3.III.23

- Whitney approx gave δ-close smooth approx'n to any cts $F: M \longrightarrow \mathbb{R}^k$.
- For $F: M \longrightarrow N$ cts, use Whitney embedding to view N inside $\mathbb{R}^k$.
- But now the smooth approx'n will miss $N \subseteq \mathbb{R}^k$ by a bit.
- Thus we need

Tubular neighborhoods

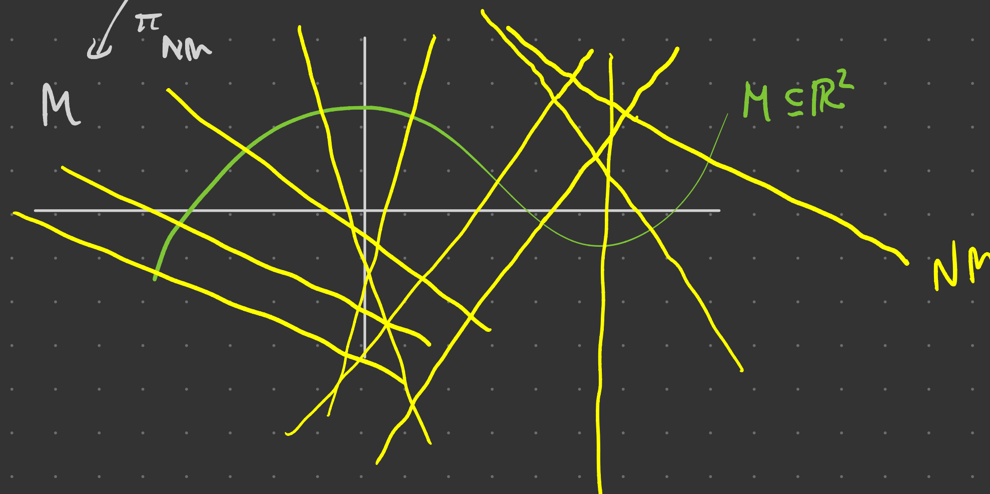
For $M \subseteq \mathbb{R}^n$ smoothly embedded, define $N_x M \subseteq T_x \mathbb{R}^n$ to be
$(T_x M)^\perp = \{(x,v) \in T_x \mathbb{R}^n \mid v \cdot w = 0 \ \forall (x,w) \in T_x M\}$

(Here using the canonical id'n $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$.)

The **normal bundle** $NM = \{(x,v) \in T\mathbb{R}^n \mid x \in M,\ v \in N_x M\}$

$$\downarrow \pi_{NM}$$

M



Thm If $M \subseteq \mathbb{R}^n$ is an emb n-dim'l smooth mfld, then NM is an embedded n-dim'l submfld of $T\mathbb{R}^n$.

__Pf__ $x_0 \in M$, (U, φ) slice chart for M in $\mathbb{R}^n$ centered at x_0

$$\begin{tikzcd} M & U \arrow[r, "\varphi"] \arrow[dl, phantom, "\subseteq" sloped] & \hat{U} \subseteq \mathbb{R}^n \\ \mathbb{R}^n \arrow[u, phantom, "\cup" description] & & \end{tikzcd}$$

$\hat{U}$: $u^1, \dots, u^n$ $\qquad$ $\mathbb{R}^n$: $x^1, \dots, x^n$

$$M \cap U = \{u^{m+1} = \dots = u^n = 0\}$$

For $x \in U$, $E_j|_x := (d\varphi_x)^{-1}\left(\partial/\partial u^j|_{\varphi(x)}\right)$ form a basis for $T_x\mathbb{R}^n$.

Write $E_j|_x = \sum_i E_j^i(x) \dfrac{\partial}{\partial x^i}\Big|_x$ for $E_j^i(x)$ smooth in x.

Define $\Phi : U \times \mathbb{R}^n \longrightarrow \hat{U} \times \mathbb{R}^n$

$$(x, v) \longmapsto (u^1(x), \dots, u^n(x), v \cdot E_1|_x, \dots, v \cdot E_n|_x)$$

Then $J\Phi_{(x,v)} = \left(\begin{array}{c|c} \frac{\partial u^i}{\partial x^j}(x) & 0 \\ \hline * & E_j^i(x) \end{array}\right)$ which is invertible, so Φ is a local diffeo.

If $\Phi(x,v) = \Phi(x',v')$, then $x = x'$ b/c φ inj,

and $v \cdot E_i|_x = v' \cdot E_i|_x \ \forall i \Rightarrow v - v' \perp \operatorname{span}\{E_1|_x, \dots, E_n|_x\}$

$\Rightarrow v - v' = 0$.

Thus Φ is injective $\Rightarrow$ defines smooth coord chart on $U \times \mathbb{R}^n$.

Have $(x,v) \in NM \cap U \times \mathbb{R}^n$ iff $\Phi(x,v) \in \left\{ (y,z) \in \mathbb{R}^n \times \mathbb{R}^n \;\middle|\; \begin{array}{l} y^{m+1} = \cdots = y^n = 0, \\ z^1 = \cdots = z^m = 0 \end{array} \right\}$

$\Rightarrow \Phi$ is a slice chart for NM in $T\mathbb{R}^n$. $\square$

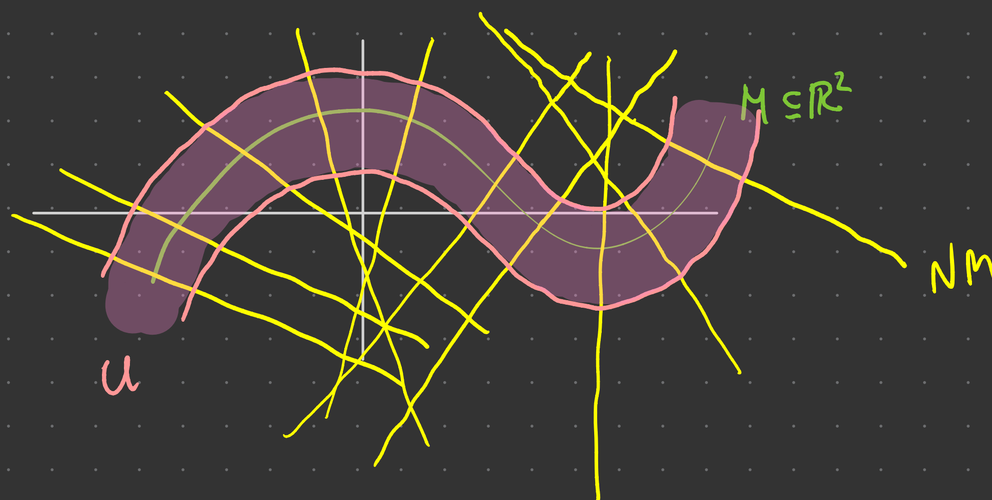
Define $E : NM \longrightarrow \mathbb{R}^n$ so that $E(\{x\} \times N_xM) =$ affine space thru x, $\perp T_xM$.

$(x,v) \longmapsto x + v$

A *tubular neighborhood* of $M \subseteq \mathbb{R}^n$ is a nbhd U of M in $\mathbb{R}^n$

s.t. $U = E\underbrace{\{(x,v) \in NM \mid |v| < \delta(x)\}}_{V}$ for some cts $\delta : M \to \mathbb{R}_{>0}$,

E a diffeo $V \to U$.



Thm Every $M \subseteq \mathbb{R}^n$ embedded submfld has a tubular neighborhood.

Pf pp. 139–140. $\square$

Recall that a retraction of a space X onto a subspace $M \subseteq X$ is $r : X \longrightarrow M$ cts s.t. $r|_M = id_M$.

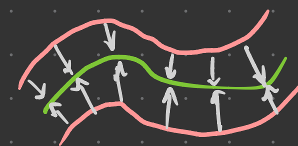
Prop $M \subseteq \mathbb{R}^n$ emb submfld, U a tubular nbhd of M. Then $\exists$ a smooth map $r : U \longrightarrow M$ that is a retraction & a smooth submersion.

Pf Define $M_0 \subseteq NM \subseteq T\mathbb{R}^n$ by $M_0 = \{(x,0) \mid x \in M\}$.

Have $M_0 \subseteq \underset{\text{open}}{V} \subseteq NM$ with $E : V \longrightarrow U$ diffeo.

Define $r : U \longrightarrow M$. Then r is smooth by comp'n.

$$\begin{tikzcd} U \arrow[rr, "r"] \arrow[dr, "E^{-1}"'] & & M \\ & V \arrow[ur, "\pi"'] & \end{tikzcd}$$



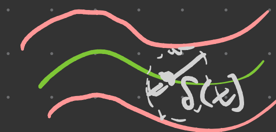
For $x \in M$, $r(x) = \pi E^{-1}(x) = \pi(x, 0) = x$, so r is a retraction.

π smooth submersion + E^{-1} diffeo $\Longrightarrow$ r smooth submersion. $\square$

Thm (Whitney Approximation) N a smooth mfld w/ or w/o ∂, M a smooth mfld (no ∂), and $F : N \longrightarrow M$ cts. Then F is homotopic to a smooth map. If F is already smooth on $A \subseteq N$ closed, then the htpy can be taken rel A.

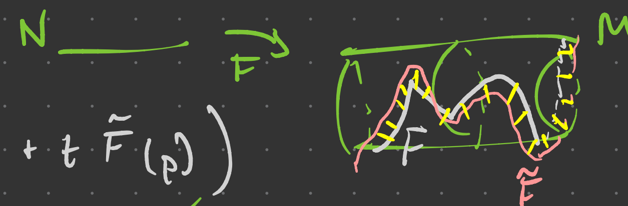
Pf By Whitney embedding, may assume $M \subseteq \mathbb{R}^n$ properly embedded. Take $\mathcal{U}$ a tubular nbhd of M in $\mathbb{R}^n$, and let $r : \mathcal{U} \longrightarrow M$ be the smooth retraction of the prop'n.

For $x \in M$, let $\delta(x) = \sup\{\varepsilon \leq 1 \mid B_\varepsilon(x) \subseteq \mathcal{U}\}$.



δ is cts (Δ-ineq....) as a fn $M \longrightarrow \mathbb{R}_{>0}$.

Let $\tilde{\delta} = \delta \circ F : N \longrightarrow \mathbb{R}_{>0}$. Know $\exists$ smooth $\tilde{F} : N \longrightarrow \mathbb{R}^n$ that is $\tilde{\delta}$-close to F & equal to F on A.



Let $H : N \times I \longrightarrow M$

$$(p,t) \longmapsto r\Big(\underbrace{(1-t)F(p) + t\tilde{F}(p)}_{\text{straight line htpy } F(p) \rightsquigarrow \tilde{F}(p)}\Big)$$

For $p \in N$, $|\tilde{F}(p) - F(p)| < \tilde{\delta}(p) = \delta(F(p)) \Longrightarrow \tilde{F}(p) \in B_{\delta(F(p))}(F(p))$

$\subseteq U \Longrightarrow$ whole line segment $F(p) \rightsquigarrow \tilde{F}(p) \subseteq U$.

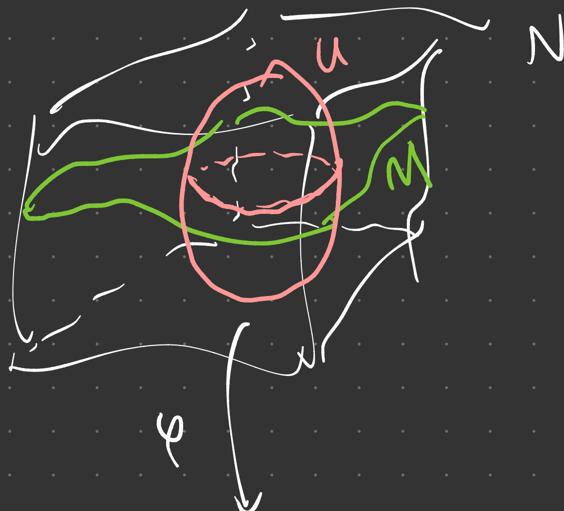
Thus H is well-defined htpy $F \Rightarrow \underbrace{r \circ \tilde{F}}$

smooth by comp'n.

For $p \in A$, $H(p,t) = F(p)$ since $\tilde{F}|_A = F$. $\square$

See pp. 142-143 for how to use these ideas to convert htpies b/w smooth maps into smooth htpies.

$\underset{m}{M} \subseteq \underset{n}{N}$ emb submfld



$\hat{U} \subseteq \mathbb{R}^n$

$u^1, \ldots, u^n$

$M \cap U = \{ u^{m+1} = \cdots = u^n = 0 \}$

$$\begin{array}{ccc}
M & \hookrightarrow & N \\
\cup\!\! \mid & & \cup\!\! \mid \\
M \cap U & & U \\
{\scriptstyle \varphi|}\downarrow & & \downarrow{\scriptstyle \varphi} \\
\widehat{M \cap U} & \hookrightarrow & \hat{U} \subseteq \mathbb{R}^n
\end{array}$$

$(u^1, \ldots, u^m) \longmapsto (u^1, \ldots, u^m, 0, \ldots, 0)$

Fig 8 immersion which is not an embedding

$$\beta : (-\pi, \pi) \longrightarrow \mathbb{R}^2$$
$$t \longmapsto (\sin(2t), \sin t)$$

$\mathrm{Im}(\beta) =$ 

In subspace topology,

$\mathrm{Im}(\beta) \subseteq \mathbb{R}^2$ is __not__ loc Euclidean

so not a mfld.

$$\begin{tikzcd} F : M \arrow[r] \arrow[dr] & N \\ & S \arrow[u, phantom, sloped, "\subseteq"] \end{tikzcd}$$ smooth

smooth $\iff$ $M \longrightarrow S$ is cts

$$\begin{tikzcd} G : \mathbb{R} \arrow[r] & \mathbb{R}^2 \\ (-\pi, \pi) \arrow[u] \arrow[ur, "\beta"'] & \end{tikzcd}$$

$G : \mathbb{R} \longrightarrow \operatorname{Im} \beta$

no longer cts !!

immersed submfld — not w/ subspace topology