1. III. 23

Whitney embedding

(Abstract) smooth mflds:
top'l mfld M
+ smooth structure (atlas)

=

Embedded smooth mflds:
$M \subseteq \mathbb{R}^N$
+ $\forall x \in M\ \exists W \subseteq \mathbb{R}^N$ nbhd of x
s.t. $W \cap M \approx \underset{\text{open}}{U} \subseteq \mathbb{R}^n$

Whitney (including bounds on N in terms of n)

Thm ((weak) Whitney embedding) Any smooth mfld of dimn n admits an embedding into $\mathbb{R}^{2n+1}$.

(strong version improves this to $\mathbb{R}^{2n}$ for $n \geq 2$
(why knot $n=1$?))

Strategy:
- M embeds in $\mathbb{R}^N$ for some $N \gg n$.
- If an n-mfld embeds in $\mathbb{R}^N$ for $N > 2n+1$, then it also embeds in $\mathbb{R}^{N-1}$.

In class we will cover the "easy" case, M compact. See text for extension to all n-mflds. With M compact, it suffices to produce an immersion $M \hookrightarrow \mathbb{R}^{2n+1}$.

Lemma Any compact smooth mfld M admits an injective immersion into $\mathbb{R}^N$ for $N \gg 0$.

Pf Take $\{(U_i, \varphi_i) \mid 1 \le i \le k\}$ a finite set of smooth charts covering M. Let $\{\rho_i \mid 1 \le i \le k\}$ be a part'n of unity subordinate to $\{U_i \mid 1 \le i \le k\}$. Define

$$\Phi : M \longrightarrow \mathbb{R}^{k(n+1)}$$

$$p \longmapsto \big(\underbrace{\rho_1(p)\varphi_1(p), \dots, \rho_k(p)\varphi_k(p)}_{\in (\mathbb{R}^n)^k}, \underbrace{\rho_1(p), \dots, \rho_k(p)}_{\in \mathbb{R}^k}\big)$$

First check Φ is injective:

If $\Phi(p) = \Phi(q)$, take index i s.t. $\rho_i(p) = \rho_i(q) \neq 0$. Then $p, q \in \operatorname{supp}(\rho_i) \subseteq U_i \implies \varphi_i(p) = \varphi_i(q)$. Thus $p = q$ since φ_i is bijective. ✓

Now check $d\Phi_p$ injective $\forall p \in M$:

For $v \in T_pM$, the Leibniz rule implies

$$d\Phi_p(v) = \big(v(\rho_1)\varphi_1(p) + \rho_1(p)(d\varphi_1)_p(v), \dots,$$
$$v(\rho_k)\varphi_k(p) + \rho_k(p)(d\varphi_k)_p(v),\ v(\rho_1), \dots, v(\rho_k) \big).$$

Thus if $d\Phi_p(v) = 0$, then $v(\rho_i) = 0$ for $1 \le i \le k \implies \rho_i(p)(d\varphi_i)_p(v) = 0$ for $1 \le i \le k$. Pick i s.t. $\rho_i(p) \neq 0$. Then $(d\varphi_i)_p(v) = 0$. Since φ_i is a diffeo, get $v = 0$ so $d\Phi_p$ injective. $\square$

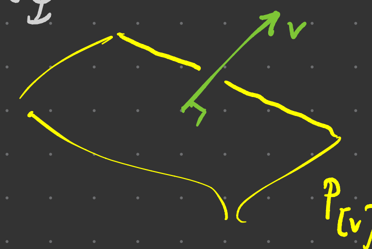
Note Above proof works for any smooth mfld covered by fin many smooth charts.

Lemma If a smooth mfld M of dimension n admits an injective immersion into $\mathbb{R}^N$ for some $N > 2n+1$, then it admits an immersion into $\mathbb{R}^{N-1}$.

Pf Suppose $\Phi : M \longrightarrow \mathbb{R}^N$ an injective immersion with $N > 2n+1$.

Idea: Look at all linear projections $\pi : \mathbb{R}^N \longrightarrow \mathbb{R}^{N-1}$. Use Sard to show almost all compose with Φ to give an injective immersion into $\mathbb{R}^{N-1}$.

For $[v] \in \mathbb{RP}^{N-1}$, let $P_{[v]} := \{ u \in \mathbb{R}^N \mid u \cdot v = 0 \} \cong \mathbb{R}^{N-1}$ be the hyperplane $\perp [v]$. Let $\pi_{[v]} : \mathbb{R}^N \longrightarrow P_{[v]}$ denote orthogonal proj^n onto $P_{[v]}$ and set $\Phi_{[v]} := \pi_{[v]} \circ \Phi$



Claim $\{ [v] \in \mathbb{RP}^{N-1} \mid \Phi_{[v]}$ not an immersion$\}$ has measure 0.

(1) For which $[v]$ is $\Phi_{[v]}$ not injective?

If so, $\exists\, p \neq q \in M$ s.t. $\Phi_{[v]}(p) = \Phi_{[v]}(q) \Rightarrow \Phi(p) - \Phi(q) \in [v]$, i.e. $[v] = [\Phi(p) - \Phi(q)]$. Thus $[v]$ is in the image of smooth map $\alpha : (M \times M) \setminus \Delta_M \longrightarrow \mathbb{RP}^{N-1}$, $\alpha(p, q) = [\Phi(p) - \Phi(q)]$.

diagonal $\{ (p,p) \mid p \in M \}$

Domain of α is a smooth mfld of dimn $2n < N-1$, so by Sard's theorem, $\mu(\operatorname{im}\alpha) = 0$. Thus $\mu(\{[v] \mid \Phi_{[v]} \text{ not inj}\}) = 0$.

$$\{[v] \mid \Phi_{[v]} \text{ not inj}\} \subseteq \operatorname{im}\alpha.$$

(2) Now consider $[v]$ with $\Phi_{[v]}$ having non-injective diffl.

Then $\exists\, p \in M$ and $0 \neq w \in T_pM$ s.t. $(d\Phi_{[v]})_p(w) = 0$, i.e.

$$\underbrace{(d\pi_{[v]})_{\Phi(p)}}_{= \pi_{[v]} \text{ b/c } \pi_{[v]} \text{ linear}}\Big((d\Phi_p)(w)\Big) = 0$$

Thus $0 \neq d\Phi_p(w) \in [v]$, i.e. $[v] = [d\Phi_p(w)]$.

It follows that $[v]$ is in the image of smooth

$$\beta : TM \setminus 0_M \longrightarrow \mathbb{RP}^{N-1}$$

$$(p, w) \longmapsto [d\Phi_p(w)]$$

image of 0-section $\{(p,0) \mid p \in M\}$

so $TM \setminus 0_M$ open submfld of TM, dimn $2n$

Since $\dim TM \setminus 0_M = 2n < N-1$, Sard's theorem implies

$$\mu(\operatorname{im}\beta) = 0 \Rightarrow \mu(\{[v] \mid \Phi_{[v]} \text{ has a non-inj diffl}\}) = 0.$$

$$\bigcap\!\!\!\!\!\!\; \text{im}\,\beta$$

The union of measure 0 sets has measure 0, so $\Phi_{[v]}$ is an immersion for $\underbrace{\text{almost all}}_{\text{outside set of measure } 0}$ $[v]$! $\square$

Whitney Approximation

$\delta : M \to \mathbb{R}_{>0}$ cts

call $F, \tilde{F} : M \to \mathbb{R}^k$ **δ-close** if $|F(x) - \tilde{F}(x)| < \delta(x)$ $\forall x \in M$.

Thm (Whitney approximation for fns) $F : M \to \mathbb{R}^k$ cts.
smooth mfld

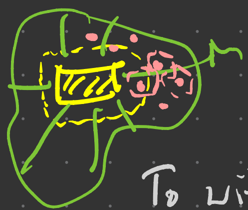
Given $\delta : M \to \mathbb{R}_{>0}$ cts, $\exists$ smooth $\tilde{F} : M \to \mathbb{R}^k$ that is δ-close to F. If F is smooth on $A \subseteq M$ closed, then $\tilde{F}$ can be chosen to be equal to F on A.

Pf If F smooth on $A \subseteq M$ closed, $\exists F_0 : M \to \mathbb{R}^k$ smooth and agreeing with F on A.

Let $U_0 = \{y \in M \mid |F_0(y) - F(y)| < \delta(y)\}$. Then $A \subseteq U_0 \underset{\text{open}}{\subseteq} M$.

<u>Claim</u> $\exists$ countably many pts $\{x_i\}_{i=1}^{\infty}$ in $M \setminus A$ and nbhds $U_i \subseteq M \setminus A$ of x_i s.t. $\{U_i\}_{i=1}^{\infty}$ is an open cover of $M \setminus A$ and

$$|F(y) - F(x_i)| < \delta(y) \quad \forall y \in U_i.$$



To wit, for any $x \in M \setminus A$ let U_x be a nbhd of x contained in $M \setminus A$ s.t.

$$\delta(y) > \tfrac{1}{2}\delta(x) \text{ and } |F(y) - F(x)| < \tfrac{1}{2}\delta(x) \quad \forall y \in U_x$$

(using continuity of δ, F). Then $\{U_x \mid x \in M \setminus A\}$ is an open cover of $M \setminus A$ with countable subcover. ✓claim

Now let $\{\varphi_0\} \cup \{\varphi_i\}_{i=1}^{\infty}$ be a smooth POU sub to $\{U_0\} \cup \{U_i\}_{i=1}^{\infty}$.

Define $\tilde{F} : M \longrightarrow \mathbb{R}^k$

$$y \longmapsto \varphi_0(y) F_0(y) + \sum_{i=1}^{\infty} \varphi_i(y) F(x_i)$$

$\tilde{F}$ is smooth & equal to F on A. For $y \in M$, $\sum_{i \geq 0} \varphi_i = 1 \Longrightarrow$

$$\left| \tilde{F}(y) - F(y) \right| = \left| \varphi_0(y) F_0(y) + \sum_{i \geq 1} \varphi_i(y) F(x_i) - \underbrace{\left(\varphi_0(y) + \sum_{i \geq 1} \varphi_i(y) \right)}_{1} F(y) \right|$$

$$\leq \varphi_0(y) \left| F_0(y) - F(y) \right| + \sum_{i \geq 1} \varphi_i(y) \left| F(x_i) - F(y) \right|$$

$$< \varphi_0(y) \delta(y) + \sum_{i \geq 1} \varphi_i(y) \delta(y) = \delta(y). \quad \square$$

Hassler Whitney
1907-89

14 years old
in Swiss Alps

Whitney-Gilman Ridge
Cannon Mtn, NH