\Rightarrow dF_{(p,s)} T_{(p,s)} W = T_q X.$$

For $w \in T_q M$ arbitrary, need $v \in T_q X$, $y \in T_p N$ s.t.

$$w \overset{\circledast}{=} v + d(F_s)_p(y).$$

By ①, $\exists\, v_1 \in T_q X$, $(y_1, z_1) \in T_p N \times T_s S \cong T_{(p,s)}(N \times S)$ s.t.

$$w = v_1 + dF_{(p,s)}(y_1, z_1)$$

Choose (by ②), $(y_2, z_2) \in T_{(p,s)} W$ with $d\pi_{(p,s)}(y_2, z_2) = z_1$.

Have $z_2 = z_1$ since π is proj'n. By linearity,

$$dF_{(p,s)}(y_1, z_1) = dF_{(p,s)}(y_2, z_1) + dF_{(p,s)}(y_1 - y_2, z_1)$$

Claim ⊛ satisfied by $v = v_1 + dF_{(p,s)}(y_2, z_1)$

$$y = y_1 - y_2 .$$

(check this!) $\square$

Thm (Transverse up to homotopy) M, N sm mflds, $X \subseteq M$ emb submfld. Every smooth map $f: N \to M$ is htpic to a smooth map $g: N \to M$ that is transverse to X.

Pf Suffices to construct $F: N \times \mathbb{B}_k \to M$ transverse to X with $F_0 = f$. (Why?)

By Whitney embedding, $M \underset{\text{emb}}{\hookrightarrow} \mathbb{R}^k$ for some k.

Let U be a tubular nbhd of M in $\mathbb{R}^k$, $r: U \to M$ smooth retraction + submersion. Take $\delta: M \to \mathbb{R}_{>0}$

$$x \longmapsto \sup\{\varepsilon \leq 1 \mid B_\varepsilon(x) \subseteq U\}$$

$\exists$ smooth $e: N \to \mathbb{R}_{>0}$ with $0 < e(p) < \delta(f(p))$ $\forall p \in N$.

unit ball in $\mathbb{R}^k$

Define $F: N \times \mathbb{B}_k \to M$

$$(p, s) \longmapsto r(\underbrace{f(p) + e(p)s}_{})$$

$\in \mathcal{U}$ b/c $|e(p)s| < e(p) < \delta(f(p))$

F is smooth and $F_0 = f$ b/c r is a retraction

For $p \in N$, $F|_{\{p\} \times B_k}$ is the local diffeo $s \mapsto f(p) + e(p)s$

followed by submersion r hence it is transverse to X. $\square$